Ioana-Claudia Lazăr

The Study of Simplicial Complexes of Non-Positive Curvature

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Contents

Introduction

The book represents the author's PhD thesis entitled "The study of simplicial complexes of non-positive curvature" written under the guidance of Prof. Dr. Dorin Andrica at the Faculty of Mathematics and Computer Science, "Babes-Bolyai" University, Cluj-Napoca, Romania. The PhD thesis was sustained in public on the 11th of December 2009.

The book is naturally divided into two parts. In the first part we introduce the notion of discrete Morse-Smale characteristic of a finite simplical complex and we define exact and $\mathbb{F}$-perfect discrete Morse functions on a finite simplicial complex, $\mathbb{F}$ being any coefficient field. Further we give a few examples of $\mathbb{F}$-perfect discrete Morse functions on finite simplicial complexes (see [4], [30]).

In the second part of the book we investigate metric and combinatorial conditions which guarantee the collapsibility of a finite cell complex of dimension 2 and 3.

Namely, we find combinatorial Cartan-Hadamard theorems on finite cell complexes of dimension 2 and 3. The classical version of the theorem states that simply connected, non-positively curved spaces are contractible. Replacing the smooth setting by the discrete one, the notion of contractibility is replaced by the one of collapsibility. The two notions are not equivalent. Although collapsible cell complexes are always contractible, contractible cell complexes are not necessarily collapsible. Bing's house with two rooms, for instance, is a contractible cell complex because it is a triangulation of the 3-ball, but it is not collapsible since every edge is contained in at least two 2-cells.

One metric condition we consider is given by the CAT(0) inequality. A geodesic metric space is a CAT(0) space if geodesic triangles are thinner than comparison triangles in Euclidean space (see [2], [3], [12], [13], [14], [23], [10], [11]). Another metric condition we have in mind is given by the strongly convex metric (see [35], [12], [33], [34]).

The combinatorial condition we consider is given by the 6-property of a simplicial complex (see [16], [17], [31], [21], [22]). A 2- dimensional simplicial complex has the 6-property if the link of each vertex is a graph of girth at least 6. The *girth* of a graph is defined as the minimum number of edges in a circuit. The 8-property for 2-dimensional square complexes and the 12-property for 2-dimensional hexagon complexes can be defined similarly. In higher dimensions, the 6-property is called local 6-largeness (see [24], [25]).

In dimension 2, the 6-property (8-property, 12-property) coincides with the CAT(0) property of the standard piecewise Euclidean metric on a simplicial (cubical, hexagonal) complex. This holds since a two-dimensional polyhedral space is a space of nonpositive curvature if and only if the link of each vertex does not contain a subspace isometric to a circle of length less than 2π (see [13], chapter 4.2, page 113; see [12], chapter II.5, page 207). Hence the standard piecewise Euclidean metric structure on a 2-dimensional simplicial (cubical, hexagonal) complex is nonpositively curved if and only if the link of each vertex of the complex has girth at least 6 (8, 12). We emphasize that the equivalence no longer holds in higher dimensions (see [24]).

The collapsibility of finite simplicial complexes was studied before. W. White showed in [35] that finite, strongly convex 2-complexes collapse to a point and that finite, strongly convex 3-complexes collapse to a 2-dimensional spine. His proof is based on the definition of an elementary collapse. So in dimension 2, the collapsibility of a finite simplicial complex can be ensured by a metric condition. In dimension 3, however, the metric condition alone does not guarantee the collapsibility of the complex to a point. Still, we will show that, by adding a combinatorial curvature condition, called local 6-largeness, a finite, strongly convex 3-complex can also be simplicially collapsed to a point. A proof of this result is one of the paper's objects (see [7]). An important step in our proof is to investigate whether the subcomplex obtained by performing an elementary collapse on a finite, 6-large complex of dimension 2 (3), remains 6-large.

Besides W. White, D. Rolfsen also studied spaces that admit a strongly convex metric (see [33], [34]). Namely, he proved that any finite, strongly convex and without ramification 3-complex, is homeomorphic to the 3-ball.

The collapsibility of a finite 2-complex can be ensured by another metric condition, different from the one in [35]. No combinatorial condition is necessary to prove that a finite 2-complex with a CAT(0) metric, is collapsible. A proof of this result is another of the paper's objects (see [28]). A key step in our proof is to show that any finite, CAT(0) 2-complex retracts to a point through subspaces which are, at each step, CAT(0) spaces. Our proof relies on the definition of an elementary collapse.

J. Corson and B. Trace proved that a finite, simply connected, 2-dimensional simplicial complex that has the 6-property, collapses to a point (see [17]). Their proof uses van Kampen diagrams (see [31]). Hence a combinatorial condition also suffices to show that finite, 2-dimensional simplicial complexes collapse to a point.

K. Crowley proved in [18] that a finite simplicial complex of dimension 3 or smaller endowed with the standard piecewise Euclidean metric that is nonpositively curved, simplicially collapses to a point, under a technical hypothesis. Namely, every 2-simplex in the complex must be the face of at most two 3-simplices in the complex. The main tool used in her proof is discrete Morse theory (see [19], [20]). The condition which guarantees the collapsibility of the 2-complex in [18] seems metric, although it is in fact combinatorial. Namely, endowing the 2-complex with the corresponding standard piecewise Euclidean metric such that each interior vertex of the complex has degree at least 6, the naturally associated piecewise Euclidean metric on the 2-complex becomes CAT(0).

The results we obtain in the last two sections of the book are based on regular tessellations of the Euclidean plane. There are only three regular tessellations of the Euclidean plane: by triangles, squares, and hexagons. Let p and q be positive real numbers such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$. As is well known, the only positive integral solutions this equation has, are $(3,6)$, $(4,4)$, and $(6,3)$. It is hence possible to subdivide the Euclidean plane into regular p-gons such that each vertex is q-valent if and only if (p,q) is one of the pairs $(3,6)$, $(4,4)$, or $(6,3)$.

J. Corson and B. Trace used in [17] the first possibility of subdividing the Euclidean plane (by triangles) when proving the collapsibility of a finite, simply connected, 2-dimensional simplicial complex with the 6-property. K. Crowley used in [18] the same method of subdividing the Euclidean plane when showing the collapsibility of a finite, CAT(0) simplicial complex of dimension 3 or less. We focus in this paper on the other two possibilities of subdividing the Euclidean plane (by squares, and hexagons) and obtain similar results (see [29], [9]).

Namely, in section 4 we prove that a finite, simply connected, 2-dimensional square (hexagon) complex satisfying a combinatorial condition, collapses to a point (see [29], [27]). The combinatorial condition we consider is given by the 8-property (12-property) of a square (hexagon) complex. Besides, we show that a locally finite, simply connected, 2-dimensional square (hexagon) complex with the 8-property (12-property) is an infinite ascending union of finite collapsible subcomplexes (see [29], [27]). Our proof uses van Kampen diagrams.

In the last section of the book we prove that a finite, CAT(0) cubical complex of dimension 3 or less endowed with the standard piecewise Euclidean metric, is collapsible, under the same technical assumption as in [18] (see [26]). We also show that a finite, CAT(0) hexagonal complex of dimension 2 endowed with the standard piecewise Euclidean metric, collapses to a point (see [9]). The CAT(0) 2-dimensional cubical (hexagonal) complex is constructed by endowing the 2-complex with the standard piecewise Euclidean metric such that each interior vertex of the complex has degree at least 4 (3). The naturally associated piecewise Euclidean metric on the 2-complex becomes then CAT(0). Our proof follows by applying discrete Morse theory.

It is interesting to note, however, that the collapsibility of a finite, simply connected simplicial (cubical, hexagonal) 2-complex with the 6-property (8-property, 12-property) implies the existence of a strongly convex metric on the 2-complex. This holds since a collapsible simplicial (cubical, hexagonal) 2-complex admits a strongly convex metric (see [35], [5], [6]). We emphasize that the implication was already clear for simplicial (cubical, hexagonal) 2-complexes with the 6-property (8-property, 12-property) endowed with the standard piecewise Euclidian metric. Still, due to their collapsibility, all finite, simply connected simplicial (cubical, hexagonal) 2-complexes with the 6-property (8-property, 12-property) admit a strongly convex metric, not only those endowed with the standard piecewise Euclidian metric (see [35], [5],

[6]).

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1 Preliminaries

We present in this section the notions we shall work with and the results we shall refer to.

In the first part of the section we introduce the fundamental concepts we shall be concerned with throughout this paper and give a few results regarding them.

In the second part of the section we define the CAT(0) spaces and we study some of their basic properties. Roughly speaking, CAT(0) spaces are geodesic metric spaces whose definition is based on a comparison with the Euclidean space.

The third part of the section presents van Kampen diagrams, a useful tool for proving the collapsibility of finite 2-dimensional cell complexes. Van Kampen diagrams enable us to study a cell complex by associating to each closed edge-path in the complex a diagram in the Euclidean plane which contains all the essential information about the closed edge-path. Besides, we introduce in this subsection a family of conditions on a simplicial complex which characterizes locally the simplicial non-positive curvature of the complex. We call these conditions local 6-largeness except that in dimension 2, local 6-largeness is called the 6-property of the complex.

1.1 Basic concepts

We present in this subsection basic facts about geometric notions such as distance, geodesic, angle, curvature and elementary collapse.

Definition 1.1.1. *Let (X, d) be a metric space. If x, m, y are three points in X such that $d(x, m) + d(m, y) = d(x, y)$, then we say that m* lies between *x and y. We call m the* midpoint *of x and y if $d(x, m) = d(m, y) = \frac{1}{2}d(x, y)$.*

Definition 1.1.2. *Let (X, d) be a metric space. X is a* convex *metric space if for any two points x, y in X, there exists at least one midpoint m. X is a* strongly convex *metric space if for any two points x, y in X, there exists exactly one midpoint m. X is a* without ramifications *metric space if for any three points x, y, x' in X, no midpoint of x and y is also a midpoint of x' and y, unless $x = x'$.*

Definition 1.1.3. *Let (X, d) be a metric space and let $c : [a, b] \to X$ be a path in X. The* length $l(c)$ *of c is defined by:*

$$l(c) = \sup_{a=t_0 \leq t_1 \leq \ldots \leq t_n = b} \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1})),$$

where the supremum is taken over all possible partitions with $a = t_0 \leq t_1 \leq \ldots \leq t_n = b$.

Theorem 1.1.4. *Let (X, d) be a metric space and let $c : [a, b] \to X$ be a path of length $l_1 < \infty$. Then the function $\lambda : [a, b] \to [0, l_1]$ defined by $\lambda(t) = l(c|_{[a,t]})$ is a continuous weakly monotonic function.*

Proof. See [12], chapter I.1, page 13.

□

Definition 1.1.5. *Let (X, d) be a metric space and let x, y be two distinct points in X. A* segment *$c : [a, b] \to X$ in X connecting x to y is a path which has, among all path joining x to y in X, the shortest length.*

Theorem 1.1.6. *Let (X, d) be a metric space. Let x and y be two distinct points in X.*

1. *A subset S of X containing x and y is a segment joining x to y if there exists a closed real line interval $[a, b]$ and an isometry $c : [a, b] \to X$ such that $c(a) = x$ and $c(b) = y$.*
2. *A path $c : [a, b] \to X$ joining x to y is a segment from x to y if and only if $l(c) = d(x, y)$.*

Proof. See [2], chapter II.2, page 76.

□

Theorem 1.1.7. *Let (X, d) be a complete metric space. There exists a segment in X (which is not necessarily unique) between any two distinct points x, y in X if and only if X is a convex metric space.*

Proof. See [32].

□

Theorem 1.1.8. *Let (X, d) be a complete metric space. There exists a unique segment in X between any two distinct points x, y in X if and only if X is a strongly convex metric space.*

Proof. See [32].

□

Definition 1.1.9. *Let (X, d) be a metric space. A* geodesic path *joining $x \in X$ to $y \in X$ is a path $c : [a, b] \to X$ such that $c(a) = x$, $c(b) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [a, b]$. The image α of c is called a* geodesic segment *with endpoints x and y.*

A *geodesic metric space* (X, d) is a metric space in which every pair of points can be joined by a geodesic segment. We denote any geodesic segment from a point x to a point y in X, by $[x, y]$. We emphasize that any such geodesic segment is not determined by its endpoints. Thus, without further assumptions on X, there may be many geodesic segments joining x to y.

A *geodesic triangle* in X consists of three points $p, q, r \in X$, called *vertices*, and a choice of three geodesic segments $[p, q], [q, r], [r, p]$ joining them, called *sides*. Such a geodesic triangle is denoted by $\triangle([p, q], [q, r], [r, p])$ or $\triangle(p, q, r)$. If a point $x \in X$ lies in the union of $[p, q], [q, r]$ and $[r, p]$, then we write $x \in \triangle$. A triangle $\overline{\triangle} = \triangle(\overline{p}, \overline{q}, \overline{r})$ in $\mathbb{R}^2$ is called a *comparison triangle* for $\triangle = \triangle(p, q, r)$ if $d(p, q) = d_{\mathbb{R}^2}(\overline{p}, \overline{q})$, $d(q, r) = d_{\mathbb{R}^2}(\overline{q}, \overline{r})$ and $d(r, p) = d_{\mathbb{R}^2}(\overline{r}, \overline{p})$. A point $\overline{x} \in [\overline{q}, \overline{r}]$ is called a *comparison point* for $x \in [q, r]$ if $d(q, x) = d_{\mathbb{R}^2}(\overline{q}, \overline{x})$. The interior angle of $\overline{\triangle} = \triangle(\overline{p}, \overline{q}, \overline{r})$ at $\overline{p}$ is called the *comparison angle* between q and r at p and it is denoted by $\overline{\angle}_p(q, r)$ (the comparison angle is well-defined provided q and r are both distinct from p).

We give further Alexandrov's definition of the angle between geodesics issuing from a common point in an arbitrary metric space.

Definition 1.1.10. *Let (X, d) be a metric space and let $c : [0, a] \to X$ and $c' : [0, a'] \to X$ be two geodesic paths with $c(0) = c'(0)$. Given $t \in (0, a]$ and $t' \in (0, a']$, we consider the comparison triangle $\overline{\triangle}(c(0), c(t), c'(t'))$ in $\mathbb{R}^2$ and the comparison angle $\overline{\angle}_{c(0)}(c(t), c'(t'))$. The* Alexandrov angle *or the* upper angle *between the geodesic paths c and c' is the number $\angle(c, c') \in [0, \pi]$ defined by:*

$$\angle(c,c') := \limsup_{t,t'\to 0} \overline{\angle}_{c(0)}(c(t),c'(t')) = \lim_{\varepsilon\to 0} \sup_{0<t,t'<\varepsilon} \overline{\angle}_{c(0)}(c(t),c'(t')).$$

In the above definition of angle, it is important that one takes a lim sup; in general the limit $\lim_{t,t'\to 0} \overline{\angle}_{c(0)}(c(t),c'(t'))$ does not exist. In the metric spaces we study in this paper, namely those with curvature bounded above, this limit always exists.

The Alexandrov angle between two geodesic segments which have a common endpoint, is defined to be the Alexandrov angle between the unique geodesics which issue from this point and whose images are the given segments. Alexandrov angles in $\mathbb{R}^2$ are equal to the usual Euclidean angles.

The Alexandrov angle between the geodesic paths $c : [0,a] \to X$ and $c' : [0,a'] \to X$ in a metric space (X,d) depends only on the germs of these paths at 0. If $c'' : [0,a''] \to X$ is any geodesic path for which there exists $\varepsilon > 0$ such that $c'' |_{[0,\epsilon]} = c' |_{[0,\epsilon]}$, then the Alexandrov angle between c and c'' is the same as that between c and c'.

We outline the importance of the above remark in this paper, since we shall use it frequently when showing one of our main results.

Definition 1.1.11. *Let (X,d) be a convex metric space. Let $c : [0,a] \to X$ and $c' : [0,a'] \to X$ be two geodesic paths with $c(0) = c'(0) = p$ which have no other common points in the neighborhood of p. The geodesic paths c and c' divide a sufficiently small neighborhood of p into two* sectors *U and V. We consider in U the geodesic paths $c_1, c_2, ..., c_n$, numbered according to their position relative to c and c'. We denote by $\alpha_0, \alpha_1, ..., \alpha_n$ the Alexandrov angles between c and c_1, c_1 and c_2, ..., c_n and c'. The upper limit of the sum $\alpha_0 + \alpha_1 + ... + \alpha_n$ for any geodesic paths c_i in U, $1 \leq i \leq n$, is called the* Alexandrov angle of the sector *U.*

Definition 1.1.12. *Let (X,d) be a convex metric space and let p be a point in X. Let $U_1, ..., U_n$ be sectors around p which form a full neighborhood of p. We call the sum of the Alexandrov angles of the sectors $U_1, ..., U_n$ in X,* the full angle around the point *p in X.*

We define further the curvature of a triangle and the curvature of a point in a convex metric space.

Definition 1.1.13. *Let (X,d) be a convex metric space. Let $\triangle(p,q,r)$ be a geodesic triangle in X. Let α,β,γ be the Alexandrov angles between the sides of $\triangle$. The* curvature of the geodesic triangle $\triangle$ *is defined by $\omega(\triangle) = \alpha+\beta+\gamma-\pi$.*

Definition 1.1.14. *Let (X,d) be a convex metric space. Let p be a point of X. Let θ be the full angle around the point p in X. The* curvature at the point p *is defined by $\omega(p) = 2\pi - \theta$.*

The following theorems give important characterizations of triangles of zero curvature in a convex metric space.

Theorem 1.1.15. *Let (X,d) be a convex metric space and let $\triangle(p,q,r)$ be a geodesic triangle in X whose curvature equals zero. Then $\triangle(p,q,r)$ is isometric to its comparison triangle $\overline{\triangle}(\overline{p},\overline{q},\overline{r})$ in $\mathbb{R}^2$.*

Proof. See [2], chapter V.6, page 218.

□

Theorem 1.1.16. *Let (X,d) be a convex metric space and let G be a subspace of X. G is locally isometric to the Euclidean plane (i.e. each point has a neighborhood isometric to a subspace of the Euclidean plane) if and only if the curvature of each geodesic triangle in G equals zero.*

Proof. See [2], chapter V.6, page 219.

□

Alexandrov's lemma, given below, will be referred to frequently in the third section of the paper.

Lemma 1.1.17. *Consider four distinct points A,B,B',C in the Euclidean plane. Suppose that B and B' lie on opposite sides of the line through A and C.*

Consider the geodesic triangles $\triangle = \triangle(A,B,C)$ and $\triangle' = \triangle(A,B',C)$. Let α,β,γ (α',β',γ') be the angles of $\triangle$ $(\triangle')$ at the vertices A,B,C (A,B',C).

If $\gamma+\gamma' \geq \pi$ then,

$$(1)\ d(B,C)+d(B',C) \leq d(B,A)+d(B',A).$$

Let $\overline{\triangle}$ be a triangle in $\mathbb{R}^2$ with vertices $\overline{A}, \overline{B}, \overline{B}'$ such that $d(\overline{A}, \overline{B}) = d(A, B)$, $d(\overline{A}, \overline{B}') = d(A, B')$ and $d(\overline{B}, \overline{B}') = d(B, C) + d(C, B')$. Let $\overline{C}$ be the point of $[\overline{B}, \overline{B}']$ with $d(\overline{B}, \overline{C}) = d(B, C)$. Let $\overline{\alpha}, \overline{\beta}, \overline{\beta}'$ be the angles of $\overline{\triangle}$ at the vertices $\overline{A}, \overline{B}, \overline{B}'$. Then,

$$(2)\ \overline{\alpha} \geq \alpha + \alpha', \overline{\beta} \geq \beta, \overline{\beta}' \geq \beta' \text{ and } d(\overline{A}, \overline{C}) \geq d(A, C).$$

If $\gamma + \gamma' \leq \pi$ then,

$$(3)\ d(B, C) + d(B', C) \geq d(B, A) + d(B', A).$$

Also,

$$(4)\ \overline{\alpha} \leq \alpha + \alpha', \overline{\beta} \leq \beta, \overline{\beta}' \leq \beta' \text{ and } d(\overline{A}, \overline{C}) \leq d(A, C).$$

Any one equality is equivalent to the others, and occurs if and only if $\gamma + \gamma' = \pi$.

Proof. See [12], chapter I.2, page 25.

□

The *unit n-cube* I^n is the n-fold product $[0, 1]^n$; it is isometric to a cube in $\mathbb{E}^n$ with edges of length one. By convention, I^0 is a point.

We define a cubical complex by mimicking the definition of a simplicial complex, using unit cubes instead of simplices. Cubical complexes are more rigid objects than simplicial complexes and in many ways they are easier to work with.

Definition 1.1.18. *An n-dimensional* cubical complex K *is the quotient of a disjoint union of cubes $X = \bigcup_\Lambda I^{n_\lambda}$ by an equivalence relation $\sim$. The restrictions $p_\lambda : I^{n_\lambda} \to K$ of the natural projection $p : X \to K = X|_\sim$ are required to satisfy:*

1. *for every $\lambda \in \Lambda$, the map p_λ is injective;*
2. *if $p_\lambda(I^{n_\lambda}) \bigcap p_{\lambda'}(I^{n_{\lambda'}}) \neq \emptyset$, then there is an isometry $h_{\lambda,\lambda'}$ from a face $T_\lambda \subset I^{n_\lambda}$ onto a face $T_{\lambda'} \subset I^{n_{\lambda'}}$ such that $p_\lambda(x) = p_{\lambda'}(x')$ if and only if $x' = h_{\lambda,\lambda'}(x)$.*

In other words, K is a cubical complex if and only if each of its cells C_λ is isometric to a cube I^{n_λ}, each of the maps p_λ is injective, and the intersection of any two cells in K is empty or a single face.

There are many interesting examples of cubical complexes all of whose cells are cubes, but which do not satisfy all the conditions of the above definition (see [1], [12]). We use the term *cubed complex* to describe this larger class of complexes, except that in the 2-dimensional case we use the term *square complex*.

The *unit* 2-*hexagon* J^2 is isometric to a regular hexagon in $\mathbb{E}^2$ with edges of length one. By convention, J^0 is a point.

We define a 2-dimensional hexagonal complex by mimicking the definition of a 2-dimensional simplicial complex, using unit hexagons instead of simplices.

Definition 1.1.19. *A* 2-*dimensional* hexagonal complex *K is the quotient of a disjoint union of hexagons $X = \bigcup_\Lambda J^{2_\lambda}$ by an equivalence relation $\sim$. The restrictions $p_\lambda : J^{2_\lambda} \to K$ of the natural projection $p : X \to K = X|_\sim$ are required to satisfy:*

1. *for every $\lambda \in \Lambda$, the map p_λ is injective;*
2. *if $p_\lambda(J^{2_\lambda}) \bigcap p_{\lambda'}(J^{2_{\lambda'}}) \neq \emptyset$, then there is an isometry $h_{\lambda,\lambda'}$ from a face $T_\lambda \subset J^{2_\lambda}$ onto a face $T_{\lambda'} \subset J^{2_{\lambda'}}$ such that $p_\lambda(x) = p_{\lambda'}(x')$ if and only if $x' = h_{\lambda,\lambda'}(x)$.*

There are many interesting examples of 2-dimensional hexagonal complexes all of whose cells are hexagons, but which do not satisfy all the conditions of the above definition. We use the term *hexagon complex* to describe this larger class of complexes.

We find in this paper necessary and sufficient conditions for the collapsibility of a finite cell complex. We define further the notion of collapsing a cell complex.

Definition 1.1.20. *Let K be a cell complex and let α be an i-cell of K. If β is a k-dimensional face of α but not of any other cell in K, then we say there is an* elementary collapse *from K to $K \setminus \{\alpha, \beta\}$. If $K = K_0 \supseteq K_1 \supseteq \ldots \supseteq K_n = L$ are cell complexes such that there is an elementary collapse from K_{j-1} to K_j, $1 \leq j \leq n$, then we say that K* collapses *to L.*

Definition 1.1.21. *Let K be an n-dimensional cell complex. A k-dimensional subcomplex K' of K is called a* spine *of K if K collapses to K', $k < n$.*

Let K be a finite, connected cell complex endowed with the standard piecewise Euclidean metric. We define the *standard piecewise Euclidean metric* on $|K|$ by taking the distance between any two points x, y in $|K|$ to be the infimum over the lengths of all paths in $|K|$ from x to y. Each cell of K is isometric with a regular Euclidean cell of the same dimension with side lengths equal 1.

Let K be a finite, connected cell complex endowed with the standard piecewise Euclidean metric. A finite sequence of vertices $[v = v_1, v_2, ..., v_{k+1} = v']$ in K such that any two consecutive vertices v_i, v_{i+1} span an edge for $1 \leq i \leq k$, is called a *combinatorial path* between the vertices v and v' in K. Such a path has *length* k and is *closed* if $v = v'$. If there exists a combinatorial path from v to v' of length k, but there does not exist a combinatorial path from v to v' of length less than k, then we call any combinatorial path of length k joining v to v', a *combinatorial geodesic*. The *combinatorial distance* between any two vertices v, v' in K, denoted by $d_c(v, v')$, is the length of any combinatorial geodesic joining v to v'. We call the vertex v a *neighbor* of v' if $d_c(v, v') = 1$.

Definition 1.1.22. *Let K be a cell complex and let v be a vertex of K. The* degree *or* valence *of v, denoted by* deg v*, is the number of edges issuing from v.*

Definition 1.1.23. *Let K be a cell complex and let σ be a cell of K. The* link *of K at σ, denoted $Lk(\sigma, K)$, is the subcomplex of K consisting of all cells which are disjoint from σ and which together with σ span a cell of K.*

Definition 1.1.24. *Let K be a cell complex and let σ be a cell of K. The (closed)* star *of σ in K, denoted $St(\sigma, K)$, is the union of all cells of K that contain σ. The star $St(\sigma, K)$ is naturally the join of σ and the link $Lk(\sigma, K)$.*

Definition 1.1.25. *A point of an n-dimensional cell complex K is an* interior point *if it is contained in a neighborhood U where U is homeomorphic to the unit n-ball B^n, and $U \subseteq K$. Otherwise, the point is an* exterior point *of K.*

Definition 1.1.26. *Let D be a disk. The* area *of D is given by the number of 2-cells of D.*

The following theorem will play an important role in the last section of the paper. We mention that the same result was proven in [36] on finite simplicial complexes.

Theorem 1.1.27. *Let K and M be finite cell complexes, and let L be a subcomplex of K. Let $f : |K| \to |M|$ be a continuous map such that the restriction $f|L$ is a cell map from L to M. Then there exists an integer r, a subdivision K_r of K and a cell map $g : K_r \to M$ such that $g|L = f|L$ and g is homotopic to f keeping L fixed.*

Proof. See [15], chapter II.4, page 146.

□

1.2 CAT(0) spaces

We define in this subsection CAT(0) spaces and present some of their basic properties.

Definition 1.2.1. *Let (X, d) be a metric space. Let $\triangle(p, q, r)$ be a geodesic triangle in X. Let $\overline{\triangle}(\overline{p}, \overline{q}, \overline{r}) \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. The metric d is* CAT(0) *if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$, the CAT(0) inequality holds:*

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}).$$

Definition 1.2.2. *A metric space X is called a* CAT(0) space *if it is a metric space all of whose geodesic triangles satisfy the CAT(0) inequality.*

Definition 1.2.3. *A metric space X is said to be* of curvature ≤ 0 *(or* non-positively curved*) if it is locally a CAT(0) space, i.e. for every $x \in X$, there exists $r_x > 0$ such that the ball $B(x, r_x)$, endowed with the induced metric, is a CAT(0) space.*

The CAT(0) spaces have the following important properties.

Theorem 1.2.4. *Let X be a CAT(0) space.*

1. *The balls in X are convex (i.e., any two points in such a ball are joined by a unique geodesic segment and this segment is contained in the ball) and contractible;*
2. *(Approximate midpoints are close to midpoints.) For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if m is the midpoint of a geodesic segment $[x, y] \subset X$ and if*

$$\max\{d(x,m'),d(y,m')\} \le \frac{1}{2}d(x,y)+\delta,$$

then $d(m,m') < \varepsilon$.

Proof. See [12], chapter II.1, page 160.

□

The notion of angle makes perfect sense in spaces of curvature bounded above. Namely, in CAT(0) spaces, angles exist in the following strong sense.

Theorem 1.2.5. *Let X be a CAT(0) space and let $c : [0,a] \to X$ and $c' : [0,a'] \to X$ be two geodesic paths issuing from the same point $c(0) = c'(0)$. Given $t \in (0,a]$ and $t' \in (0,a']$, let $\overline{\triangle}(c(t),c(0),c'(t'))$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle(c(t),c(0),c'(t'))$. The comparison angle $\overline{\angle}_{c(0)}(c(t),c'(t'))$ is a non-decreasing function of both $t,t' \ge 0$ and the Alexandrov angle $\angle(c,c')$ is equal to $\lim_{t,t'\to 0}\overline{\angle}_{c(0)}(c(t),c'(t')) = \lim_{t\to 0}\overline{\angle}_{c(0)}(c(t),c'(t))$. Hence*

$$\angle(c,c') = \lim_{t\to 0} 2\arcsin \tfrac{1}{2t}d(c(t),c'(t)).$$

Proof. See [12], chapter II.3, page 184.

□

Let p,x,y be points of a metric space X such that $p \ne x$ and $p \ne y$. If there are unique geodesic segments $[p,x]$ and $[p,y]$, then we write $\angle_p(x,y)$ to denote the Alexandrov angle between these segments.

Theorem 1.2.6. *Let X be a CAT(0) space. For fixed $p \in X$, the function $(x,y) \mapsto \angle_p(x,y)$ is continuous.*

Proof. See [12], chapter II.3, page 185.

□

The following theorem gives important characterizations of CAT(0) spaces.

Theorem 1.2.7. *Let X be a metric space. The following conditions are equivalent:*

1. *X is a CAT(0) space.*

2. *For every geodesic triangle $\triangle(p,q,r)$ in X and for every point $x \in [q,r]$, the following inequality is satisfied by the comparison point $\overline{x} \in [\overline{q},\overline{r}] \subset \overline{\triangle}(p,q,r) \subset \mathbb{R}^2$:*

$$d(p,x) \leq d(\overline{p},\overline{x}).$$

3. *The Alexandrov angle between the sides of any geodesic triangle in X with distinct vertices is no greater than the angle between the corresponding sides of its comparison triangle in $\mathbb{R}^2$.*

Proof. See [12], chapter II.1, page 161.

□

We present further other important properties of CAT(0) spaces.

Theorem 1.2.8. *Any CAT(0) space is contractible; in particular it is simply connected.*

Proof. See [12], chapter II.1, page 161.

□

Theorem 1.2.9. *Let (X,d) be a CAT(0) space. Then the distance function $d : X \times X \to \mathbb{R}$ is convex and strongly convex.*

Proof. See [12], chapter II.2, page 176 and chapter II.1, page 160.

□

The following basic theorem states in particular that a CAT(-1) space is a CAT(0) space.

Theorem 1.2.10. *If X is a CAT(k') space for every $k' > k$ real numbers, then it is a CAT(k) space.*

Proof. See [12], chapter II.1, page 165.

□

The following theorem relates the local to the global context, representing a generalization of the Cartan-Hadamard theorem to the context of complete geodesic metric spaces.

Theorem 1.2.11. *Let X be a complete connected metric space. If X is simply connected and of curvature ≤ 0, then X is a CAT(0) space.*

Proof. See [12], chapter II.4, page 194.

□

1.3 Van Kampen diagrams. Simplicial nonpositive curvature

Van Kampen diagrams constitute a significant part of small cancellation theory which deals mostly with dimension 2 (see [31]). It turns out that van Kampen diagrams are an adequate tool for studying the collapsibility of finite cell complexes of dimension 2. In the first part of the subsection we introduce this notion and we present some important and typical results dealing with it.

In the second part of the subsection we introduce a family of conditions on a simplicial complex that we call local 6-largeness (see [24]). They are simply stated, combinatorial and easily checkable. Local 6-largeness describes the curvature of a simplicial complex in combinatorial terms; it represents a combinatorial analogue of metric non-positive curvature. Local 6-largeness in dimension 2 is called the 6-property of the simplicial complex.

Let K be a cell complex. A *closed edge* is an edge together with its endpoints. An *oriented edge* of K is an oriented 1-cell of K, $e = [v_0, v_1]$. We denote by $i(e) = v_0$, the *initial* vertex of e, by $t(e) = v_1$, the *terminus* of e, and by $e^{-1} = [v_1, v_0]$, the *inverse* of e. A finite sequence $\alpha = e_1 e_2 ... e_n$ of oriented closed edges in K such that $t(e_i) = i(e_{i+1})$ for all $1 \leq i \leq n-1$, is called an *edge-path* in K. The *inverse* of α is the edge-path $\alpha^{-1} = e_n^{-1} ... e_2^{-1} e_1^{-1}$. If $t(e_n) = i(e_0)$, then we call α a *closed edge-path* or *cycle*. We denote by $|\alpha|$ the number of 1-cells contained in α and we call $|\alpha|$ the *length* of α. A *region* is a 2-cell of K.

Definition 1.3.1. *A* combinatorial map $f : K_1 \to K_2$ *between cell complexes is a continuous map such that each open cell of K_1 is mapped homeomorphically onto an open cell of K_2.*

Definition 1.3.2. *A* combinatorial 2-complex K *is a* 2*-dimensional cell complex such that the* 2*-cells are attached through continuous maps from* S^1 *to* $K^{(1)}$.

Definition 1.3.3. *A* combinatorial disk *is a combinatorial* 2*-complex homeomorphic to a disk.*

Definition 1.3.4. *Let K be a cell complex. Let $\alpha = e_0 e_1 ... e_n$ be a closed edge-path in K. A* van Kampen diagram *for α is a pair (D, ϕ). D is a finite, simply connected combinatorial disk embedded in the plane, bounded by the closed edge-path $\beta = f_0 f_1 ... f_n$. $\phi : D \to K$ is a combinatorial map assigning to each oriented edge f_i of β in D an oriented edge $\phi(f_i) = e_i$ of α in K such that $\phi(f_i^{-1}) = \phi(f_i)^{-1}$ for all $0 \leq i \leq n$.*

Theorem 1.3.5. *Let K be a* 2*-dimensional cell complex and let α be a closed edge-path in K. α is null-homotopic if and only if there exists a van Kampen diagram for α.*

Proof. See [31], chapter V, page 237-239.

□

Definition 1.3.6. *Let K be a cell complex. Let α be a closed edge-path in K. Let (D, ϕ) be a van Kampen diagram for α. Let D_1, D_2 be regions (not necessarily distinct) of D with an edge $e \subseteq \partial D_1 \bigcap \partial D_2$. Let $e\delta_1$ and $\delta_2 e^{-1}$ be boundary cycles of D_1 and D_2, respectively. Let $\phi(\delta_1) = f_1$ and $\phi(\delta_2) = f_2$. The diagram (D, ϕ) is called* reduced *if one never has $f_2 = f_1^{-1}$.*

Definition 1.3.7. *Let K be a cell complex. Let $\alpha = e_0 e_1 ... e_n$ be a closed edge-path in K and let (D, ϕ) be a van Kampen diagram for α. The* area *of the diagram is given by the number of regions of D.*

Theorem 1.3.8. *Let K be a* 2*-dimensional cell complex and let α be a closed edge-path in K. Let (D, ϕ) be a van Kampen diagram for α. If (D, ϕ) is a van Kampen diagram of minimal area, then the diagram is reduced.*

Proof. See [31], chapter V, page 241.

□

Definition 1.3.9. *Let K be a cell complex. A subcomplex L in K is called* full *(in K) if any cell of K spanned by a set of vertices in L is a cell of L. A* full cycle *in K is a cycle that is full as subcomplex of K.*

Definition 1.3.10. *Let K be a cell complex. We define the* systole *of K by*

$$sys(K) = \min\{|\alpha| : \alpha \textit{ is a full cycle in } K\}.$$

In graph theory, we use instead of the term systole, the one of girth *of the graph.*

Definition 1.3.11. *A 2- dimensional cell complex has the k-*property *if the link of each vertex is a graph of girth at least k, $k \in \{6, 8, 12\}$.*

Theorem 1.3.12. *Let K be a 2-dimensional cell complex with the k-property and let α be a closed edge-path in K. If (D, ϕ) is a reduced van Kampen diagram for α, then D also has the k-property, $k \in \{6, 8, 12\}$.*

Proof. See [31], chapter V, page 242.

□

In the remaining part of this subsection we introduce a combinatorial curvature condition on simplicial complexes, called local 6-largeness and we give a few results concerning this condition.

Definition 1.3.13. *Let K be a simplicial complex. We call K* 6-large *if $sys(K) \geq 6$ and $sys(Lk(\sigma, K)) \geq 6$, for each simplex σ of K. We call K* locally 6-large *if the star of every simplex of K is 6-large. We call K* 6-systolic *if it is connected, simply connected and locally 6-large.*

An important result of the paper relies on the following fact.

Theorem 1.3.14. *Let K be a finite, simply connected, 2-dimensional simplicial complex. If K is locally 6-large, then K is collapsible.*

Proof. See [17].

□

The following remark will be of crucial importance when proving an important result of the paper.

A simplicial complex K is 6-large if every cycle of length less than 6 has some two consecutive edges contained in a common 2-simplex of K.

The following theorem can be viewed as a combinatorial analogue of the Cartan-Hadamard theorem proven on geodesic metric spaces in [12] (see chapter II.4, page 193).

Proposition 1.3.15. *If K is a 6-systolic simplicial complex, then K is 6-large.*

Proof. See [24], chapter 1, page 10.

□

Definition 1.3.16. *The* homotopical systole *of a simplicial complex K is the minimal length of a cycle that is homotopically nontrivial in K. We denote homotopical systole of K by $sys_h(K)$.*

The following corollary states that the converse of sorts of the above result also holds.

Corollary 1.3.17. *A simplicial complex is 6-large if and only if it is locally 6-large and $sys_h(K) \geq 6$.*

Proof. See [24], chapter 1, page 10.

□

2 Elements of discrete Morse theory

We introduce in this section fundamental notions and results in discrete Morse theory. Besides we define the discrete Morse-Smale characteristic of a finite simplical complex and the exact and $\mathbb{F}$-perfect discrete Morse functions on a finite simplicial complex, $\mathbb{F}$ being any coefficient field. We also study examples of $\mathbb{F}$-perfect discrete Morse functions on finite simplicial complexes.

2.1 Discrete Morse theory for cell complexes

The goal of this subsection is to present an overview of the subject of discrete Morse theory sufficient to apply the theory when proving the collapsibility of finite cell complexes of dimension 2 and 3. We will introduce the main notions in discrete Morse theory and we will give a few basic results in discrete Morse theory. For further details on the subject, see [20] and [19].

Let K be a cell complex. We denote by $\alpha^{(i)}$ an i-dimensional cell of K and by $\alpha < \beta$ the fact that α is a face of β.

Definition 2.1.1. *Let K be a cell complex. A function $f : K \to \mathbb{R}$ is a* discrete Morse function *if for every $\alpha^{(i)} \in K$*

1. $\sharp\{\beta^{(i+1)} > \alpha \mid f(\beta) \leq f(\alpha)\} \leq 1$ *and*

2. $\sharp\{\gamma^{(i-1)} < \alpha \mid f(\gamma) \geq f(\alpha)\} \leq 1.$

Definition 2.1.2. *Let K be a cell complex and let $f : K \to \mathbb{R}$ be a discrete Morse function. A cell $\alpha^{(i)}$ is* critical *if*

1. $\sharp\{\beta^{(i+1)} > \alpha \mid f(\beta) \leq f(\alpha)\} = 0$ *and*

2. $\sharp\{\gamma^{(i-1)} < \alpha \mid f(\gamma) \geq f(\alpha)\} = 0.$

Definition 2.1.3. *Let K be a cell complex and let $f : K \to \mathbb{R}$ be a discrete Morse function. For $c \in \mathbb{R}$, we define the* level subcomplex

$$K(c) = \bigcup_{\alpha \in |K|, f(\alpha) \leq c} \bigcup_{\beta \leq \alpha} \beta.$$

That is, $K(c)$ denotes the subcomplex of K consisting of all cells α with $f(\alpha) \leq c$, together with all of their faces.

We prove further one of the fundamental results in discrete Morse theory. Its proof is based on the following two lemmas.

Lemma 2.1.4. *Let K be a cell complex, let $f : K \to \mathbb{R}$ be a discrete Morse function and let α be an i-cell of K. Then α is not critical if and only if either of the following conditions holds:*

1. $\exists \beta^{(i+1)} > \alpha$ *such that* $f(\beta) \leq f(\alpha)$;
2. $\exists \gamma^{(i-1)} < \alpha$ *such that* $f(\gamma) \geq f(\alpha)$.

The above conditions cannot both be true.

Proof. See [19], chapter 2, page 103. □

Lemma 2.1.5. *Let K be a cell complex, let $f : K \to \mathbb{R}$ be a discrete Morse function and let α be an i-cell of K. Suppose that $\beta > \alpha$. Then there is an $(i+1)$-cell $\overline{\beta}$ such that $\alpha < \overline{\beta} \leq \beta$ and $f(\overline{\beta}) \leq f(\beta)$.*

Proof. See [19], chapter 3, page 104. □

We following theorem represents one of the main results in discrete Morse theory.

Theorem 2.1.6. *Let K be a cell complex and let $f : K \to \mathbb{R}$ be a discrete Morse function. If $a < b$ are real numbers such that $[a, b]$ contains no critical values of f, then $K(a)$ collapses to $K(b)$.*

Proof. Note that if $\beta^{(i+1)} > \alpha^{(i)}$ satisfies $f(\beta) \leq f(\alpha)$, then we may perturb f by replacing $f(\beta)$ by $f(\beta)-\varepsilon$, or $f(\alpha)$ by $f(\alpha)+\varepsilon$, for $\varepsilon \geq 0$ small enough, without changing which cells are critical. If $\alpha^{(i)}$ satisfies $f(\beta^{(i+1)}) \neq f(\alpha) \neq f(\gamma^{(i-1)})$, then we may perturb f by changing $f(\alpha)$ by $f(\alpha) \pm \varepsilon$, or $f(\alpha)$ by $f(\alpha) + \varepsilon$, for $\varepsilon \geq 0$ small enough, without changing which cells are critical. Combining such operations, we may perturb f slightly without changing $K(b)$ or $K(a)$ so that $f : K \to \mathbb{R}$ is injective.

If $f^{-1}([a, b]) = \emptyset$, then $K(a) = K(b)$ and so there is nothing to prove. Otherwise, by partitioning $[a, b]$ into smaller intervals if necessary, we may assume that there is a single noncritical cell α with $f(\alpha) \in [a, b]$. According to Lemma 2.1.4 exactly one of the following holds:

1. $\exists \beta^{(i+1)} > \alpha$ such that $f(\beta) \leq f(\alpha)$;
2. $\exists \gamma^{(i-1)} < \alpha$ such that $f(\gamma) \geq f(\alpha)$.

In the first case, we must have $f(\beta) < a$. Thus $\beta \subseteq K(a)$. Because α is a face of β, we have $\alpha \subseteq K(a)$ so that $K(a) = K(b)$ and again there is nothing to prove.

In the second case, Lemma 2.1.4 implies that the first case cannot be true. So, for all $\beta^{(i+1)} > \alpha$, we have $f(\beta) > f(\alpha)$. In particular, $f(\beta) > b$. So, by Lemma 2.1.5, for any $\beta > \alpha$, $f(\beta) > b$. Therefore,

$$\alpha \bigcap K(a) = \emptyset.$$

We have assumed that there is $\gamma^{(i-1)} < \alpha$ such that $f(\gamma) \geq f(\alpha)$, so that, in particular, $f(\gamma) \geq b$. If $\overline{\gamma}^{(i-1)} \neq \gamma$ is any other $(i-1)$-face of α we must have

$$f(\overline{\gamma}) < f(\alpha)$$

so that

$$f(\overline{\gamma}) < a.$$

Hence $\overline{\gamma}$ and all of its faces are contained in $K(a)$.

Let $\overline{\alpha}^{(i)} \neq \alpha$ be any other (i)-face of K with $\overline{\alpha} > \gamma$. Then, because f is a discrete Morse function on K, $f(\overline{\alpha}) > f(\alpha) > b$. According to Lemma 2.1.5, if $\overline{\alpha}$ is any face of any dimension such that $\overline{\alpha} > \gamma$, then $f(\overline{\alpha}) < b$ so that

$$\gamma \bigcap K(a) = \emptyset.$$

Altogether, it follows that $K(b)$ can be expressed as a disjoint union

$$K(b) = K(a) \bigcup \alpha \bigcup \gamma$$

where γ is a free face of α. Therefore

$$K(b) \searrow K(a).$$

$\square$

Let K be a cell complex. Associated to a discrete Morse function $f : K \to \mathbb{R}$ is a *gradient vector field* $V : K \to K \bigcup \{0\}$. We define $V(\alpha) = \beta$, if $\alpha^{(i)} < \beta^{(i+1)}$ such that $f(\alpha) \geq f(\beta)$. We define $V(\alpha) = 0$ for all cells α for which there is no such β. It is often easier to work with the gradient vector field associated to a discrete Morse function rather than the function itself.

Definition 2.1.7. *Let K be a cell complex. A* discrete vector field *is a map $W : K \to K \bigcup \{0\}$ such that for each $\alpha^{(i)}$*

1. *there is at most one cell γ in K with $W(\gamma) = \alpha$;*

2. *$W(\alpha) = 0$ or α is a codimension-one face of $W(\alpha)$;*

3. *if $\alpha \in ImageW$, then $W(\alpha) = 0$.*

Definition 2.1.8. *A sequence $\alpha_0^{(i)}, \beta_0^{(i+1)}, \alpha_1^{(i)}, \beta_1^{(i+1)}, \alpha_2^{(i)}, \beta_2^{(i+1)}, ..., \beta_r^{(i+1)}, \alpha_{r+1}^{(i)}$ of cells is a W-*path *if $W(\alpha_j) = \beta_j$ for $0 \leq j \leq r$ and $\beta_j > \alpha_{j+1} \neq \alpha_j$. Such a path is* nontrivial *if $r \geq 0$ and* closed *if $\alpha_0 = \alpha_{r+1}$.*

We will refer to the following characterization of gradient vector fields frequently.

Theorem 2.1.9. *Let K be a cell complex and let $f : K \to \mathbb{R}$ be a discrete Morse function. A discrete vector field W defined on K is the gradient vector field of the discrete Morse function f if and only if it has no nontrivial closed W-paths.*

Proof. See [19], chapter 9, page 131.

$\square$

The proof of main results in the last section of the paper also relies on the following theorem.

Theorem 2.1.10. *Let K be a cell complex that admits a discrete Morse function f. If there exists a critical 3-cell β and a critical 2-cell α with a unique gradient path from the boundary of β to α, then K admits a new discrete Morse function g with the same critical cells as f, except that β and α are no longer critical.*

Proof. See [19], chapter 11, page 140.

□

Let K be an n-dimensional cell complex containing exactly m_i cells of dimension i, $0 \leq i \leq n$ and let $\mathbb{F}$ be any coefficient field. We denote by $C_i(K, \mathbb{F})$ the space $\mathbb{F}^{m_i}$, i.e. $C_i(K, \mathbb{F})$ denotes the free abelian group generated by the i-cells of K, each endowed with an orientation. The following is one of the fundamental results in the theory of cell complexes.

Theorem 2.1.11. *Let K be an n-dimensional cell complex. Then there are boundary maps $\partial_i : C_i(K, \mathbb{F}) \longrightarrow C_{i-1}(K, \mathbb{F})$, $0 \leq i \leq n$, such that*

$$\partial_{i-1} \circ \partial_i = 0$$

and such that the resulting differential complex

$$\mathfrak{C} : 0 \longrightarrow C_n(K, \mathbb{F}) \xrightarrow{\partial_n} C_{n-1}(K, \mathbb{F}) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_1} C_0(K, \mathbb{F}) \longrightarrow 0$$

calculates the homology of K. That is, if we define

$$H_i(\mathfrak{C}, \partial) = \frac{Ker(\partial_i)}{Im(\partial_{i+1})}$$

then for each i

$$H_i(\mathfrak{C}, \partial) \cong H_i(K, \mathbb{F})$$

where $H_i(K, \mathbb{F})$ denotes the singular homology of K.

Proof. See [19], chapter 3, page 122.

□

We call the differential complex $\mathfrak{C}$ in the above theorem the *Morse complex* of K.

Let K be an n-dimensional cell complex. For $0 \leq i \leq n$, let $\mathfrak{M}_i$ denote the span of the critical i-cells of K, i.e.

$$\mathfrak{M}_i = \{\textstyle\sum_{\sigma \in K} a_\sigma \sigma \mid a_\sigma \in \mathbb{Z}, \text{ if } a_\sigma \neq 0 \text{ then } \sigma \text{ is a critical cell of } K\}.$$

Theorem 2.1.12. *Let K be a n-dimensional cell complex. The Morse complex of K is isomorphic to*

$$\mathfrak{M} : 0 \to \mathfrak{M}_n \xrightarrow{\bar{\partial}} \mathfrak{M}_{n-1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathfrak{M}_0 \to 0.$$

Proof. See [19], chapter 8, page 124.

□

The Morse inequalities, given below, will be used frequently throughout this paper.

Theorem 2.1.13. *Let K be a finite n-dimensional cell complex with a discrete Morse function. Let m_i denote the number of critical cells of dimension i, $0 \leq i \leq n$. Let $\mathbb{F}$ be any coefficient field and let $b_i(K, \mathbb{F}) = rank\, H_i(K, \mathbb{F}) = \dim_{\mathbb{F}} H_i(K, \mathbb{F})$ denote the i^{th} Betti number with respect to $\mathbb{F}$. Let $\chi(K)$ denote the Euler characteristic of K. Then the following inequalities hold:*

1. The Weak Morse Inequalities:

a. $m_i \geq b_i$ *for* $0 \leq i \leq n$;

b. $m_0 - m_1 + m_2 - \dots + (-1)^n m_n = b_0 - b_1 + b_2 - \dots + (-1)^n b_n = \chi(K)$;

3. The Strong Morse Inequalities:

$m_i - m_{i-1} + \dots + (-1)^i m_0 \geq b_i - b_{i-1} + \dots + (-1)^i b_0$ *for* $0 \leq i \leq n+1$.

Proof. See [19], chapter 8, page 124.

□

2.2 Discrete Morse-Smale characteristic

In this subsection we introduce the notion of discrete Morse-Smale characteristic of a finite simplical complex. We define further exact and $\mathbb{F}$-perfect discrete Morse functions defined on a finite simplicial complex, $\mathbb{F}$ being any coefficient field. Besides we give a few examples of $\mathbb{F}$-perfect discrete Morse functions defined on finite simplicial complexes (see [30], [4]).

Let K be a finite n-dimensional simplicial complex. Let $\Omega(K)$ denote the set containing all discrete Morse functions defined on K. It is clear that $\Omega(K) \neq \emptyset$. Consider, for instance, the trivial example $f(\sigma) = \dim \sigma$, $\sigma \in K$. For $f \in \Omega(K)$, let $m_i(f)$ denote the number of critical simplices of dimension i of K, $0 \leq i \leq n$. We denote by $m(f)$ the total number of critical simplices of K, i.e. $m(f) = \sum_{i=0}^{n} m_i(f)$.

Definition 2.2.1. *Let K be a finite n-dimensional simplicial complex. Let $\Omega(K)$ denote the set containing all discrete Morse functions defined on K. The* discrete Morse Smale characteristic *of K is defined by $\gamma(K) = \min\{m(f) : f \in \Omega(K)\}$.*

So the discrete Morse-Smale characteristic represents the minimal number of critical simplices of all discrete Morse functions defined on K.

We denote by $\gamma_i(K)$ the minimal numbers of critical simplices of dimension i of all discrete Morse functions defined on K. So $\gamma_i(K) = \min\{m_i(f) : f \in \Omega(K)\}$, $0 \leq i \leq n$.

Let L be a finite n-dimensional simplicial complex and let $\psi : L \to K$ be a simplicial isomorphism. We consider the discrete Morse functions $f : K \to \mathbb{R}$ and $g : L \to \mathbb{R}$ such that the following diagram commutes, i.e. $g = f \circ \psi$.

$$\begin{tikzcd} L \arrow[rr, "\psi"] \arrow[dr, "g"'] & & K \arrow[dl, "f"] \\ & \mathbb{R} & \end{tikzcd} \qquad g = f \circ \psi$$

We define the sets $C(f) = \{\alpha^{(i)} \mid \alpha \text{ is a critical simplex of } f, 0 \leq i \leq n\}$ and $C(g) = \{\beta^{(i)} \mid \beta \text{ is a critical simplex of } g, 0 \leq i \leq n\}$. Because ψ is a simplicial isomorphism, $C(f) = \psi(C(g))$. Besides we notice that the simplicial isomorphism ψ preserves the dimension of critical simplices.

Let K and L be two isomorphic simplicial complexes. The above remarks imply that $\gamma(K) = \gamma(L)$ and $\gamma_i(K) = \gamma_i(L)$, $0 \leq i \leq n$. The numbers $\gamma(K)$ and $\gamma_i(K)$ represent hence isomorphic invariants of the simplicial complex K.

We define further the exact discrete Morse function defined on a finite simplicial complex.

Definition 2.2.2. *Let K be a finite n-dimensional simplicial complex. Let $\Omega(K)$ denote the set that contains all discrete Morse functions defined on K. The discrete Morse function $f \in \Omega(K)$ is called* exact *(or* minimal*) if $m_i(f) = \gamma_i(K)$, $0 \leq i \leq n$.*

So the number of critical simplices of any dimension of an exact discrete Morse function defined on a finite simplicial complex is always minimal.

We introduce next the $\mathbb{F}$-perfect discrete Morse function defined on a finite simplicial complex, $\mathbb{F}$ being any coefficient field.

Definition 2.2.3. *Let K be a finite n-dimensional simplicial complex. Let $\Omega(K)$ denote the set that contains all discrete Morse functions defined on K and let $\mathbb{F}$ be any coefficient field. A discrete Morse function $f \in \Omega(K)$ is called* $\mathbb{F}$-perfect *if $m_i(f) = b_i(K, \mathbb{F})$, $0 \leq i \leq n$.*

Considering the weak Morse inequalities, we have:

$$m_i(f) \geq \min\{m_i(f) : f \in \Omega(K)\} = \gamma_i(K) \geq b_i(K, \mathbb{F}).$$

The above inequality will be used frequently throughout this subsection.

The following result outlines the relation that exists between the $\mathbb{F}$-perfect discrete Morse functions defined on a finite simplicial complex and the discrete Morse Smale characteristic of the complex.

Theorem 2.2.4. *Let K be a finite n-dimensional simplicial complex. One can define on K an $\mathbb{F}$-perfect discrete Morse functions if and only if $\gamma(K) = b(K, \mathbb{F})$ where $b(K, \mathbb{F}) = \sum_{i=0}^{n} b_i(K, \mathbb{F})$ is the total Betti number of K with respect to $\mathbb{F}$.*

Proof. We start by proving the direct implication. Let $f \in \Omega(K)$ be a fixed $\mathbb{F}$-perfect discrete Morse function defined on K. The weak Morse inequalities imply

$$m(f) = \textstyle\sum_{i=0}^{n} m_i(f) \geq \sum_{i=0}^{n} b_i(K, \mathbb{F}) = b(K, \mathbb{F}).$$

Thus

$$\gamma(K) = \min\{m(f) : f \in \Omega(K)\} \geq b(K, \mathbb{F}).$$

Because f is an $\mathbb{F}$-perfect discrete Morse function defined on K, we have $m_i(f) = b_i(K, \mathbb{F})$ for $0 \leq i \leq n$. Hence

$$\gamma(K) = \min\{m(f) : f \in \Omega(K)\} \leq b(K, \mathbb{F}).$$

Altogether the above inequalities imply $\gamma(K) = b(K, \mathbb{F})$.

To prove the converse implication, let $f \in \Omega(K)$ be a discrete Morse function defined on K. The hypothesis $\gamma(K) = b(K, \mathbb{F})$ implies

$$\textstyle\sum_{i=0}^{n}[m_i(f) - b_i(K, \mathbb{F})] = 0.$$

According to the weak Morse inequalities, we have

$$m_i(f) - b_i(K, \mathbb{F}) \geq 0 \text{ for } 0 \leq i \leq n.$$

Hence

$$m_i(f) = b_i(K, \mathbb{F}) \text{ for } 0 \leq i \leq n.$$

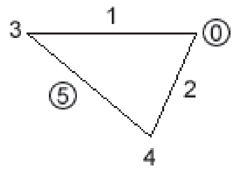
So f is an $\mathbb{F}$-perfect discrete Morse function. □

Further we give a few examples of $\mathbb{Z}$-perfect discrete Morse functions defined on finite simplicial complexes.

Example 2.2.5. We consider the circle S^1. The singular homology of S^1 is

$$H_i(S^1, \mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } i = 0; \\ \mathbb{Z} \text{ if } i = 1. \end{cases}$$

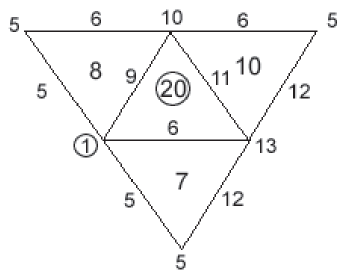
Since $b_0(S^1, \mathbb{Z}) = 1$ and $b_1(S^1, \mathbb{Z}) = 1$, the total Betti number with respect to $\mathbb{Z}$ is $b(S^1, \mathbb{Z}) = 2$. Since $\gamma(S^1) = b(S^1, \mathbb{Z}) = 2$, Theorem 2.2.4 implies that one can define on S^1 an $\mathbb{Z}$-perfect discrete Morse function, $f : S^1 \to \mathbb{R}$ with exactly two critical simplices. We notice that f is also an exact discrete Morse function defined on S^1. The above figure illustrates an $\mathbb{Z}$-perfect discrete Morse function defined on S^1. The encircled values are the ones attached to the critical simplices.



An $\mathbb{Z}$-perfect discrete Morse function with two critical simplices defined on S^1

Example 2.2.6. The figure below illustrates a triangulation of the sphere S^2. The singular homology of S^2 is

$$H_i(S^2, \mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } i = 0; \\ 0 \text{ if } i = 1; \\ \mathbb{Z} \text{ if } i = 2. \end{cases}$$

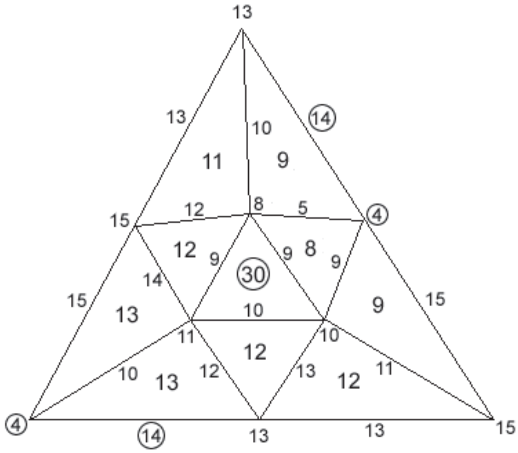


An $\mathbb{Z}$-perfect discrete Morse function with two critical simplices defined on S^2

Since $b_0(S^2, \mathbb{Z}) = 1$, $b_1(S^2, \mathbb{Z}) = 0$ and $b_2(S^2, \mathbb{Z}) = 1$, the total Betti number with respect to $\mathbb{Z}$ is $b(S^2, \mathbb{Z}) = 2$. Since $\gamma(S^2) = b(S^2, \mathbb{Z}) = 2$, Theorem 2.2.4 implies that one can define on S^2 an $\mathbb{Z}$-perfect discrete Morse function $f : S^2 \to \mathbb{R}$ with exactly two critical simplices. We notice that f is also an exact discrete Morse function defined on S^2. The figure below illustrates an $\mathbb{Z}$-perfect discrete Morse function defined on S^2. The encircled values are the ones attached to the critical simplices.

Example 2.2.7. The figure below illustrates a triangulation of the real projective plane $\mathbb{P}^2$. The singular homology of $\mathbb{P}^2$ is

$$H_i(\mathbb{P}^2, \mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } i = 0; \\ \mathbb{Z}_2 \text{ if } j = 1; \\ 0 \text{ if } j = 2. \end{cases}$$



An $\mathbb{Z}$-perfect discrete Morse function with three critical simplices defined on $\mathbb{P}^2$

Because $b_0(\mathbb{P}^2, \mathbb{Z}) = 1$, $b_1(\mathbb{P}^2, \mathbb{Z}) = 2$ and $b_2(\mathbb{P}^2, \mathbb{Z}) = 0$, the total Betti number with respect to $\mathbb{Z}$ is $b(\mathbb{P}^2, \mathbb{Z}) = 3$. Since $\gamma(\mathbb{P}^2) = b(\mathbb{P}^2, \mathbb{Z}) = 3$, Theorem 2.2.4 implies that one can define on $\mathbb{P}^2$ an $\mathbb{Z}$-perfect discrete Morse function $f : \mathbb{P}^2 \to \mathbb{R}$ with exactly three critical simplices. We notice that f is also an exact discrete Morse function defined on $\mathbb{P}^2$. The above figure illustrates an $\mathbb{Z}$-perfect discrete Morse function defined on $\mathbb{P}^2$. The encircled values are the ones attached to the critical simplices.

3 Collapsing simplicial complexes by using an elementary collapse

This section provides metric characterizations of collapsible simplicial complexes of dimension 2. Besides, we find metric and combinatorial conditions which guarantee the collapsibility of a finite simplicial complex of dimension 3. Namely, we show that finite 2-complexes that admit a CAT(0) metric are collapsible (see [28]). Similar results are obtained in [35] on finite, strongly convex 2-complexes. Namely, W. White showed that finite, strongly convex 2-complexes collapse to a point, and that finite, strongly convex 3-complexes collapse to a 2-dimensional spine. So in dimension 3, the metric condition alone does not ensure the collapsibility of the complex to a point. Still, by adding a combinatorial curvature condition, called local 6-largeness, we will show that finite, strongly convex 3-complexes can also be simplicially collapsed to a point (see [7]). As in [35], our proofs rely on the definition of an elementary collapse.

In subsection 3.1 we show that any finite, CAT(0) 2-complex retracts to a point through subspaces which are, at each step, CAT(0) spaces (see [28]). An important step in the proof is to show that the subcomplex obtained by performing an elementary collapse on a finite, CAT(0) 2-complex remains a CAT(0) space. In subsection 3.2 we study strongly convex simplicial complexes of dimension 2 and 3 (see [35]). In subsection 3.3 we recall the basic steps in W. White's proof when showing the collapsibility of finite, strongly convex 2-complexes (see [35]). In subsection 3.4 we prove that any finite, locally 6-large, 3-dimensional simplicial complex is collapsible, when endowed with a strongly convex metric (see [7]). An essential step in the proof is to study whether the subcomplex obtained by performing an elementary collapse on a finite, 6-large simplicial complex of dimension 2 (3), remains 6-large.

3.1 Collapsing a CAT(0) 2-dimensional simplicial complex

In this subsection we show that finite, CAT(0) 2-complexes are collapsible (see [28]). We emphasize that the 2-dimensional simplicial complexes in this section are not endowed with the standard piecewise Euclidean metric and their interior vertices do not necessarily have degree at least 6, as in [18]. Still, they can be simplicially collapsed to a point.

The collapsibility of finite, CAT(0) 2-complexes is a consequence of the fact that finite, strongly convex 2-complexes also collapse to a point (see [35]). This holds since a CAT(0) complex has a strongly convex metric. We present in this subsection a second proof for the collapsibility of finite, CAT(0) 2-complexes showing that these spaces retract to a point through CAT(0) subspaces.

We start investigating the collapsibility of finite, CAT(0) 2 -complexes by proving that they have a 2-simplex with a free 1-dimensional face. Besides, the following proposition characterizes the curvature at the interior points of a CAT(0) 2-complex.

Proposition 3.1.1. *Let K be a finite, 2-dimensional simplicial complex. If $|K|$ admits a CAT(0) metric d, then:*

1. *K has a 2-simplex with a free 1-dimensional face;*

2. *$|K|$ has curvature ≤ 0 at any of its interior points.*

Proof. 1. We argue by contradiction: suppose that K contains no 2-simplex with a free 1-dimensional face.

Let e be a 1-simplex of K. There exist at least two 2-simplices σ_1 and σ_2 in K such that $e < \sigma_1$ and $e < \sigma_2$. Let $\triangle(a, b, c)$ be the 2-simplex σ_1 and let d be a point on the edge $e = [b, c]$.

Let $\triangle(a', b', d')$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle(a, b, d)$ and let $\triangle(a', d', c')$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle(a, d, c)$. We place the comparison triangles $\triangle(a', b', d')$ and $\triangle(a', d', c')$ in different half-planes with respect to the line $a'd'$ in $\mathbb{R}^2$.

Because any geodesic triangles in $|K|$ satisfies the CAT(0) inequality and $d \in [b, c]$, $\pi = \angle_d(b, c) \leq \angle_d(b, a) + \angle_d(a, c) \leq \angle_{d'}(b', a') + \angle_{d'}(a', c')$. So $\angle_{d'}(b', a') + \angle_{d'}(a', c') \geq \pi$. According to Alexandrov's lemma, we have $d_{\mathbb{R}^2}(a', d') \leq d(a, d)$.

But $\triangle(a', b', d')$ is a comparison triangle for $\triangle(a, b, d)$ and hence $d_{\mathbb{R}^2}(a', d') = d(a, d)$. Because one equality in Alexandrov's lemma implies the others, the following equalities hold $\angle_{d'}(b', a') + \angle_{d'}(a', c') = \pi$, $\angle_b(a, d) = \angle_{b'}(a', d')$, $\angle_c(a, d) = \angle_{c'}(a', d')$ and $\angle_a(b, d) + \angle_a(d, c) = \angle_{a'}(b', c')$. So the sum of the angles between the sides of σ_1 equals π. Therefore, since $|K|$ has a convex metric, the curvature of the 2-simplex σ_1 equals $\omega(\sigma_1) = \pi - \pi = 0$. It similarly follows that each 2-simplex in K has curvature zero. Each 2-simplex in K is therefore isometric to its comparison triangle in $\mathbb{R}^2$.

Each 1-simplex in K is a face of at least two 2-simplices in K whose 1-simplices are further faces of at least two 2-simplices in K and so on. Each 2-simplex in K is isometric to its comparison triangle in $\mathbb{R}^2$ and K has no 2-simplex with a free 1-dimensional face. Since K is finite, this implies a contradiction. So K has a 2-simplex with a free 1-dimensional face.

2. Let e be a 1-simplex of K that is the face of at least two 2-simplices in K, σ_1 and σ_2. Let $\triangle(a, b, c)$ be the 2-simplex σ_1 and let $\triangle(b, c, f)$ be the 2-simplex σ_2. Let d be a point on the edge $e = [b, c]$.

Let $\triangle(a', b', d')$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle(a, b, d)$ and let $\triangle(a', d', c')$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle(a, d, c)$. We place the comparison triangles $\triangle(a', b', d')$ and $\triangle(a', d', c')$ in different half-planes with respect to the line $a'd'$ in $\mathbb{R}^2$. Let $\triangle(b'', f'', d'')$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle(b, f, d)$ and let $\triangle(f'', d'', c'')$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle(f, d, c)$. We place the comparison triangles $\triangle(b'', f'', d'')$ and $\triangle(f'', d'', c'')$ in different half-planes with respect to the line $d''f''$ in $\mathbb{R}^2$.

Because $d_{\mathbb{R}^2}(a', d') = d(a, d)$ and $d_{\mathbb{R}^2}(d'', f'') = d(d, f)$, Alexandrov's lemma implies that $\angle_{d'}(b', a') + \angle_{d'}(a', c') = \pi$ and $\angle_{d''}(b'', f'') + \angle_{d''}(f'', c'') = \pi$. So $\angle_d(b, a) + \angle_d(a, c) = \pi$ and $\angle_d(b, f) + \angle_d(f, c) = \pi$. Hence, because the 1-simplex e is contained in at least two 2-simplices in K, the full angle around the point d equals at least 2π. We denote by θ the full angle around the point d in $|K|$. Because $|K|$ has a convex metric, the curvature at the point d is $\omega(d) = 2\pi - \theta \leq 0$. It similarly follows that the full angle around any interior point in $|K|$ equals at least 2π. $|K|$ has thus curvature ≤ 0 at any of its interior points.

$\square$

We show further that the subcomplex K' obtained by performing an elementary collapse on a finite, CAT(0) 2-complex K, remains non-positively curved. We treat only the general case when K' is obtained by pushing in an entire 2-simplex with a free 1-dimensional face, by starting at its free face. We emphasize that the same result holds for any deformation retract of a finite, CAT(0) 2-complex K obtained by pushing in any geodesic triangle δ in $|K|$ such that one side of δ belongs to a free 1-simplex of K. First we need a lemma.

Lemma 3.1.2. *Let K be a CAT(0) space. Then any path $c : [0,1] \to K$ in K has a unique midpoint.*

Proof. Let $t \in [0,1]$ such that $l(c|_{[0,t]}) = l(c|_{[t,1]}) = \frac{1}{2}l(c|_{[0,1]})$. Because K is CAT(0), Theorem 1.2.4 implies that for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that if

$$l(c|_{[0,t']}) = l(c|_{[t',1]}) = \frac{1}{2}l(c|_{[0,1]}) \leq \frac{1}{2}l(c|_{[0,1]}) + \delta,$$

$t' \in [0,1]$, then $d(c(t), c(t')) < \varepsilon$. So, because $d(c(t), c(t')) < \varepsilon$ for every $\varepsilon > 0$, $d(c(t), c(t')) = 0$. The path c has therefore a unique midpoint.

$\square$

Proposition 3.1.3. *Let K be a finite, CAT(0), 2-dimensional simplicial complex and let σ be a 2-simplex in K with a free 1-dimensional face e. Then the subcomplex $K' = K \setminus \{e, \sigma\}$ is non-positively curved.*

Proof. Let $\triangle(a, b, c)$ be the 2-simplex σ and let $e = [b, c]$ be its free 1-dimensional face. We denote by $r := \max\{d(a, b), d(a, c)\}$. We consider in $|K|$ a neighborhood of a homeomorphic to a closed ball of radius r, $U = \{x \in |K| \mid d(a, x) \leq r\}$. U endowed with the induced metric, is a CAT(0) space. Because U is complete and it has a strongly convex metric, any two points in U are joined by a unique geodesic segment which is contained in U. So any geodesic triangle with vertices at any three points in U, belongs to U and it satisfies the CAT(0) inequality.

We consider in $|K'|$ a neighborhood of a homeomorphic to a closed ball of radius r, $U' = \{x \in |K'| \mid d'(a,x) \leq r\}$, endowed with the induced metric d'. We notice that $U' = U \setminus \{e, \sigma\}$. We find next the geodesic segments in U' with respect to d'.

Let p, q be two distinct points in U that do not belong to the interior of σ such that the geodesic segment $[p,q]$ intersects σ and it does not pass through a. We denote by m the intersection point of $[p,q]$ with $[a,b]$ and by n the intersection point of $[p,q]$ with $[a,c]$.

Lemma 3.1.4. *Let $c : [0,1] \to U$ be a path in U joining p to q that does not intersect σ. Then there exists a point s_0 on c such that the geodesic segments $[p,s_0]$ and $[q,s_0]$ do not intersect σ and for whom the following inequality holds: $d'(p,a) + d'(a,q) < d'(p,s_0) + d'(s_0,q)$.*

Proof. We call the points on c such that the segment $[q,c(t)]$ intersects σ and the segment $[p,c(t)]$ does not intersect σ, points of type I.

We call the points on c such that the segment $[p,c(t)]$ intersects σ, points of type II. Notice that if $c(t)$ is a point of type II, then the segment $[q,c(t)]$ might also intersect σ.

We call the points on c such that the segments $[q,c(t)]$ and $[p,c(t)]$ do not intersect σ, points of type III.

Suppose that there are no points of type III on c. Any point on the path c is hence either a point of type I or a point of type II. Thus, for any $t \in [0,1]$, at least one of the segments $[q,c(t)]$ and $[p,c(t)]$ intersects σ.

We define the mapping $m : c[0,1] \times c[0,1] \to c[0,1]$ by $\forall t_1, t_2 \in [0,1]$, $m(c(t_1), c(t_2)) = c(t)$, $t \in [0,1]$, where $l(c|_{[t_1,t]}) = l(c|_{[t,t_2]}) = \frac{1}{2} l(c|_{[t_1,t_2]})$. Since K is CAT(0), Lemma 5.1.3 implies that the path c has a unique midpoint. The mapping m is therefore well-defined.

We define the sequence $(s_n)_{n\in\mathbb{N},n\to\infty}$ of tuples (s'_n, s''_n) as follows:

– the elements s'_n are points of type I;

– the elements s''_n are points of type II;

– s_0=(s'_0, s''_0)=(p, q);

– s_1=(s'_1, s''_1)=$\begin{cases} (s'_0, m(s'_0, s''_0)), \text{if } m(s'_0, s''_0) \text{ is a point of type II;} \\ \\ (m(s'_0, s''_0), s''_0), \text{if } m(s'_0, s''_0) \text{ is a point of type I;} \end{cases}$

...

– s_n=(s'_n, s''_n)=$\begin{cases} (s'_{n-1}, m(s'_{n-1}, s''_{n-1})), \text{if } m(s'_{n-1}, s''_{n-1}) \text{ is a point of type II;} \\ \\ (m(s'_{n-1}, s''_{n-1}), s''_{n-1}), \text{if } m(s'_{n-1}, s''_{n-1}) \text{ is a point of type I.} \end{cases}$

Let $s'_n = c(t'_n)$ be a point of type I on c and let $s''_n = c(t''_n)$ be a point of type II on c, $n \geq 1$.

Since s_n is a point of type I on c, while s'_n is a point of type II on c, the geodesic segments $[q, s'_n]$ and $[p, s''_n]$ intersect σ. Because these geodesic segments are unique and because the value of any Alexandrov angle lies, by its definition, between 0 and π, we have

$$\angle_a(q,p) + \angle_a(p, s'_n) \leq \pi$$

and

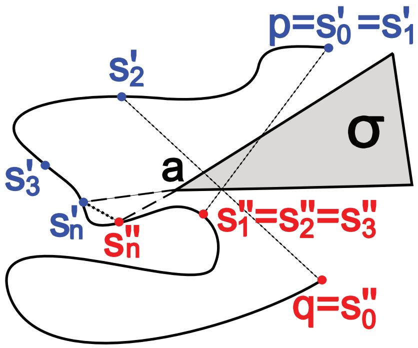
$$\angle_a(p,q) + \angle_a(q, s''_n) \leq \pi.$$

Adding the above inequalities, we get

$$2 \cdot \angle_a(q,p) + \angle_a(p, s'_n) + \angle_a(q, s''_n) \leq 2\pi.$$

Further, because the curvature at the point a in U is ≤ 0,

$$\angle_a(q,p) + \angle_a(p, s'_n) + \angle_a(s'_n, s''_n) + \angle_a(s''_n, q) \geq 2\pi.$$



The path c connecting p to q in U.
The blue points s'_k on the path c are points of type I, $0 \le k \le n$.
The red points s''_k on the path c are points of type II, $0 \le k \le n$.

$$\begin{array}{lcl}
(s'_0, s''_0) & = & (\mathrm{p}, \mathrm{q}); \\
(s'_1, s''_1) & = & (s'_0, m(s'_0, s''_0)); \\
(s'_2, s''_2) & = & (m(s'_1, s''_1), s''_1); \\
(s'_3, s''_3) & = & (m(s'_2, s''_2), s''_2); \\
\quad \dots & & \\
(s'_n, s''_n) & = & (s'_{n-1}, m(s'_{n-1}, s''_{n-1})) \text{ or } (m(s'_{n-1}, s''_{n-1}), s''_{n-1}).
\end{array}$$

The above relations imply that $\angle_a(s'_n, s''_n) \ge \angle_a(p, q)$. Hence, since $\angle_a(p, q) \ge \angle_a(b, c) \ne 0$, we have

$$\lim_{n\to\infty} \angle_a(s'_n, s''_n) \ne 0. \tag{1}$$

Further, since

$$\begin{cases} l(c|_{[t'_n, t''_n]}) = \frac{1}{2^n} l(c|_{[0,1]}), & \text{if } t'_n \le t''_n, \\ \\ l(c|_{[t''_n, t'_n]}) = \frac{1}{2^n} l(c|_{[0,1]}), & \text{if } t''_n \le t'_n, \end{cases}$$

it follows that

$$\begin{cases} \lim_{n\to\infty} l(c|_{[t'_n, t''_n]}) = 0, & \text{if } t'_n \le t''_n, \\ \\ \lim_{n\to\infty} l(c|_{[t''_n, t'_n]}) = 0, & \text{if } t''_n \le t'_n. \end{cases}$$

There exists a unique geodesic segment in U joining $s'_n = c(t'_n)$ to $s''_n = c(t''_n)$

whose length equals $d(s'_n, s''_n)$. Because

$$\begin{cases} 0 \leq d(s'_n, s''_n) \leq l(c|_{[t'_n, t''_n]}), & \text{if } t'_n \leq t''_n, \\ \\ 0 \leq d(s'_n, s''_n) \leq l(c|_{[t''_n, t'_n]}), & \text{if } t''_n \leq t'_n, \end{cases}$$

we get

$$\lim_{n\to\infty} d(s'_n, s''_n) = 0. \tag{2}$$

Because the path c does not pass through a, the points s'_n and s''_n are both distinct from a. The angle of the geodesic triangle $\triangle(a, s'_n, s''_n)$ at a is hence the Alexandrov angle between the geodesic segments $[a, s'_n]$ and $[a, s''_n]$ issuing from a. We consider the comparison triangle $\overline{\triangle}(a, s'_n, s''_n)$ in $\mathbb{R}^2$ for $\triangle(a, s'_n, s''_n)$. The comparison angle $\overline{\angle}_a(s'_n, s''_n)$ is well-defined since s'_n and s''_n both differ from a. By (2), it follows that

$$\lim_{n\to\infty} \overline{\angle}_a(s'_n, s''_n) = 0.$$

Since U is CAT(0),

$$0 \leq \angle_a(s'_n, s''_n) \leq \overline{\angle}_a(s'_n, s''_n)$$

and hence

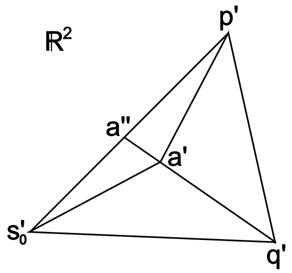
$$\lim_{n\to\infty} \angle_a(s'_n, s''_n) = 0.$$

Due to (1), the above relation implies a contradiction. So there exist points of type III on the path c. Let s_0 be a point of type III on c.

Because the geodesic segment $[p, q]$ intersects σ and it does not pass through a, $\angle_a(p, q) < \pi$. So, since the curvature at the point a in U is ≤ 0, we have

$$\angle_a(p, s_0) + \angle_a(s_0, q) > \pi. \tag{3}$$

Let $\triangle(p', a', s'_0)$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle(p, a, s_0)$ and let $\triangle(q', a', s'_0)$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle(q, a, s_0)$. We place the comparison triangles $\triangle(p', a', s'_0)$ and $\triangle(q', a', s'_0)$ in different half-planes with respect to the line $a's'_0$ in $\mathbb{R}^2$.



Comparison triangles in $\mathbb{R}^2$

Because any geodesic triangle in U satisfies the CAT(0) inequality, by (3) we have $\angle_{a'}(p', s_0') + \angle_{a'}(s_0', q') > \pi$. So the comparison triangles $\triangle(p', a', s_0')$ and $\triangle(q', a', s_0')$ in $\mathbb{R}^2$ are placed one with respect to the other as in the figure from above. Because the curvature at any point in $\mathbb{R}^2$ equals zero, while any Euclidean triangle has curvature zero, we get: $\angle_{p'}(a', q') < \angle_{p'}(s_0', q')$ and $\angle_{q'}(a', p') < \angle_{q'}(s_0', p')$. The point a' lies hence in the interior of the Euclidean triangle $\triangle(p', s_0', q')$. We consider a point a'' on $[p', s_0']$ such that a' lies on $[a'', q']$. Thus

$$d_{\mathbb{R}^2}(p', a') + d_{\mathbb{R}^2}(a', q') <$$

$$< d_{\mathbb{R}^2}(p', a'') + d_{\mathbb{R}^2}(a'', a') + d_{\mathbb{R}^2}(a', q') =$$

$$= d_{\mathbb{R}^2}(p', a'') + d_{\mathbb{R}^2}(a'', q').$$

Further

$$d_{\mathbb{R}^2}(p', a'') + d_{\mathbb{R}^2}(a'', q') <$$

$$< d_{\mathbb{R}^2}(p', a'') + d_{\mathbb{R}^2}(a'', s_0') + d_{\mathbb{R}^2}(s_0', q') =$$

$$= d_{\mathbb{R}^2}(p', s_0') + d_{\mathbb{R}^2}(s_0', q').$$

Hence

$$d_{\mathbb{R}^2}(p', a') + d_{\mathbb{R}^2}(a', q') < d_{\mathbb{R}^2}(p', s_0') + d_{\mathbb{R}^2}(s_0', q').$$

The above inequality implies that in U we have

$$d(p,a) + d(a,q) < d(p,s_0) + d(s_0,q),$$

for any point s_0 such that the geodesic segments $[p,s_0]$ and $[s_0,q]$ do not intersect σ. The geodesic segments $[p,s_0]$ and $[s_0,q]$ in U belong therefore to U'. So the same inequality holds in U'

$$d'(p,a) + d'(a,q) < d'(p,s_0) + d'(s_0,q).$$

□

Lemma 3.1.5. *The geodesic segment $[p,q]$ in U' with respect to d', is the union of the geodesic segments $[p,a]$ and $[a,q]$.*

Proof. We denote by $c : [0,1] \to U'$ the path obtained by concatenating the segments $[p,a]$ and $[a,q]$. Among all paths joining p to q in U' which pass through a, the path c has the shortest length.

Suppose that there exists a path $c_0 : [0,1] \to U'$ connecting p to q in U' that does not pass through a and whose length is less or equal to the length of the path c. Because the path c_0 does not intersect σ, there exists, according to Lemma 3.1.4, a point s_0 on c_0 such that the geodesic segments $[p,s_0]$ and $[s_0,q]$ do not intersect σ. The geodesic segments $[p,s_0]$ and $[s_0,q]$ in U belong therefore to U'. So

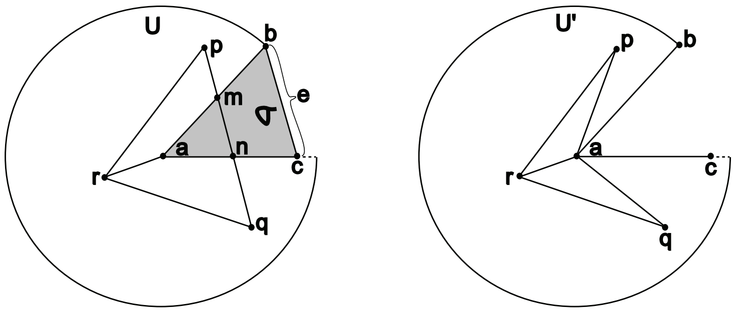
$$d'(p,s_0) + d'(s_0,q) \leq l(c_0) \leq l(c) = d'(p,a) + d'(a,q)$$

which is, by Lemma 3.1.4, a contradiction. Thus any path in U' joining p to q and which does not pass through a, is longer than c.

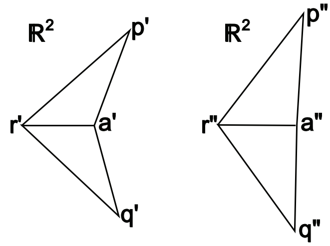
Altogether, it follows that the geodesic segment joining p to q in U' with respect to d' is the union of the geodesic segments $[p,a]$ and $[a,q]$.

□

Lemma 3.1.6. *Let r be a point in U such that the geodesic segments $[r,p]$ and $[r,q]$ do not intersect σ and such that the quadrilaterals ramp and ranq are convex. Then, the geodesic triangle $\triangle(p,r,q)$ in U' satisfies the CAT(0) inequality.*



The subcomplex K' is non-positively curved



Comparison triangles in $\mathbb{R}^2$

Proof. By Lemma 3.1.5, $d'(p,q) = d'(p,a) + d'(a,q)$.

Let $\triangle(p'',r'',q'')$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle(p,r,q)$ in U'. Let $\triangle(r',p',a')$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle(r,p,a)$ in U and let $\triangle(r',q',a')$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle(r,q,a)$ in U. We place the comparison triangles $\triangle(r',p',a')$ and $\triangle(r',q',a')$ in different half-planes with respect to the line $r'a'$ in $\mathbb{R}^2$.

Because U is CAT(0), the following inequalities hold: $\angle_p(r,a) \leq \angle_{p'}(r',a')$, $\angle_r(p,a) \leq \angle_{r'}(p',a')$, $\angle_q(r,a) \leq \angle_{q'}(r',a')$, $\angle_r(q,a) \leq \angle_{r'}(q',a')$.

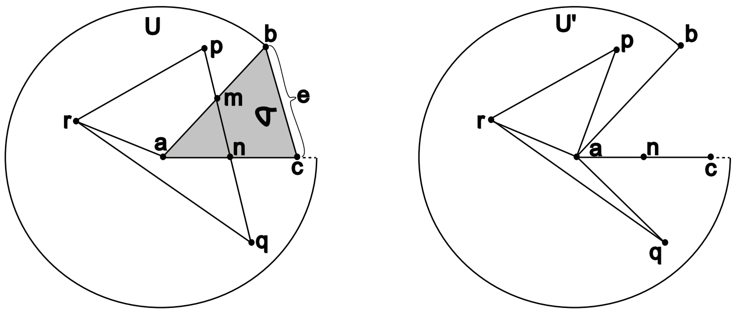
We consider $a'' \in [p'',q'']$ such that $d_{\mathbb{R}^2}(p'',a'') = d_{\mathbb{R}^2}(p',a')$. Since $\angle_{a'}(r',p') + \angle_{a'}(r',q') \geq \pi$, Alexandrov's lemma implies that $\angle_{r'}(p',a') + \angle_{r'}(q',a') \leq \angle_{r''}(p'',q'')$, $\angle_{p'}(r',a') \leq \angle_{p''}(r'',a'')$ and $\angle_{q'}(r',a') \leq \angle_{q''}(r'',a'')$.

So the following inequalities hold: $\angle_p(r,a) \leq \angle_{p'}(r',a') \leq \angle_{p''}(r'',a'')$, $\angle_q(r,a) \leq \angle_{q'}(r',a') \leq \angle_{q''}(r'',a'')$, $\angle_r(p,q) \leq \angle_r(p,a) + \angle_r(q,a) \leq \angle_{r'}(p',a') + \angle_{r'}(q',a') \leq \angle_{r''}(p'',q'')$. The geodesic triangle $\triangle(p,r,q)$ in U' satisfies thus the CAT(0) inequality.

□

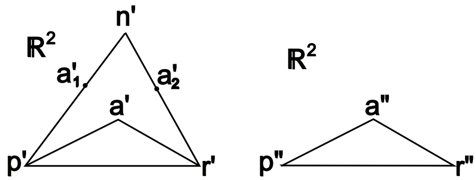
Lemma 3.1.7. *Let r be a point in U such that the geodesic segments $[r, p]$ and $[r, q]$ do not intersect σ and such that the quadrilateral ramp is convex, while the quadrilateral ranq is concave. Then, the geodesic triangle $\triangle(p, n, r)$ in U' satisfies the CAT(0) inequality.*

Proof. By Lemma 3.1.5, $d'(p, n) = d'(p, a) + d'(a, n)$ and $d'(r, n) = d'(r, a) + d'(a, n)$.



The subcomplex K' is non-positively curved

Let $\triangle(p', n', r')$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle(p, n, r)$ in U' and let $\triangle(p'', a'', r'')$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle(p, a, r)$ in U.



Comparison triangles in $\mathbb{R}^2$

Because U is CAT(0), the following inequalities hold: $\angle_p(a, r) \leq \angle_{p''}(a'', r'')$ and $\angle_r(a, p) \leq \angle_{r''}(a'', p'')$.

We consider a point a' in the interior of the Euclidean triangle $\triangle(p', n', r')$ such that $d_{\mathbb{R}^2}(p', a') = d'(p, a)$ and $d_{\mathbb{R}^2}(r', a') = d'(r, a)$. Thus $\angle_{p'}(a', r') \leq$

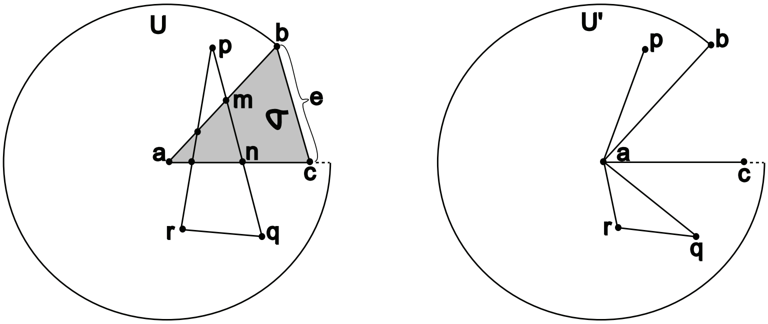
$\angle_{p'}(n', r')$ and $\angle_{r'}(a', p') \leq \angle_{r'}(n', p')$. Since the Euclidean triangles $\triangle(p', a', r')$ and $\triangle(p'', a'', r'')$ are congruent to each other, $\angle_{p'}(a', r') \equiv \angle_{p''}(a'', r'')$ and $\angle_{r'}(a', p') \equiv \angle_{r''}(a'', p'')$.

So $\angle_p(n, r) \leq \angle_{p'}(n', r')$ and $\angle_r(n, p) \leq \angle_{r'}(n', p')$. Since the angle between the geodesic segments $[n, p]$ and $[n, r]$ in U' equals zero, the geodesic triangle $\triangle(p, n, r)$ in U' satisfies the CAT(0) inequality.

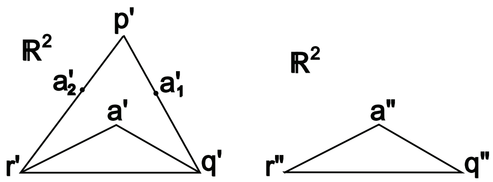
□

Lemma 3.1.8. *Let r be a point in U such that the geodesic segment $[r, p]$ intersects σ. Then, the geodesic triangle $\triangle(p, q, r)$ in U' satisfies the CAT(0) inequality.*

Proof. By Lemma 3.1.5, $d'(p, q) = d'(p, a) + d'(a, q)$ and $d'(p, r) = d'(p, a) + d'(a, r)$.



The subcomplex K' is non-positively curved



Comparison triangles in $\mathbb{R}^2$

Let $\triangle(p', q', r')$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle(p, q, r)$ in U' and let $\triangle(r'', a'', q'')$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle(r, a, q)$ in U.

Because U is CAT(0), the following inequalities hold: $\angle_r(a,q) \leq \angle_{r''}(a'',q'')$ and $\angle_q(a,r) \leq \angle_{q''}(a'',r'')$.

We consider a point a' in the interior of the Euclidean triangle $\triangle(p',q',r')$ such that $d_{\mathbb{R}^2}(r',a') = d'(r,a)$ and $d_{\mathbb{R}^2}(q',a') = d'(q,a)$. Thus $\angle_{r'}(a',q') \leq \angle_{r'}(p',q')$ and $\angle_{q'}(a',r') \leq \angle_{q'}(p',r')$. Since the Euclidean triangles $\triangle(r',a',q')$ and $\triangle(r'',a'',q'')$ are congruent to each other, $\angle_{r'}(a',q') \equiv \angle_{r''}(a'',q'')$ and $\angle_{q'}(a',r') \equiv \angle_{q''}(a'',r'')$.

In conclusion, the geodesic triangle $\triangle(p,q,r)$ in U' verifies the CAT(0) inequality.

□

Lemma 3.1.9. *Every point in $|K'|$ has a neighborhood which is a CAT(0) space.*

Proof. Let u,v,w be three distinct points in U chosen such that they do not belong to σ and such that the geodesic segments $[u,v]$, $[u,w]$ and $[v,w]$ in U do not intersect σ. Because the geodesic triangle $\triangle(u,v,w)$ in U' satisfies the CAT(0) inequality and considering the Lemmas 3.1.6, 3.1.7 and 3.1.8, we may conclude that any geodesic triangle in U' fulfills the CAT(0) inequality. So U' is a CAT(0) space.

Let y be a point in $|K|$ that does not belong to σ. Let U_y be a neighborhood of y homeomorphic to a closed ball of radius r_y, $U_y = \{x \in |K| \mid d(y,x) \leq r_y\}$. The radius r_y is chosen small enough such that U_y does not intersect σ. For any y in $|K'|$ that does not lie on $[a,b]$ or $[a,c]$, we consider a neighborhood U'_y that coincides with U_y. U'_y is hence a CAT(0) space.

So every point in $|K'|$ has a neighborhood which is a CAT(0) space.

□

We are now in the position to show the main result of the subsection: any finite, CAT(0) 2-complex retracts to a point through subspaces which remain, at each step, CAT(0) spaces.

Theorem 3.1.10. *Let K be a finite, CAT(0),* 2*-dimensional simplicial complex. Then K collapses to a point through CAT(0) subspaces $|K'|$.*

Proof. Proposition 3.1.1 implies that K has a 2-simplex with a free 1-dimensional face. We fix a point p in the interior of a 2-simplex of K. We define the map $R : |K| \times [0,1] \to |K|$ which associates for any $x \in |K|$ and for any $t \in [0,1]$, to (x,t) the point a distance $t \cdot d(p,x)$ from x along the geodesic segment $[p,x]$. Because $|K|$ has a strongly convex metric, the map R is a continuous retraction of $|K|$ to p. $R(|K| \times [0,1])$ is therefore contractible and then simply connected. Let a, b, c be any three distinct points in $R(|K| \times [0,1])$ such that the unique geodesic segment $[b,c]$ belongs to a 1-simplex that is the face of a single 2-simplex in the complex. For each $\delta = \triangle(a,b,c)$, we deformation retract $R(|K| \times [0,1])$ by pushing in δ starting at $[b,c]$. We obtain each time a subspace $|K'| = R(|K| \times [0,1])$ which remains simply connected and, by Proposition 3.1.3, non-positively curved. So $|K'|$ is a CAT(0) space implying that any two points in $|K'|$ are joined by a unique geodesic segment in $|K'|$. If at a certain step we delete the point p, we fix another point in the interior of a 2-simplex of K', define the map R as before and retract the space by CAT(0) subspaces further. Since K is finite, we reach, after a finite number of steps, a 1-dimensional spine L. Since $|L|$ is also a CAT(0) space, it is contractible. Taking into account that a contractible 1-complex is collapsible, the result follows.

□

Since a CAT(k) space is a CAT(0) space, for any real number $k \leq 0$, the above result implies the following corollary.

Corollary 3.1.11. *Let K be a finite, CAT(k),* 2*-dimensional simplicial complex,* $k \leq 0$ *being a real number. Then K collapses to a point through CAT(k) subspaces* $|K'|$.

3.2 Strongly convex simplicial complexes of dimension 2 and 3

We study in this subsection finite, strongly convex simplicial complexes of dimension 2 and 3. Namely, considering W. White's proofs, we show that these spaces are contractible and locally contractible and we obtain further results necessary to prove that such spaces are even collapsible (see [35]).

In the following lemma we investigate the contractibility and local contractibility of strongly convex simplicial complexes.

Lemma 3.2.1. *Let K be a finite simplicial complex that admits a strongly convex metric. Then K is contractible and locally contractible.*

Proof. Because K is a complete strongly convex metric space, any two points x, y in K can be joined by a unique segment in K. So, for each $t \in [0, 1]$, there exists a unique point z in $|K|$ such that $d(x, z) = t{\cdot}d(x, y)$ and $d(z, y) = (1-t){\cdot}d(x, y)$. We denote the interval $[0, 1]$ by I and the point z by $P(x, y, t)$. We prove further that the function $P : |K|\times|K|\times I \to |K|$ is continuous. Let $x_i, x, y_i, y \in |K|$ and let $t_i, t \in I$ such that $x_i \to x$, $y_i \to y$ and $t_i \to t$. Because K is compact, the sequence $p_i = P(x_i, y_i, t_i)$ has a subsequence $p_{n_i} = P(x_{n_i}, y_{n_i}, t_{n_i})$ that converges, say to p. Since $p_{n_i} = P(x_{n_i}, y_{n_i}, t_{n_i})$, the definition of the function P implies that $d(x_{n_i}, p_{n_i}) = t_{n_i}{\cdot}d(x_{n_i}, y_{n_i})$ and $d(p_{n_i}, y_{n_i}) = (1-t_{n_i}){\cdot}d(x_{n_i}, y_{n_i})$. Letting i converge to infinity, we get $d(x, p) = t\cdot d(x, y)$ and $d(p, y) = (1-t)\cdot d(x, y)$. The definition of P implies that $p = P(x, y, t)$. So, since $p_{n_i} \to p$, we have $P(x_{n_i}, y_{n_i}, t_{n_i}) \to P(x, y, t)$, for any subsequence $\{x_{n_i}\}$ of $\{x_i\}$, for any subsequence $\{y_{n_i}\}$ of $\{y_i\}$ and for any subsequence $\{t_{n_i}\}$ of $\{t_i\}$. Hence, since $P(x_i, y_i, t_i) \to P(x, y, t)$, P is a continuous function.

Let x_0 be a fixed point of $|K|$. We consider the homotopy $H : |K| \times [0, 1] \to |K|$, $H(x, t) = P(x, x_0, t)$: $H(x, 0) = x$, $\forall x \in |K|$, $H(x, 1) = x_0$, $\forall x \in |K|$ and $H(x, t) = z$, $\forall x \in |K|$, $\forall t \in [0, 1]$ where $z \in [x, x_0]$. So, since H shrinks $|K|$ along segments towards x_0, K is contractible.

Let A be a subset of $|K|$ and let a be a fixed point in A. Because A has a strongly convex metric, we can define the homotopy $H : A \times [0, 1] \to A$ that shrinks A along segments towards a. So K is locally contractible.

□

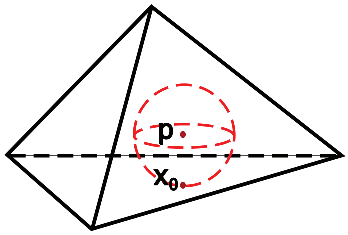
The proof of the collapsibility of finite strongly convex (, locally 6-large) simplicial complexes of dimension 2 (3) uses the following result.

Lemma 3.2.2. *Let K be a finite 3-complex that admits a strongly convex metric. Let L be a subcomplex of K that contains 3-simplices of K and their faces. Let α be a 3-simplex of L and let p be a fixed point in the interior of α. Then there exists a point x_0 of $|L|$ such that:*

1. *x_0 is not a vertex of K;*
2. *for any $y \in |L|$ such that $x_0 \in [p, y]$, the points x_0 and y coincide.*

Furthermore $|Lk(x_0, L)|$ can be contracted to a point in $|Lk(x_0, K)|$.

Proof. We define the contraction $H : |K| \times [0, 1] \to |K|$ such that $H(x, 0) = x, \forall x \in |K|$, $H(x, 1) = p, \forall x \in |K|$ and $H(x, t) = z, \forall x \in |K|, \forall t \in [0, 1]$ where $z \in [x, p]$.



We choose ε small enough such that the set $S_\varepsilon(p) = \{y \in |L| : d(p, y) = \varepsilon\}$ separates the 3-simplex α. Because $S_\varepsilon(p) \cap H(K^{(0)} \times [0, 1])$ is a finite set, the set $S_\varepsilon(p) \setminus H(K^{(0)} \times [0, 1])$ differs from the empty set. Let x be a point in $S_\varepsilon(p) \setminus H(K^{(0)} \times [0, 1])$. We define the set $F = \{y \in |L| : x \in [p, y]\}$. Because K is compact, F is also compact and hence it is bounded. So there exists a point x_0 in F such that for any $y \in F$, $d(y, p) \leq d(x_0, p)$. According to the hypothesis, for any $y \in |L|$, $x_0 \in [p, y]$. So $d(y, p) \geq d(x_0, p)$. In conclusion $d(y, p) = d(x_0, p)$.

Because K is a complete strongly convex metric space, there exists a unique segment joining p to y and there exists a unique segment joining p to x_0. Thus, since $x_0 \in [p, y]$ and $d(y, p) = d(x_0, p)$, the points y and x_0 coincide.

Altogether, there exists a point x_0 in $|L|$ that is not a vertex of K such that for any $y \in |L|$ with $x_0 \in [p, y]$, the points x_0 and y coincide.

Because $St(x_0, K)$ has the structure of a cone, $|St(x_0, K)| = x_0|Lk(x_0, K)|$, we can define the continuous map $\pi : |St(x_0, K)| \setminus \{x_0\} \to |Lk(x_0, K)|$. We choose $\varepsilon > 0$ small enough such that $S_\varepsilon(x_0) \subset |St(x_0, K)|$. We define the continuous map $\pi_\varepsilon : |Lk(x_0, K)| \to S_\varepsilon(x_0)$. We consider further the mapping $\pi H(\cdot, t)\pi_\varepsilon : |Lk(x_0, L)| \to |Lk(x_0, K)|$. This map is well-defined if there exists a neighborhood N_0 of x_0 such that $H(N_0, t) \subset int|St(x_0, K)|, \forall t \in [0, 1]$. Furthermore, because $|Lk(x_0, K)|$ is locally contractible, the neighborhood N_0 of x_0

must be chosen such that $\pi H(N_0, t)$ can be contracted to a point in $|Lk(x_0, K)|$. Besides ε must be chosen small enough such that $S_\varepsilon(x_0) \subset N_0$. If these conditions are fulfilled, the map $\pi H(\cdot, t)\pi_\varepsilon$ is well-defined. Since $\pi H(\cdot, t)\pi_\varepsilon$ is continuous, it preserves the structure of $|Lk(x_0, K)|$ of being contractible to a point in $|Lk(x_0, K)|$. Hence $|Lk(x_0, L)|$ can also be contracted to a point in $|Lk(x_0, K)|$.

□

3.3 Collapsing a strongly convex 2-dimensional simplicial complex

We give in this subsection a metric characterization of collapsible 2-complexes. Namely, considering W. White's proof, we show that finite, strongly convex 2-complexes collapse to a point (see [35]).

Theorem 3.3.1. *Let K be a finite, strongly convex, 2-dimensional simplicial complex. Then K has a 1-dimensional spine.*

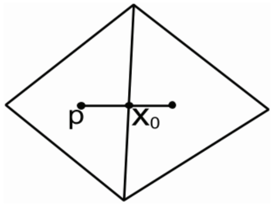
Proof. Let α_1 be a 2-simplex of K and let p be a fixed vertex in the interior of α_1. Lemma 3.2.2 implies that there exists a point x_0 in $|K|$ such that:

1. x_0 is not a vertex of K;
2. for any $y \in |K|$ such that $x_0 \in [p, y]$, the points x_0 and y must coincide.

Let β be a 1-dimensional face of α_1 such that x_0 lies on β.

We show that K has a 2-simplex with a free 1-dimensional face. Suppose, on the contrary, that K has no 2-simplex with a free 1-dimensional face. So β is contained in at least two 2-simplices α_1 and α_2. Because x_0 lies on β and p is a point in the interior of α_1, we can prolong the segment $[p, x_0]$ towards the interior of α_2. But in that case the second property of the point x_0 does no longer hold, which is a contradiction. So K has a 2-simplex with a free 1-dimensional face.

Performing an elementary collapse on K, we obtain a subcomplex K'. It similarly follows that K' has a 2-simplex with a free 1-dimensional face so it also admits an elementary collapse. Because K is finite, there exists a finite



K contains a 2-simplex with a free 1-dimensional face

sequence of elementary collapses starting with K which eventually terminates in a 1-dimensional spine.

□

We prove further the main result of the subsection.

Corollary 3.3.2. *Let K be a finite, strongly convex, 2-dimensional simplicial complex. Then K is collapsible.*

Proof. Because K admits a strongly convex metric, it has a 1-dimensional spine L. Since K is contractible, L remains contractible. A contractible 1-complex being collapsible, the corollary follows.

□

3.4 Collapsing a locally 6-large, strongly convex 3-dimensional simplicial complex

We prove in this subsection that finite, locally 6-large, strongly convex 3-dimensional simplicial complexes are collapsible (see [7]). The proof has two steps. Firstly, following W. White's proof given in [35], we show that these spaces collapse to a 2-dimensional spine. The spine does not inherit the metric condition, but it inherits the combinatorial curvature condition and it remains simply connected. Secondly, we refer to a result proven by J. Corson and B. Corson (see [17]) to conclude that the combinatorial condition ensures the collapsibility of the simply connected, 2-dimensional spine (see [7]).

We start by investigating whether the subcomplex obtained by performing an elementary collapse on a finite, 6-large simplicial complex of dimension

3 and 2, remains 6-large.

Lemma 3.4.1. *Let K be a finite, 6-large 2-complex that has a 2-simplex α with a free 1-dimensional face β. Then the subcomplex $K' = K \setminus \{\beta, \alpha\}$ is 6-large.*

Proof. Let γ be a full cycle in K of length 7 with one edge e_1 which coincides with β and with another edge e_2 which coincides with another face of α. So the edges e_1 and e_2 are consecutive edges of the 2-simplex α. By performing an elementary collapse on K, the cycle γ in K becomes a cycle γ' in K' which has length 6. So K' remains 6-large.

Let γ be a full cycle in K of length $6, 5$ or 4 with one edge e_1 that coincides with β and with another edge e_2 that coincides with another 1-dimensional face of α. So the edges e_1 and e_2 are consecutive edges of the 2-simplex α. By performing an elementary collapse on K, the cycle γ becomes a cycle γ' of length $5, 4$ or 3. Since γ' is a full cycle in K as well, it must have some two consecutive edges contained in a common 2-simplex of K. So, since γ' also has some two consecutive edges contained in a common 2-simplex of K', K' remains 6-large.

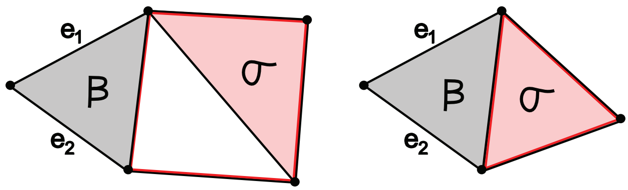
□

Lemma 3.4.2. *Let K be a finite, 6-large 3-complex that has a 3-simplex α with a free 2-dimensional face β. Then the subcomplex $K' = K \setminus \{\beta, \alpha\}$ is 6-large.*

Proof. Let γ be a full cycle in K of length 6 which has some two consecutive edges e_1, e_2 which are faces of β. Since by performing an elementary collapse on K, we do not delete any edges of K, there exists a cycle γ' in K' that coincides with γ in K and has therefore length 6. Thus K' remains 6-large.

Let γ be a full cycle in K of length 5 or 4 such that it has some two consecutive edges e_1, e_2 which are faces of β (see the figure below). By performing an elementary collapse on K we do not delete any edges of K. So there exists a cycle γ' in K' that coincides with γ in K. Because K is 6-large and since γ has length less than 6, γ must have some two consecutive edges contained in a common 2-simplex of K. In case this 2-simplex is β, whom we delete, we consider another full cycle γ'' in K which contains the third edge of β (that differs from e_1 and e_2) and the other edges of γ (which differ from e_1 and e_2).

Because γ'' has length less than 6 and since K is 6-large, γ'' must have some two consecutive edges contained in a common 2-simplex σ of K. So γ has some two consecutive edges contained in a common 2-simplex σ of K. Thus γ' also has some two consecutive edges contained in a common 2-simplex σ of K'. So K' remains 6-large.



K' remains 6-large

□

Lemma 3.4.3. *Let K be a finite,* 6*-large* 3*-complex that has a* 3*-simplex* α *with a free* 1*-dimensional face* β*. Then the subcomplex* $K' = K \setminus \{\beta, \alpha\}$ *is* 6*-large.*

Proof. Let γ be a full cycle in K of length 6 or 5 such that one of its 1-simplices is β. By performing an elementary collapse on K, we delete β and the two 2-simplices of α which have β as a face. We obtain a cycle γ' with length 1 greater than γ. So γ' has length 7 or 6 implying that K' remains 6-large.

Let γ be a full cycle in K of length 4 such that one of its 1-simplices is β. By performing an elementary collapse on K, we delete β and the two 2-simplices of α which have β as a face. We obtain a cycle γ' with length 1 greater than γ. Because γ' is a full cycle in K and since K is 6-large, γ' must have some two consecutive edges contained in a common 2-simplex of K. Thus K' remains 6-large.

□

We show further that any finite, strongly convex 3-complex has a 2-dimensional spine.

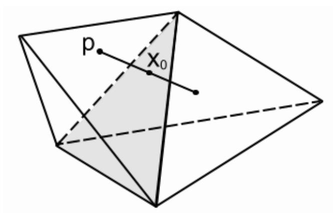
Theorem 3.4.4. *Let K be a finite* 3*-complex that admits a strongly convex metric. Then K has a* 2*-dimensional spine.*

Proof. Let L be a subcomplex of K that contains 3-simplices and their faces. Let α_1 be a 3-simplex of L. Let p be a fixed point in the interior of α_1. Lemma 3.2.2 implies that there exists a point x_0 in $|L|$ such that:

1. x_0 is not a vertex of K;
2. for any $y \in |L|$ for whom $x_0 \in [p, y]$, x_0 coincides with y.

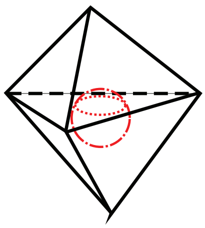
We denote by β the simplex that contains x_0. Suppose that β is 2-dimensional.

We show that K has a 3-simplex with a free 2-dimensional face. Suppose that, on the contrary, K has no 3-simplex with a free 2-dimensional face. So there exist two 3-simplices, one of them being α_1, the other one, say α_2, such that $\beta < \alpha_1$ and $\beta < \alpha_2$. Because x_0 is contained in β, we can prolong the segment $[p, x_0]$ towards the interior of α_2. But in that case the second property of the point x_0 does no longer hold which is a contradiction. So K has a 3-simplex with a free 2-dimensional face.



K contains a 3-simplex with a free 2-dimensional face

Suppose that β is 1-dimensional. We consider the link of β in L, $Lk(\beta, L) = \{\mu^{(1)} \in L | \beta\mu \in L\}$. We consider the link of x_0 in L. We notice that $Lk(x_0, L)$ is the suspension of $Lk(\beta, L)$.



K contains a 3-simplex with a free 1-dimensional face

We show that K contains a 3-simplex with a free 1-dimensional face. Suppose, on the contrary, that K has no 3-simplex with a free 1-dimensional face. So $Lk(x_0, L)$ might contain a 2-sphere. Because $Lk(x_0, L)$ is the suspension of $Lk(\beta, L)$, $Lk(\beta, L)$ contains in that case a closed curve. Since $|Lk(x_0, L)|$ can be contracted, according to Lemma 3.2.1, to a point in $|Lk(x_0, K)|$, $|Lk(x_0, L)|$ can not contain a 2-sphere. $Lk(\beta, L)$ can therefore not contain a closed curve. So $Lk(\beta, L)$ has a 1-simplex e with a free vertex v. Since $e \in Lk(\beta, L)$, βe is a 3-simplex in L. Because $v^{(0)} < e^{(1)}$ and v is a free vertex of e in $Lk(\beta, L)$, it follows that $[\beta v]^{(1)} < [\beta e]^{(3)}$. So L has a 3-simplex βe with a free 1-dimensional face βv.

Because K contains a 3-simplex with a free 2-dimensional face and a 3-simplex with a free 1-dimensional face, there exists a finite sequence of elementary collapses, starting with K, which eventually terminates in a 2-dimensional spine K'.

□

We present one of the main results of the paper.

Theorem 3.4.5. *Let K be a finite, locally 6-large, 3-dimensional simplicial complex that admits a strongly convex metric. Then K is collapsible.*

Proof. Because K admits a strongly convex metric, it is contractible and then simply connected. Since K has a convex metric, it is connected. So K is 6-systolic and then, according to Proposition 1.3.15, 6-large.

Since K admits a strongly convex metric, Theorem 3.4.4 implies that K has a 3-simplex with a free face. By performing an elementary collapse on K, we obtain a subcomplex K' which remains simply connected. According to Lemma 3.4.2 and Lemma 3.4.3, K' also remains 6-large. So, by Corollary 1.3.17, K' is locally 6-large.

Since K is finite, a finite sequence of elementary collapses starting with K eventually terminates in a 2-dimensional spine L. Since L is finite, simply connected and locally 6-large, the result follows by applying Theorem 1.3.14.

□

4 Collapsing cell complexes by using van Kampen diagrams

This section provides combinatorial characterizations of collapsible 2-dimensional cell complexes. We prove that finite, simply connected, 2-dimensional square (hexagon) complexes with the 8- (12-) property are collapsible (see [29], [27]). Besides, we describe the combinatorial structure of locally finite, simply connected, 2-dimensional square (hexagon) complexes with the 8- (12-) property (see [29], [27]). J. Corson and B. Corson obtained in [17] similar results. Namely, they proved that finite, simply connected, 2-dimensional simplicial complexes that have the 6-property, collapse to a point. They showed further that locally finite, simply connected, 2-dimensional simplicial complexes with the 6-property, possess a simple combinatorial structure. As in [17], our proof uses van Kampen diagrams.

4.1 Collapsing a 2-dimensional square complex with the 8-property

We consider in this section finite, 2-dimensional, square complexes that have the 8-property and prove that they are collapsible. We also analyze the combinatorial structure of infinite, 2-dimensional square complexes with the 8-property.

Definition 4.1.1. *A* 2*-dimensional square complex* K *has the* 8-property *if the link of each vertex of* K *is a graph of girth at least* 8*.*

The following lemma gives an important property of square disks whose interior vertices have degree at least 4.

Lemma 4.1.2. *Let* D *be a square disk whose interior vertices have degree at least* 4*. Then*

$$\sum_{v \in \partial D} (3 - \deg v) \geq 4,$$

summing over the boundary vertices of D*.*

Proof. We denote the set of interior vertices of D by $int(D)$. We denote by V, V_{int}, V_{ext}, E, E_{ext} and F, the number of vertices, interior vertices, exterior vertices, edges, exterior edges and 2-cells of D, respectively. The following relations hold in any square disk: $1 = V - E + F$ (Euler's characteristic), $2E - E_{ext} = 4F$, $V_{ext} = E_{ext}$, $\sum_v \deg v = 2E$. Using these relations, we obtain

$$\begin{aligned}
6 &= 6(V - E + F) = \\
&= 6V - \tfrac{3}{2} \cdot 2E - \tfrac{3}{2}V_{ext} = \\
&= 6V - \tfrac{3}{2}\Big(\sum_{v\in int(D)} \deg v + \sum_{v\in\partial D} \deg v\Big) - \tfrac{3}{2}V_{ext} = \\
&= \tfrac{3}{2}\Big(4V_{int} - \sum_{v\in int(D)} \deg v\Big) + \tfrac{3}{2}\Big(3V_{ext} - \sum_{v\in\partial D} \deg v\Big).
\end{aligned}$$

Thus

$$4 = \sum_{v\in int(D)} (4 - \deg v) + \sum_{v\in\partial D} (3 - \deg v).$$

Since D is a square disk whose interior vertices have degree at least 4, $\sum\limits_{v\in int(D)} (4 - \deg v) \leq 0$. Using the above relation, it follows

$$\sum_{v\in\partial D} (3 - \deg v) \geq 4.$$

□

Let K be a square complex. We define the following elementary operations on edge-paths in K.

1. *Free reduction*: Let α be an edge-path containing a subpath of the form ee^{-1} and let β be the edge-path obtained by deleting this subpath. We call the passage from α to β a *free reduction*.

2. *Short-cut*: Let v_0, v_1, v_2, v_3 be the vertices of a 2-cell in K. The passage from one edge-path to another, obtained

a. either by replacing an occurrence of a path $[v_0, v_1][v_1, v_2][v_2, v_3]$ by the single edge $[v_0, v_3]$,

b. or by replacing an occurrence of a path $[v_0, v_1]$ by the edge-path $[v_0, v_3]$ $[v_3, v_2]$ $[v_2, v_1]$,

is called a *short-cut*.

3. *Elementary exchange*: Let v_0, v_1, v_2, v_3 be the vertices of a 2-cell in K. The passage from one edge-path to another, obtained by replacing an occurrence of a path $[v_0, v_1][v_1, v_2]$ by the path $[v_0, v_3][v_3, v_2]$, is called an *elementary exchange*.

One can notice that none of the elementary operations alter the endpoints of an edge-path. Free reductions and short-cuts change the length of an edge-path, whereas elementary exchanges do not. This will be important.

If we can pass from one edge-path α to another edge-path β via a finite sequence of elementary exchanges, then we call the edge-paths α and β *exchangeable* and we write $\alpha \equiv \beta$. An edge-path α is *strongly reduced* if for any edge-path β, exchangeable with α, β does not admit a free reduction or short-cut.

If there exists a finite sequence of elementary operations passing from an edge-path α to another edge-path β, then α and β are path-homotopic. The converse affirmation also holds for 2-dimensional square complexes with the 8-property.

Theorem 4.1.3. *Let K be a 2-dimensional square complex with the 8-property. Let β be a strongly reduced edge-path in K. If α is any edge-path, path-homotopic to β, then there exists a finite sequence of elementary operations passing from α to β.*

Proof. We apply elementary operations on α until it becomes strongly reduced. We must prove the existence of a finite sequence of elementary exchanges from α to β.

Since the edge-path $\alpha\beta^{-1}$ is null-homotopic, there exists a van Kampen diagram (D, ϕ) for $\alpha\beta^{-1}$. D is a square disk. We choose a van Kampen diagram for $\alpha\beta^{-1}$ of minimal area. (D, ϕ) is thus a reduced van Kampen diagram for

$\alpha\beta^{-1}$. Since the edge-paths α and β in K are strongly reduced, no boundary vertex of D has degree smaller than 2.

Let v_0, v_1 and v_2 be boundary vertices of D. Because K has the 8-property and (D, ϕ) is a reduced van Kampen diagram for a closed edge-path in K, D also has the 8-property. Thus, since $(3 - \deg v_i) \leq 1$, $0 \leq i \leq 2$, Lemma 4.1.2 implies

$$\begin{aligned} 4 &\leq (3 - \deg v_0) + (3 - \deg v_1) + (3 - \deg v_2) + \\ &\quad + \sum_{v \in \partial D, v \notin \{v_0, v_1, v_2\}} (3 - \deg v) \leq \\ &\leq 3 + \sum_{v \in \partial D, v \notin \{v_0, v_1, v_2\}} (3 - \deg v). \end{aligned}$$

So D has an exterior vertex $v \notin \{v_0, v_1, v_2\}$ such that $\deg v \leq 2$. Since $\deg v = 2$, the open star of v in D contains one 2-cell.

By deleting in D the open star of v, we perform either a short-cut or an elementary exchange on α or β and we construct a disk $D' = D \setminus \mathrm{St}\, v$. Because α and β are strongly reduced, we can not perform short-cuts on them. So, the disk D' is obtained by performing an elementary exchange on, say α. We reach hence another edge-path γ. Thus $\gamma \equiv \alpha$.

Because elementary exchanges preserve the length of edge-paths, the diagram (D', ϕ) is a van Kampen diagram for $\alpha\beta^{-1}$ as well. But the disk D' has one 2-cell less than D, which is a contradiction since D has minimal area. So, by performing an elementary exchange either on α or on β, we reach either β or α. Hence $\alpha \equiv \beta$.

$\square$

Let K be a 2-dimensional square complex. Let e, e' be two directed edges in K such that $i(e) = i(e') = v$. We denote by $\rho(e, e')$ the length of a shortest edge-path in $\mathrm{Lk}(v, K)$ joining $t(e)$ and $t(e')$. We define $\rho(e, e')$ to be infinite, if there does not exist an edge-path joining $t(e)$ and $t(e')$ in K. We call an edge-path $e_1 ... e_n$ in K a *locally geodesic* if $\rho(e_i^{-1}, e_{i+1}) \geq 4$ for all $1 \leq i < n$. The term 'locally' does not have to its traditional meaning. Instead, it suggests that such an edge-path can not be deleted by any elementary operations on K. So there exists an edge-path between any two points in K joined by a locally geodesic, no matter the elementary operations we perform on the complex.

By its definition, a locally geodesic edge-path does not admit any elementary operations. So, if K is a 2-dimensional square complex with the 8-property, any closed locally geodesic edge-path in K is, by Theorem 4.1.3, essential.

The following results represent applications of Theorem 4.1.3. The first proves the collapsibility of finite, simply connected, 2-dimensional square complexes with the 8-property. The second gives a combinatorial description of a locally finite, simply connected, 2-dimensional square complexes with the 8-property.

Corollary 4.1.4. *Let K be a finite, simply connected,* 2*-dimensional square complex with the* 8*-property. Then K is collapsible.*

Proof. Suppose that there exists a connected subcomplex of K, L, that has more than one vertex and that does not admit any elementary collapses. If L is 1-dimensional, its fundamental group is nontrivial. If L is 2-dimensional, each 0- and 1-cell of L is contained in at least two cells of L. Because a locally geodesic edge-path does not permit any elementary collapses, by choosing edges in succession, we can construct in L a locally geodesic edge-path α. L being finite, α is eventually closed. Hence, because L inherits from K the 8-property, Theorem 4.1.3 implies that α is not null-homotopic in L. The fundamental group of L is therefore nontrivial. But $|K|$ is simply connected; the fundamental group of any connected subcomplex of K is therefore trivial. This implies a contradiction. So any connected subcomplex of K admits an elementary collapse. K being finite, there exists a finite sequence of elementary collapses, starting with K, which eventually terminates in a one-point subcomplex. So K collapses to a point.

□

Corollary 4.1.5. *Let K be a locally finite, simply connected,* 2*-dimensional square complex with the* 8*-property. Then K is a monotone union of a sequence of collapsible subcomplexes.*

Proof. Let v_0 be a fixed vertex in K. For each integer n, let L_n be the full subcomplex of K generated by the vertices that can be joined to v_o by an edge-path of length at most n. Thus, for each n, L_n is finite and connected. Since $K = \bigcup_{n=1}^{\infty} L_n$, the corollary follows due to the above result, once we have shown that, for each n, $|L_n|$ is simply connected.

Let $\alpha = e_1 ... e_n$ be a closed edge-path in L_n with endpoints at v_0. We denote by $v_i = t(e_i) = i(e_{i+1})$ and by γ_i an edge-path in K from v_o to v_i of minimal length. We consider the edge-path $\gamma = e_1\gamma_1^{-1}\gamma_1 e_2\gamma_2^{-1}\gamma_2 ...\ e_{n-1}\gamma_{n-1}^{-1}\gamma_{n-1}e_n$ in L_n that freely reduces to α.

Because $|K|$ is simply connected, the edge-paths $\gamma_i e_{i+1}$ and γ_{i+1} are path-homotopic. Because K has the 8-property, Theorem 4.1.3 implies that there exists, for each i, a finite sequence of elementary operations passing from $\gamma_i e_{i+1}$ to γ_{i+1}. Each edge-path in this sequence lies in L_n. The edge-path γ is therefore null-homotopic in L_n and so is α. $|L_n|$ is hence simply connected.

□

Since 2-dimensional square complex with the 8-property are collapsible, the weaker condition of contractibility does also characterize these spaces.

Corollary 4.1.6. *Let K be a locally finite, simply connected, 2-dimensional square complex with the 8-property. Then K is contractible.*

Since collapsible square 2-complexes admit a strongly convex metric, we may conclude that a combinatorial curvature condition on a square 2-complex, given by the 8-property, guarantees the existence of a strongly convex metric on the complex.

Corollary 4.1.7. *Let K be a locally finite, simply connected, 2-dimensional square complex with the 8-property. Then K admits a strongly convex metric.*

4.2 Collapsing a 2-dimensional hexagon complex with the 12-property

This subsection gives a second combinatorial characterization of collapsible 2-complexes. Namely, we prove that finite, 2-dimensional hexagon complexes with the 12-property, collapse to a point.

Definition 4.2.1. *A* 2*-dimensional hexagon complex K has the* 12-property *if the link of each vertex of K is a graph of girth at least* 12*.*

The following lemma gives an important property which holds for any hexagon disk whose interior vertices have degree at least 3.

Lemma 4.2.2. *Let D be a hexagon disk whose interior vertices have degree at least* 3*. Then*

$$\sum_{v\in\partial D}\left(\tfrac{5}{2}-\deg v\right)\geq 3,$$

summing over the boundary vertices of D.

Proof. We denote the set of interior vertices of D by $int(D)$. We denote by V, V_{int}, V_{ext}, E, E_{ext} and F, the number of vertices, interior vertices, exterior vertices, edges, exterior edges and 2-cells of D, respectively. The following relations hold in any hexagon disk: $1 = V - E + F$ (Euler's characteristic), $2E - E_{ext} = 6F$, $V_{ext} = E_{ext}$, $\sum_v \deg v = 2E$. Using these relations, we obtain

$$6 = 6(V-E+F) =$$

$$= 6V - 4E - V_{ext} =$$

$$= \left(6V_{int} - 2\sum_{v\in int(D)}\deg v\right) + \left(5V_{ext} - 2\sum_{v\in\partial D}\deg v\right).$$

So

$$3 = \sum_{v\in int(D)}(3-\deg v) + \sum_{v\in\partial D}\left(\tfrac{5}{2}-\deg v\right).$$

Because D is a hexagon disk whose interior vertices have degree at least 3, we have $\sum_{v\in int(D)}(3-\deg v)\leq 0$. The above relation further implies

$$\sum_{v\in\partial D}\left(\tfrac{5}{2}-\deg v\right)\geq 3.$$

□

Let K be a hexagon complex. We introduce the following elementary operations on edge-paths in K.

1. *Free reduction*: Let α be an edge-path containing a subpath of the form ee^{-1}. Let β be the edge-path obtained by deleting this subpath. We call the passage from α to β a *free reduction*.

2. *Short-cut*: Let $v_0, v_1, v_2, v_3, v_4, v_5$ be the vertices of a 2-cell in K. The passage from one edge-path to another, obtained

a. either by replacing an occurrence of an edge-path $[v_0, v_1]$ by the edge-path $[v_0, v_5]$ $[v_5, v_4]$ $[v_4, v_3]$ $[v_3, v_2]$ $[v_2, v_1]$,

b. or by replacing an occurrence of an edge-path $[v_0, v_1]$ $[v_1, v_2]$ by the edge-path $[v_0, v_5]$ $[v_5, v_4]$ $[v_4, v_3]$ $[v_3, v_2]$,

c. or by replacing an occurrence of an edge-path $[v_0, v_1]$ $[v_1, v_2]$ $[v_2, v_3]$ $[v_3, v_4]$ by the edge-path $[v_0, v_5]$ $[v_5, v_4]$,

d. or by replacing an occurrence of an edge-path $[v_0, v_1]$ $[v_1, v_2]$ $[v_2, v_3]$ $[v_3, v_4]$ $[v_4, v_5]$ by the edge-path $[v_0, v_5]$,

is called a *short-cut*.

5. *Elementary exchange*: Let $v_0, v_1, v_2, v_3, v_4, v_5$ be the vertices of a 2-cell in K. The passage from one edge-path to another, obtained by replacing an occurrence of an edge-path $[v_0, v_1][v_1, v_2][v_2, v_3]$ by the edge-path $[v_0, v_5][v_5, v_4][v_4, v_3]$, is called an *elementary exchange*.

Similar to the elementary operations defined on square complexes, the ones defined on hexagon complexes do not alter the endpoints of an edge-path either. Although free reductions and short-cuts modify the length of an edge-path, elementary exchanges do not. This will be important.

In a 2-dimensional hexagon complex with the 12-property, we can essentially pass from one edge-path to any path-homotopic edge-path, via a finite sequence of elementary operations, as proven in the following theorem.

Theorem 4.2.3. *Let K be a 2-dimensional hexagon complex with the 12-property. Let β be a strongly reduced edge-path in K. If α is any edge-path that is path-homotopic to β, then there exists a finite sequence of elementary operations passing from α to β.*

Proof. We apply elementary operations on α until it becomes strongly reduced. We must show that there exists a finite sequence of elementary exchanges from α to β.

Since α and β are path-homotopic, the edge-path $\alpha\beta^{-1}$ is null-homotopic. So there exists a van Kampen diagram (D, ϕ) for $\alpha\beta^{-1}$. D is a hexagon disk. We choose a van Kampen diagram for $\alpha\beta^{-1}$ of minimal area. Since the diagram (D, ϕ) is reduced and the edge-paths α and β in K are strongly reduced, D has no boundary vertex with degree smaller than 2.

Because K has the 12-property and (D, ϕ) is a reduced van Kampen diagram for a closed edge-path in K, D also has the 12-property. Let v_0, v_1, v_2, v_3 and v_4 be boundary vertices of D. Since $(\frac{5}{2} - \deg v_i) \leq \frac{1}{2}$, $0 \leq i \leq 4$, Lemma 4.2.2 implies that

$$3 \leq (\tfrac{5}{2} - \deg v_0) + (\tfrac{5}{2} - \deg v_1) + (\tfrac{5}{2} - \deg v_2) + (\tfrac{5}{2} - \deg v_3)+$$

$$+(\tfrac{5}{2} - \deg v_4) + \sum_{v\in\partial D, v\notin\{v_0,v_1,v_2,v_3,v_4\}} (\tfrac{5}{2} - \deg v) \leq$$

$$\leq \tfrac{5}{2} + \sum_{v\in\partial D, v\notin\{v_0,v_1,v_2,v_3,v_4\}} (\tfrac{5}{2} - deg v).$$

So D has an exterior vertex $v \notin \{v_0, v_1, v_2, v_3, v_4\}$ such that $\deg v \leq 2$. Because $\deg v = 2$, the open star of v in D contains one 2-cell.

Because α and β are strongly reduced, we can not perform short-cuts or free reductions on them. By deleting the open star of v in D, we obtain a disk $D' = D \setminus \mathrm{St}\, v$. So D' is constructed by performing an elementary exchange on, say α. We reach hence another edge-path γ. So $\gamma \equiv \alpha$.

Since elementary exchanges do not change the length of edge-paths, the diagram (D', ϕ) is a van Kampen diagram for $\alpha\beta^{-1}$ as well. But the disk D' has one 2-cell less than D, a contradiction because D has minimal area. Therefore, by performing an elementary exchange either on α or on β, we reach either β or α. So $\alpha \equiv \beta$.

$\square$

Let K be a 2-dimensional hexagon complex. Let e, e' be two directed edges in K such that $i(e) = i(e') = v$. We denote by $\rho(e, e')$ the length of a shortest edge-path in $\mathrm{Lk}(v, K)$ joining $t(e)$ and $t(e')$. We define $\rho(e, e')$ to be infinite if there does not exist any edge-path in K joining $t(e)$ to $t(e')$. An edge-path $e_1 ... e_n$ is called a *locally geodesic* if $\rho(e_i^{-1}, e_{i+1}) \geq 6$ for all $1 \leq i < n$. By its definition, a locally geodesic edge-path does not permit any elementary operations. So, if K is a 2-dimensional hexagon complex with the 12-property, any closed locally geodesic edge-path in K is, according to Theorem 4.2.3, essential.

Applying Theorem 4.2.3, the collapsibility of any finite, simply connected, 2-dimensional hexagon complex with the 12-property, follows.

Corollary 4.2.4. *Let K be a finite, simply connected, 2-dimensional hexagon complex that has the 12-property. Then K is collapsible.*

Similar to infinite, 2-dimensional square complexes with the 8-property, infinite, 2-dimensional hexagon complexes with the 12-property also possess a simple combinatorial structure.

Corollary 4.2.5. *Let K be a locally finite, simply connected, 2-dimensional hexagon complex with the 12-property. Then K is a monotone union of a sequence of collapsible subcomplexes.*

Since 2-dimensional hexagon complex with the 12-property are collapsible, the weaker condition of contractibility does also characterize these spaces.

Corollary 4.2.6. *Let K be a locally finite, simply connected, 2-dimensional hexagon complex with the 12-property. Then K is contractible.*

Because collapsible 2-dimensional hexagon complexes admit a strongly convex metric, the 12-property ensures the existence of a strongly convex metric on the complex.

Corollary 4.2.7. *Let K be a finite, simply connected, 2-dimensional hexagon complex that has the 12-property. Then K admits a strongly convex metric.*

5 Collapsing cell complexes by applying discrete Morse theory

The results we obtain in this section are based on the regular tessellations of the Euclidean plane: by triangles, squares, and hexagons. K. Crowley used in [18] the first possibility of subdividing the Euclidean plane (by triangles) when proving the collapsibility of finite, CAT(0) simplicial complexes of dimension 3 or less endowed with the standard piecewise Euclidean metric , under a technical hypothesis. We consider the other two possibilities, and obtain similar results.

We review first the basic steps in K. Crowley's proof (see [18], [8]). We show further that CAT(0) cubical complexes of dimension 2 and 3, endowed with the standard piecewise Euclidean metric, and hexagonal complexes of dimension 2 endowed with the standard piecewise Euclidean metric, also collapse to a point (see [26], [9]). As in [18], our results follow by applying discrete Morse theory.

In subsection 5.1 (5.5, 5.9) we analyze the geometry of CAT(0) triangulated (cubical, hexagonal) disks. In subsection 5.2 (5.6, 5.10) we show, by applying discrete Morse theory, that any CAT(0) simplicial (cubical, hexagonal) 2-complex is collapsible. The construction of the CAT(0) simplicial (cubical, hexagonal) 2-complexes is essential. Namely, we obtain a CAT(0) simplicial (cubical, hexagonal) 2-complex, by endowing the 2-complex with the standard piecewise Euclidean metric such that each interior vertex of the complex has degree at least 6 (4, 3). The standard piecewise Euclidean metric on the 2-complex becomes then CAT(0). In subsection 5.3 (5.7) we investigate the geometry of CAT(0) simplicial (cubical) 3-complexes. The main result of section 5.3 (5.7) states that any CAT(0) simplicial (cubical) 3-complex admits, for each closed edge-path, an immersion of a CAT(0) triangulated (cubical) disk of minimal area which maps the boundary of the disk to the edge-path in the complex. Subsection 5.4 (5.8) finds necessary and sufficient conditions for the collapsibility of a finite CAT(0) simplicial (cubical) 3-complex.

5.1 The geometry of CAT(0) triangulated disks

We present in this subsection results proven in [18] regarding the geometry of CAT(0) triangulated disks. A CAT(0) triangulated disk D is constructed by endowing it with the standard piecewise Euclidean metric such that each of its interior vertices has degree at least 6. The standard piecewise Euclidean metric on the disk becomes then CAT(0).

Lemma 5.1.1. *Let D be a triangulated disk whose interior vertices have degree at least* 6*. Then*

$$\sum_{v \in \partial D} (4 - \deg v) \geq 6,$$

summing over the boundary vertices of D.

Proof. See [18]. ☐

Definition 5.1.2. *A* geodesic disk *is a triangulated disk D that satisfies* deg $v \geq 6$ *for all interior vertices v, and whose exterior vertices lie on the combinatorial geodesics* $[v_n, v_{n-1}, ..., v_1, v_0]$ *and* $[\overline{v}_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$*. If* $v_n = \overline{v}_n$*, we call D a* geodesic disk of type I. *If* $d_c(v_n, \overline{v}_n) = 1$*, then we call D a* geodesic disk of type II.

Definition 5.1.3. *Let J be a simplicial complex whose underlying space is homeomorphic to* $\mathbb{R}^2$ *such that* $\deg v \geq 6$*, for all v in J. A connected, finite subcomplex S of J is called a* string of pearls *if it is simply connected and if its exterior vertices lie on the combinatorial geodesics* $[v_n, v_{n-1}, ..., v_1, v_0]$ *and* $[\overline{v}_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$*. If* $v_n = \overline{v}_n$*, then we call S a* string of pearls of type I. *If* $d_c(v_n, \overline{v}_n) = 1$*, then we call S a* string of pearls of type II.

Theorem 5.1.4. *Let S be a string of pearls whose exterior vertices are the vertices of the combinatorial geodesics* $[v_n, v_{n-1}, ..., v_1, v_0]$ *and* $[\overline{v}_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$*. Then each vertex v of S lies on a combinatorial geodesic either from* v_n *to* v_0 *or from* $\overline{v}_n$ *to* v_0*.*

Proof. See [18]. ☐

Corollary 5.1.5. *Let S be a string of pearls whose exterior vertices are the vertices of the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[\overline{v}_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$. Then $d_c(v, v_0) < n$, for each interior vertex v of S.*

Proof. See [18].

□

Lemma 5.1.6. *Let S be a string of pearls whose exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[\overline{v}_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$, $n \geq 2$. If $v_{n-1} \neq \overline{v}_{n-1}$, then $v_n, v_{n-1}, \overline{v}_{n-1}$ span a 2-simplex in S.*

Proof. See [18].

□

5.2 Collapsing a CAT(0) 2-dimensional simplicial complex

We give in this subsection an apparently metric although in fact combinatorial characterization of collapsible 2-dimensional simplicial complexes.

Theorem 5.2.1. *Let K be a 2-dimensional simplicial complex endowed with the standard piecewise Euclidean metric. If each interior vertex of K have degree at least 6, then K is collapsible.*

Proof. See [18].

□

Notice that the simplicial complex in the above theorem is a CAT(0) space and therefore a CAT(k) space, $k \leq 0$ being a real number.

5.3 The geometry of CAT(0) 3-dimensional simplicial complexes

We investigate in this subsection the geometry of CAT(0) simplicial complexes of dimension 3 or less.

Lemma 5.3.1. *Let K be a simplicial complex whose underlying space $|K|$ is simply connected. Let $[w_1, w_2, ..., w_k, w_1]$ be a closed combinatorial path in K. Then there exists a triangulated disk D whose k exterior vertices are the vertices of the closed combinatorial path $[v_1, v_2, ..., v_k, v_1]$. Furthermore, there exists a simplicial map $|\phi| : |D| \to |K|$ satisfying $\phi(v_i) = w_i$ for all i.*

Proof. See [18].

□

Lemma 5.3.2. *Let K be a simplicial complex. Let $w_1, w_2, ..., w_k$ be distinct vertices and let $[w_1, w_2, ..., w_k, w_1]$ be a closed combinatorial path in K. Let D be a triangulated disk whose exterior vertices are the vertices of the closed combinatorial path $[v_1, v_2, ..., v_k, v_1]$ such that $|\phi| : |D| \to |K|$ is a simplicial map with $\phi(v_i) = w_i$ for all i. If D has minimal area, then:*

1. $|\phi|$ *maps i-simplices to i-simplices, $0 \le i \le 2$;*
2. $|\phi|(\beta_1) \neq |\phi|(\beta_2)$, *whenever $\beta_1^{(2)} \neq \beta_2^{(2)}$ contain a common edge.*

Proof. See [18].

□

Lemma 5.3.3. *Let K be an n-dimensional simplicial complex ($n \geq 2$) endowed with the standard piecewise Euclidean metric. Then:*

1. for any vertex w of K, $\overline{B}(w, \sqrt{\frac{n+1}{2n}}) \subseteq \overline{St}(w)$;

2. let w be a vertex in K and let m be the midpoint of an edge e in K. If $d(w, m) \le \frac{\sqrt{3}}{2}$, then $m \in \overline{St}(w)$;

3. if w and w' are distinct vertices of K such that $d(w, w') < \sqrt{\frac{2(n+1)}{n}}$, then $d(w, w') = 1$.

Proof. See [18].

□

Lemma 5.3.4. *Let K be an n-dimensional simplicial complex ($n \geq 2$) endowed with the standard piecewise Euclidean metric. Let $[w_1, w_2, w_3, w_1]$ be a closed* *combinatorial path in K. If $|K|$ is CAT(0), then the vertices w_1, w_2 and w_3 span a 2-simplex in K.*

Proof. See [18].

□

Theorem 5.3.5. *Let K be a 3-dimensional simplicial complex. Let w_1,..., w_k be distinct vertices such that $[w_1, ..., w_k, w_1]$ is a closed combinatorial path in K. Let D be a regular piecewise Euclidean disk whose exterior vertices are the vertices of the closed combinatorial path $[v_1, ..., v_k, v_1]$ such that $|\phi| : |D| \to |K|$ is a simplicial map with $\phi(v_i) = w_i$ for all i. If D has minimal area and $|K|$ is CAT(0), then $|D|$ is CAT(0).*

Proof. See [18].

□

Lemma 5.3.6. *Let K be a simplicial complex of dimension three or less endowed with the standard piecewise Euclidean metric. Let $w_0, ..., w_n, \overline{w}_1, ..., \overline{w}_m$ be distinct vertices on the combinatorial geodesics $[w_n, w_{n-1}, ..., w_1, w_0]$ and $[\overline{w}_m, \overline{w}_{m-1}, ..., \overline{w}_1, w_0]$ satisfying $d_c(w_n, \overline{w}_m) = 1$, $m \in \{n, n-1\}$. Let D be a triangulated disk whose exterior vertices are the vertices of the combinatorial path $[v_0, v_1, ..., v_n, \overline{v}_m, \overline{v}_{m-1}, ..., \overline{v}_1]$. Let $|\phi| : |D| \to |K|$ be a simplicial map with $\phi(v_i) = w_i$ and $\phi(\overline{v}_i) = \overline{w}_i$ for all i. Let β denote the unique 2-simplex of D containing the edge $[v_n, \overline{v}_m]$. Let w be the vertex of $|\phi|(\beta)$ opposite $[w_n, \overline{w}_m]$. If D has minimal area and $|K|$ is CAT(0), then $d_c(w, w_0) = n - 1$.*

Proof. See [18].

□

5.4 Collapsing a CAT(0) 3-dimensional simplicial complex

We find in this subsection necessary and sufficient conditions for the collapsibility of a finite simplicial complex of dimension 3 or less. The result follows by applying discrete Morse theory.

Theorem 5.4.1. *Let K be a finite simplicial complex of dimension 3 or less endowed with the standard piecewise Euclidean metric. If $|K|$ is CAT(0), then K admits a discrete Morse function with no critical edges.*

Proof. See [18].

□

Corollary 5.4.2. *Let K be a finite simplicial complex of dimension 3 or less endowed with the standard piecewise Euclidean metric. If $|K|$ is CAT(0) and if it satisfies the property that every 2-simplex of K is a face of at most two 3-simplices of K, then K is collapsible.*

Proof. See [18].

□

Since a CAT(0) space is a CAT(k) space, $k \leq 0$ being a real number, the following holds.

Corollary 5.4.3. *Let K be a finite simplicial complex of dimension 3 or less endowed with the standard piecewise Euclidean metric. If $|K|$ is CAT(k), $k \leq 0$ being a real number, and if it satisfies the property that every 2-simplex of K is a face of at most two 3-simplices of K, then K is collapsible.*

5.5 The geometry of CAT(0) cubical disks

The proof of the collapsibility of finite CAT(0) cubical 3-complexes is based on results obtained on CAT(0) cubical disks immersed cellwise into cubical 3-complexes. Understanding the geometry of CAT(0) cubical disks is therefore necessary. We will show that a CAT(0) cubical disk possesses a good direction of flow along the edges of its triangulation.

Let D be a cubical disk endowed with the standard piecewise Euclidean metric such that each of its interior vertices has degree at least 4. The standard piecewise Euclidean metric on the disk becomes then CAT(0).

The following inequality holds in any cubical disk whose interior vertices have degree at least 4. The proof was given in section 4.1.

Lemma 5.5.1. *Let D be a cubical disk whose interior vertices have degree at least 4. Then*

$$\sum_{v \in \partial D} (3 - \deg v) \geq 4,$$

summing over the boundary vertices of D.

Definition 5.5.2. *A* geodesic disk *is a cubical disk D that satisfies* $\deg v \geq 4$ *for all interior vertices v, and whose exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[v_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$.*

Lemma 5.5.3. *Let D be a geodesic disk whose exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[v_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$, $n \geq 2$. Then D has an exterior vertex $v \in \{v_1, \overline{v}_1, ..., v_{n-1}, \overline{v}_{n-1}\}$ such that* $\deg v = 3$.

Proof. For $1 \leq k \leq n-1$, the degree of v_k must be at least 3. Otherwise the vertices $v_{k-1}, v_k, v_{k+1}, v_{k+2}$ would span a 2-cell in D, contradicting the fact that $[v_n, ..., v_{k-1}, v_k, v_{k+1}, v_{k+2} ..., v_0]$ is a combinatorial geodesic in D. Similarly, deg $\overline{v}_k \geq 3$, $1 \leq k \leq n-1$.

Because the boundary vertices v_0 and v_n of D have each at least one neighbor, $(3 - \deg v_0) + (3 - \deg v_n) \leq 4$. Lemma 5.5.1 further implies

$$4 \leq (3 - \deg v_0) + (3 - \deg v_n) + \sum_{v \in \partial D, v \notin \{v_0, v_n\}} (3 - \deg v) \leq$$

$$\leq 4 + \sum_{v \in \partial D, v \notin \{v_0, v_n\}} (3 - \deg v).$$

So there exists an exterior vertex $v \in \{v_1, \overline{v}_1, ..., v_{n-1}, \overline{v}_{n-1}\}$ such that $\deg v \leq 3$. D has therefore an exterior vertex $v \in \{v_1, \overline{v}_1, ..., v_{n-1}, \overline{v}_{n-1}\}$ such that $\deg v = 3$.

□

The following definition generalizes the notion of geodesic disk.

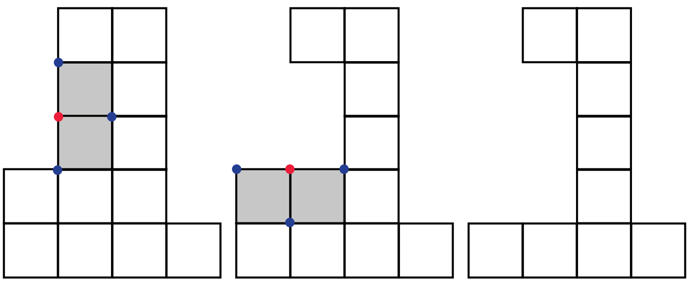
Definition 5.5.4. *Let J be a cubical complex whose underlying space is homeomorphic to $\mathbb{R}^2$ such that* $\deg v \geq 4$ *for all interior vertices v of J. A connected, finite subcomplex S of J is called a* string of pearls *if it is simply connected and if its exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[v_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$ in J.*

Let S be a string of pearls whose exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[v_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$. Each exterior vertex in S lies, by the definition of a string of pearls, on a combinatorial geodesic from v_n to v_0. We show that each vertex of S lies on a combinatorial geodesic from v_n to v_0. There exists hence a good direction of flow along the edges of a string of pearls.

Theorem 5.5.5. *Let S be a string of pearls whose exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[v_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$. Then each vertex of S lies on a combinatorial geodesic from v_n to v_0.*

Proof. Lemma 5.5.3 implies that S has an exterior vertex v_{k_1}, $1 \leq k_1 \leq n-1$ such that $\deg v_{k_1} = 3$. So the closed star of v_{k_1} in S contains two 2-cells and their faces. We consider the subcomplex $S_1 = S - \overline{St} v_{k_1}$ of S obtained by deleting the closed star of v_{k_1} in S. Because S is a string of pearls, it is simply connected. Because S deformation retracts to S_1, S_1 remains simply connected. Each interior vertex of S_1 is an interior vertex of S and has therefore at least 4 distinct neighbors. So the subcomplex S_1 is a string of pearls. Each exterior vertex of S_1 lies therefore on a combinatorial geodesic from v_n to v_0.

Because S_1 is a string of pearls, Lemma 5.5.3 implies that it has an exterior vertex v_{k_2} with 3 distinct neighbors, $1 \leq k_2 \leq n-1$. So the closed star of v_{k_2} in S_1 contains two 2-cells and their faces. Because the subcomplex $S_2 = S_1 - \overline{St} v_{k_2}$ remains a string of pearls, every exterior vertex of S_2 lies on a combinatorial geodesic from v_n to v_0.



Deleting 2-cells in a string of pearls

We retract further and obtain each time a string of pearls. Because S is finite, we reach, after a finite number of steps, a string of pearls S' with no interior vertices. Because each exterior vertex of S' lies on a combinatorial geodesic from v_n to v_0, the theorem follows.

$\square$

The following corollary proves that the combinatorial distance function measured along the edges of a string of pearls, is maximized on its boundary.

Corollary 5.5.6. *Let S be a string of pearls whose exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[v_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$. Then, for each interior vertices v of S, we have $d_c(v, v_0) < n$.*

Proof. By Theorem 5.5.5, every vertex of S lies on a combinatorial geodesic from v_n to v_0. Hence $d_c(v, v_0) < d_c(v_n, v_0) = n$.

$\square$

The following lemma provides information regarding the structure of the star of an exterior vertex in a string of pearls.

Lemma 5.5.7. *Let S be a string of pearls whose exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[v_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$, $n \geq 2$. If $v_{n-1} \neq \overline{v}_{n-1}$, then the vertices $v_n, v_{n-1}, \overline{v}_{n-1}$ and v_{n-2} span a 2-cell in S.*

Proof. For $n = 2$, the exterior vertices of S lie on the combinatorial geodesics $[v_2, v_1, v_0]$ and $[v_2, \overline{v}_1, v_0]$. Thus $\deg v_2 \geq 2$, $\deg v_1 \geq 2$, $\deg \overline{v}_1 \geq 2$ and $\deg v_0 \geq 2$. Lemma 5.5.1 further implies

$$(3 - \deg v_2) + (3 - \deg v_1) + (3 - \deg \overline{v}_1) + (3 - \deg v_0) \geq 4.$$

So the following inequalities hold

$$\deg v_2 \leq 2, \deg v_1 \leq 2, \deg \overline{v}_1 \leq 2 \text{ and } \deg v_0 \leq 2.$$

Since the vertices $v_2, v_1, \overline{v}_1$ and v_0 have degree exactly 2 in the subcomplex bounded by the closed combinatorial path $[v_2, v_1, \overline{v}_1, v_0, v_2]$, they span a 2-cell in S.

In general, let $[v_{k+1}, v_k, ..., v_1, v_0]$ and $[v_{k+1}, \overline{v}_k, ..., \overline{v}_1, v_0]$ be the combinatorial geodesics the exterior vertices of S lie on. Suppose that the vertices

v_{k+1}, v_k, v_{k-1} and $\overline{v}_k$ do not span a 2-cell in S. The vertex v_{k+1} has therefore $r > 2$ neighbors. Assume without loss of generality that $\deg v_{k+1} = 3$. We denote by v'_k the third neighbor of v_{k+1}, besides v_k and $\overline{v}_k$. Theorem 5.5.5 implies that v_{k+1} is the only neighbor of v'_k combinatorial distance $k+1$ from v_0. Theorem 5.5.5 further implies that $d_c(v_k, v_0) = d_c(\overline{v}_k, v_0) = k$. Because D is a cubical disk, v'_k has no neighbors combinatorial distance k from v_0. It follows by induction that v'_k is the only neighbor of v_{k-1} combinatorial distance k from v_0. So v_{k-1} is the only neighbor of v'_k combinatorial distance $k-1$ from v_0. Thus $\deg v'_k \leq 2$, a contradiction since v'_k is an interior vertex of a string of pearls. The vertices v_{k+1}, v_k, v_{k-1} and $\overline{v}_k$ span therefore a 2-cell in S.

□

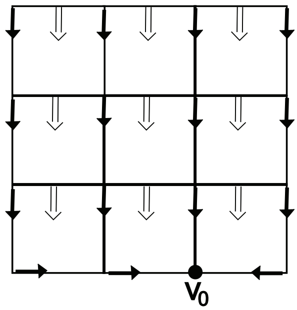
5.6 Collapsing a CAT(0) 2-dimensional cubical complex

We show in this subsection that 2-dimensional CAT(0) cubical complexes collapse to a point. The idea of the proof is to fix a vertex v of the complex and to apply discrete Morse theory to the function "combinatorial distance from v".

Theorem 5.6.1. *Let K be a 2-dimensional cubical complex endowed with the standard piecewise Euclidean metric. If each interior vertex of K has degree at least 4, then K is collapsible.*

Proof. Notice that $|K|$ is a CAT(0) space and therefore a CAT(k) space, $k \leq 0$ being a real number.

We define on K the vector field $V : K \to K \bigcup \{0\}$. We fix a vertex v_0 of K and we define $V(v_0) = 0$. For each vertex v of K different from v_0, we define $V(v) = e = [v, u]$, where $[v, u, ..., v_0]$ is a combinatorial geodesic from v to v_0. Such a combinatorial geodesic exists because $|K|$ has a convex metric. For each edge e in the image of V, we define $V(e) = 0$. For each edge e not in the image of V, we consider the string of pearls S bounded by e and the V-paths from the endpoints of e to v_0. Because e belongs to the boundary of S, there exists a unique 2-cell σ such that $e < \sigma$. We define $V(e) = \sigma$. So V is defined such that it has no critical edges and a single critical vertex.



The gradient vector field of a discrete Morse function defined on K

According to the definition of V, there exists, for each edge $e \in ImV$, a unique vertex v such that $V(v) = e$. To verify that V is a discrete vector field defined on K, we still need to check whether, for each 2-cell $\sigma \in ImV$, there exists a unique edge e such that $V(e) = \sigma$. Let e be an edge in K whose endpoints are combinatorial distance n and $n-1$ from v_0. Corollary 5.5.6 and Lemma 5.5.7 imply that e is the only edge mapped by V to σ. V is thus a discrete vector field defined on K. According to its definition, the arrows of V always point closer to v_0. So V contains no nontrivial closed V-paths. V is hence a gradient vector field defined on K. So we can associate it a discrete Morse function with no critical edges and a single critical vertex.

Because K is contractible, the weak Morse inequalities imply $1 = \chi(K) = m_0 - m_1 + m_2 = 1 - 0 + m_2$, where m_i denotes the number of critical cells of dimension i. So $m_2 = 0$. The discrete Morse function defined on K has therefore no critical cells of dimension 1 or 2 and a single critical vertex. The collapsibility of K follows.

□

5.7 The geometry of CAT(0) 3-dimensional cubical complexes

In this subsection we prove that there exists, for each knot in a cubical 3-complex, an immersion of a cubical disk of minimal area into the complex which maps the boundary of the disk to the knot. The main result of the subsection states that if the complex is CAT(0), then the disk itself is CAT(0). This fact will allow us to use results obtained in subsection 5.5 on CAT(0) cubical disks, in order to obtain similar ones on CAT(0) cubical 3-complexes.

Lemma 5.7.1. *Let K be a cubical complex whose underlying space $|K|$ is simply connected. Then, for any closed combinatorial path $[w_1, ..., w_k, w_1]$ in K, there exists a cubical disk D whose k exterior vertices are the vertices of the closed combinatorial path $[v_1, ..., v_k, v_1]$ and a cell map $|\phi| : |D| \to |K|$ satisfying $\phi(v_i) = w_i$ for all i.*

Proof. Let A be a finitely triangulated cubical disk whose k exterior vertices are the vertices of the closed combinatorial path $[v_1, ..., v_k, v_1]$. Let $\psi : (Bd|A|)^{(0)} \to K^{(0)}$ be a continuous map such that $\psi(v_i) = w_i$ for all i. Theorem 1.1.27 implies that there exists a subdivision D of A and a cell map $|\phi| : |D| \to |K|$ such that D and A have the same exterior vertices and such that $|\psi| = |\phi|$ on the boundary of A.

□

The following lemma proves that the cell map which maps the boundary of a cubical disk of minimal area to a closed edge-path in a cubical complex, is an immersion.

Lemma 5.7.2. *Let K be a cubical complex and let $[w_1, ..., w_k, w_1]$ be a closed combinatorial path in K. Let D be a cubical disk whose k exterior vertices are the vertices of the closed combinatorial path $[v_1, ..., v_k, v_1]$ such that $|\phi| : |D| \to |K|$ is a cell map satisfying $\phi(v_i) = w_i$ for all i. If D has minimal area, then*

1. *$|\phi|$ maps i-cells to i-cells, $0 \leq i \leq 2$;*
2. *$|\phi|(\beta_1) \neq |\phi|(\beta_2)$, whenever $\beta_1^{(2)} \neq \beta_2^{(2)}$ contain a common edge.*

Proof.

1. Suppose that $|\phi|$ maps a 2-cell to an edge, a 2-cell to a vertex or an edge to a vertex. $|\phi|$ maps in each case an edge to a vertex. Let $[a, b]$ be an edge in D such that $\phi(a) = \phi(b)$. Let $[a, d, c, b, e, f, a]$ be a closed combinatorial path in D such that for any closed combinatorial path $[a, d', c', b, e', f', a]$ in D, at least

one of the vertices d', c', e' or f' is interior to $[a, d, c, b, e, f, a]$. We remove from D any cell that contains a vertex interior to $[a, d, c, b, e, f, a]$ as well as any cell that contains the edge $[a, b]$. Identifying $[a, d]$ with $[b, c]$ and $[a, f]$ with $[b, e]$, we obtain a cubical disk D' whose exterior vertices coincide with the exterior vertices of D, but which has at least two 2-cells less than D, a contradiction since D has minimal area. $|\phi|$ maps therefore i-cells to i-cells, $0 \le i \le 2$.

2. Let d and c be the vertices of β_1 that are not in β_2 and let f and e be the vertices of β_2 that are not in β_1. Hence $|\phi|(\beta_1) = |\phi|(\beta_2)$ if and only if $\phi(d) = \phi(c)$ and $\phi(f) = \phi(e)$. Suppose that $\phi(d) = \phi(c)$ and $\phi(f) = \phi(e)$. As before, let $[a, d, c, b, e, f, a]$ be a closed combinatorial path in D such that, for any closed combinatorial path $[a, d', c', b, e', f', a]$ in D, at least one of the vertices d', c', e' or f' is interior to $[a, d, c, b, e, f, a]$. We remove from D all cells with vertices interior to $[a, d, c, b, e, f, a]$ as well as β_1 and β_2 and their common edge. Identifying $[a, d]$ with $[b, c]$ and $[a, f]$ with $[b, e]$, we obtain a cubical disk D' whose exterior vertices coincide with the exterior vertices of D, but which has at least two 2-cells less than D, a contradiction since D has minimal area. Thus $|\phi|(\beta_1) \neq |\phi|(\beta_2)$.

□

Lemma 5.7.3. *Let K be an n-dimensional cubical complex ($n \ge 2$) endowed with the standard piecewise Euclidean metric. Then:*

1. *for any vertex w of K, $\overline{B}(w, 1) \subseteq \overline{St}(w)$;*

2. *let w be a vertex in K and let m be the midpoint of an edge e in K. If $d(w, m) \le \frac{\sqrt{5}}{2}$, then $m \in \overline{St}(w)$;*

3. *if w and w' are distinct vertices of K such that $d(w, w') < 2$, then $d(w, w') = 1$ or $d(w, w') = \sqrt{2}$.*

Proof.

1. Because the distance from a vertex w to the opposite face in an n-cell equals 1, the result follows.

2. Suppose that $m \notin \overline{St}(w)$. Then $St(e) \bigcap St(w) = \emptyset$. Let x be a point outside the star of e. Then $d(x, m) \geq \frac{1}{2} \cdot 1$. Any geodesic in K from m to w must leave the star of e and enter the star of w. Therefore

$$d(w, m) \geq d(w, x) + d(x, m) \geq$$

$$\geq 1 + \frac{1}{2} \cdot 1 > \frac{\sqrt{5}}{2},$$

which is a contradiction. So $m \in \overline{St}(w)$.

3. Suppose that $w' \notin \overline{St}w$. So any geodesic from w to w' must leave the star of w and enter the star of w'. Thus

$$d(w, w') \geq 2 \cdot 1 = 2,$$

which is a contradiction. Thus either $d(w, w') = 1$ or $d(w, w') = \sqrt{2}$.

$\square$

Lemma 5.7.4. *Let K be an n-dimensional cubical complex ($n \geq 2$) endowed with the standard piecewise Euclidean metric. Let $[w_1, w_2, w_3, w_4, w_1]$ be a closed combinatorial path in K. If $|K|$ is CAT(0), then the vertices w_1, w_2, w_3 and w_4 span a 2-cell in K.*

Proof. Let $d(w_2, w_4) = 2l$. We will show that either $d(w_1, w_3) \leq \sqrt{2}$ or $d(w_2, w_4) \leq \sqrt{2}$.

Suppose that $d(w_2, w_4) > \sqrt{2}$. Let $\triangle_1' = (w_1', w_2', w_4')$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle_1 = (w_1, w_2, w_4)$ in K. Because $|K|$ has a strongly convex metric, there exists a unique point m on the edge $[w_2, w_4]$ such that $d(w_2, m) = d(m, w_4) = \frac{1}{2} \cdot d(w_2, w_4)$. Let m' be the point in $\triangle_1'$ corresponding to m in $\triangle_1$. Because $|K|$ is CAT(0) and $\triangle_1'$ is isosceles, we have

$$d(w_1, m) \leq d_{\mathbb{R}^2}(w_1', m') = \sqrt{1 - l^2} \leq \frac{1}{\sqrt{2}}.$$

It similarly follows that

$$d(w_3, m) \leq \frac{1}{\sqrt{2}}.$$

Hence

$$d(w_1, w_3) \leq d(w_1, m) + d(m, w_3) \leq \sqrt{2}.$$

So either $d(w_1, w_3) \leq \sqrt{2}$ or $d(w_2, w_4) \leq \sqrt{2}$. Assume without loss of generality that $d(w_1, w_3) \leq \sqrt{2}$. Because $d(w_1, m) \leq \frac{1}{\sqrt{2}}$ and $d(w_3, m) \leq \frac{1}{\sqrt{2}}$, Lemma 5.7.3 implies that $m \in \overline{St}(w_1)$ and $m \in \overline{St}(w_3)$. So the vertices w_1, w_2, w_3 and w_4 span a 2-cell in K.

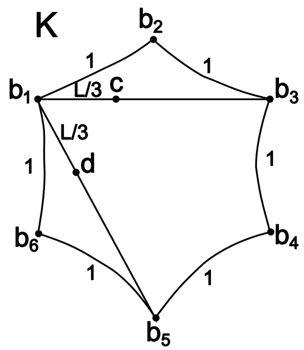
□

The next theorem gives a combinatorial analogue to a classical result in differential geometry which states that a minimal surface in $\mathbb{R}^3$ has nonpositive Gauss curvature. It represents the link between CAT(0) cubical disks and CAT(0) cubical 3-complexes and it will enable us to obtain results on CAT(0) cubical 3-complexes by referring to certain results proven on CAT(0) cubical disks.

Theorem 5.7.5. *Let K be a 3-dimensional cubical complex endowed with the standard piecewise Euclidean metric. Let w_1,..., w_k be distinct vertices such that $[w_1, ..., w_k, w_1]$ is a closed combinatorial path in K. Let D be a cubical disk endowed with the standard piecewise Euclidean metric whose exterior vertices lie on the combinatorial path $[v_1, ..., v_k, v_1]$. Let $|\phi| : |D| \to |K|$ be a cell map with $\phi(v_i) = w_i$ for all i. If D has minimal area and $|K|$ is a CAT(0) space, then $|D|$ is also a CAT(0) space.*

Proof. The piecewise Euclidean metric on D is CAT(0) if and only if each interior vertex of D has degree at least 4. Hence it suffices to show that D has no interior vertices of degree 3 or 2.

Suppose, on the contrary, that there exists an interior vertex a of D such that $\deg a = 2$. Let a_1, a_3 be vertices of D labeled such that $d(a, a_1) = 1, d(a, a_3) = 1$. Because D is a cubical disk endowed with the standard piecewise Euclidean metric, there exist vertices a_2, a_4 such that $[a_1, a_2, a_3, a_4]$ is a closed combinatorial path in Lka. Let $b_i = \phi(a_i)$ for $1 \leq i \leq 4$. Because D has minimal area, $|\phi|$ preserves the dimension of cells. So $[b_1, b_2, b_3, b_4]$ is a closed combinatorial path in K. Lemma 5.7.4 implies that the vertices b_1, b_2, b_3 and b_4 span a 2-cell in K. Let α be the 2-cell spanned by the vertices a_1, a_2, a_3 and a_4 in D. Identifying the edge $[a_2, a_1]$ with the edge $[a_4, a_1]$, and the edge $[a_2, a_3]$ with the edge $[a_4, a_3]$, we construct a disk $D' = D \setminus \{\alpha\}$. The cell map ϕ restricted to the vertices of D' induces a cell map from the $|D'|$ to $|K|$ which satisfies the conditions of Lemma 5.7.1. But D' has one fewer 2-cell than D which is a contradiction since D has minimal area. D contains therefore no interior vertices of degree 2.

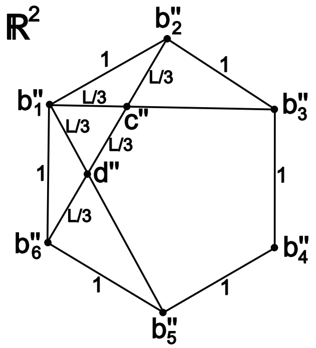


Inside K

Suppose that there exists an interior vertex a of D such that $dega = 3$. Let a_1, a_3, a_5 be vertices of D labeled such that $d(a, a_1) = 1, d(a, a_3) = 1$ and $d(a, a_5) = 1$. Because D is a cubical disk endowed with the standard piecewise Euclidean metric, there exist vertices a_2, a_4, a_6 such that $[a_1, a_2, a_3, a_4, a_5, a_6]$ is a closed combinatorial path in Lka. Let $b_i = \phi(a_i)$ for $1 \leq i \leq 4$. Because D has minimal area, $|\phi|$ preserves the dimension of cells. So $[b_1, b_2, b_3, b_4, b_5, b_6]$ is a closed combinatorial path in K.

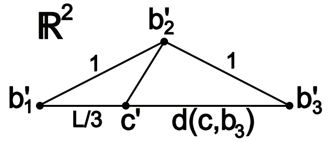
Let $L = \sqrt{3}$. We will show that at least one of $d(b_1, b_3), d(b_1, b_5)$ and $d(b_2, b_6)$ is equal to 1. Suppose that $d(b_1, b_3) > L$ and $d(b_1, b_5) > L$. Let $c \in [b_1, b_3]$ such that $d(b_1, c) = \frac{L}{3}$. Let $d \in [b_1, b_5]$ such that $d(b_1, d) = \frac{L}{3}$. Let $b_1''b_2''b_3''b_4''b_5''b_6''$ be a regular hexagon in $\mathbb{R}^2$.

Let $\triangle_1' = (b_1', b_2', b_3')$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle_1 = (b_1, b_2, b_3)$ in K. Let c' be a point on $\triangle_1'$ corresponding to c on $\triangle_1$. Because $|K|$ is



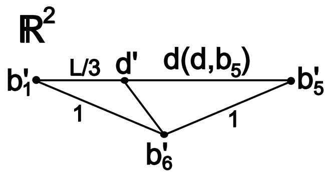
Regular hexagon in $\mathbb{R}^2$

CAT(0), $d(b_2, c) \le d_{\mathbb{R}^2}(b'_2, c')$. Properties of the Euclidean triangles imply that $d_{\mathbb{R}^2}(b'_2, c') < d_{\mathbb{R}^2}(b''_2, c'') = \frac{L}{3}$. Hence $d(b_2, c) < \frac{L}{3}$.



Comparison triangle in $\mathbb{R}^2$

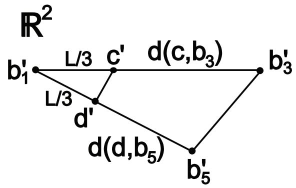
Let $\triangle'_2 = (b'_1, b'_5, b'_6)$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle_2 = (b_1, b_5, b_6)$ in K. Let d' be a point on $\triangle'_2$ corresponding to d on $\triangle_2$. Because $|K|$ is CAT(0), $d(b_6, d) \le d_{\mathbb{R}^2}(b'_6, d')$. Properties of the Euclidean triangles imply that $d_{\mathbb{R}^2}(b'_6, d') < d_{\mathbb{R}^2}(b''_6, d'') = \frac{L}{3}$. Hence $d(b_6, d) < \frac{L}{3}$.



Comparison triangle in $\mathbb{R}^2$

Let $\triangle'_3 = (b'_1, b'_3, b'_5)$ be a comparison triangle in $\mathbb{R}^2$ for $\triangle_3 = (b_1, b_3, b_5)$ in K. Let c' be a point on $\triangle'_3$ corresponding to c on $\triangle_3$. Let d' be a point on $\triangle'_3$ corresponding to d on $\triangle_3$. Because $|K|$ is CAT(0), $d(c, d) \le d_{\mathbb{R}^2}(c', d')$.

Properties of the Euclidean triangles imply that $d_{\mathbb{R}^2}(c', d') < d_{\mathbb{R}^2}(c'', d'') = \frac{L}{3}$. Hence $d(c, d) < \frac{L}{3}$. Due to the above relations, we have $d(b_2, b_6) \leq d(b_2, c) + d(c, d) + d(d, b_6) < L$.



Comparison triangle in $\mathbb{R}^2$

So either $d(b_1, b_3) \leq L$, or $d(b_1, b_5) \leq L$, or $d(b_2, b_6) \leq L$. Assume without loss of generality that $d(b_2, b_6) \leq L$. Lemma 5.7.3 implies that either $d(b_2, b_6) = 1$, or $d(b_2, b_6) = \sqrt{2}$. Assume without loss of generality that $d(b_2, b_6) = 1$.

It similarly follows that either $d(b_4, b_2) \leq L$, or $d(b_4, b_6) \leq L$, or $d(b_3, b_5) \leq L$. Assume without loss of generality that $d(b_3, b_5) \leq L$. Lemma 5.7.3 implies that either $d(b_3, b_5) = 1$, or $d(b_3, b_5) = \sqrt{2}$. Assume without loss of generality that $d(b_3, b_5) = 1$.

Lemma 5.7.4 implies that the vertices b_2, b_3, b_5 and b_6 span a 2-cell in K. Let α be the 2-cell spanned by the vertices a_2, a_3, a_5 and a_6 in D. Identifying the edge $[a_6, a_2]$ with the edge $[a_3, a_2]$, and the edge $[a_6, a_5]$ with the edge $[a_3, a_5]$, we construct a cubical disk $D' = D \setminus \{\alpha\}$. The cell map ϕ restricted to the vertices of D' induces a cell map from the $|D'|$ to $|K|$ which satisfies the conditions of Lemma 5.7.1. But D' has one fewer 2-cell than D which is a contradiction since D has minimal area. So D contains no interior vertices of degree 3. In conclusion $|D|$ is a CAT(0) space.

□

The following lemma finds information regarding the combinatorial distance function measured along the edges of a CAT(0) cubical 3-complex. The proof is based on results obtained in subsection 5.5 on CAT(0) cubical disks.

Lemma 5.7.6. *Let K be a cubical complex of dimension three or less endowed with the standard piecewise Euclidean metric. Let $w_0, ..., w_n, \overline{w}_{n-1}, ..., \overline{w}_1$ be distinct* *vertices on the combinatorial geodesics* $[w_n, w_{n-1}, ..., w_1, w_0]$ *and* $[w_n, \overline{w}_{n-1}, ..., \overline{w}_1, w_0]$. *Let D be a cubical disk whose exterior vertices lie on the combinatorial path* $[v_0, ..., v_n, \overline{v}_{n-1}, ..., \overline{v}_1]$. *Let* $|\phi| : |D| \to |K|$ *be a cell map with* $\phi(v_i) = w_i$ *for all i and* $\phi(\overline{v}_i) = \overline{w}_i$ *for* $1 \le i \le n-1$. *Let β denote the unique 2-cell of D containing the edge* $[v_{n-1}, v_n]$. *Let* w_{01}, w_{02} *be the vertices of* $|\phi|(\beta)$ *opposite* $[w_{n-1}, w_n]$. *If D has minimal area and $|K|$ is CAT(0), then* $d_c(w_{01}, w_0) = n-2$ *and* $d_c(w_{02}, w_0) = n-1$.

Proof. Let v_{01}, v_{02} be the vertices of β opposite the edge $[v_{n-1}, v_n]$. Because $|K|$ is CAT(0), Theorem 5.7.5 implies that the cubical disk D is a string of pearls. It follows by Theorem 5.5.5 that each vertex of D lies on a combinatorial geodesic from v_n to v_0. So $d_c(v_{01}, v_0) = n-2$ and $d_c(v_{02}, v_0) = n-1$.

Let $[v_{01}, v'_{n-3}, ..., v'_1, v_0]$ be any combinatorial geodesic in D from v_{01} to v_0. Because D has minimal area, the cell map $|\phi|$ preserves the dimension of cells. So $[|\phi|(v_{01}), |\phi|(v'_{n-3}), ..., |\phi|(v'_1), |\phi|(v_0)]$ is a combinatorial geodesic in K from w_{01} to w_0 of length $n-2$. Hence $d_c(w_{01}, w_0) \le n-2$. Because the vertex w_{01} is a neighbor of w_{n-1}, $d_c(w_{01}, w_0) \ge n-2$. In conclusion $d_c(w_{01}, w_0) = n-2$.

Let $[v_{02}, v''_{n-2}, ..., v''_1, v_0]$ be any combinatorial geodesic in D from v_{02} to v_0. Since $[|\phi|(v_{02}), |\phi|(v''_{n-2}), ..., |\phi|(v''_1), |\phi|(v_0)]$ is a combinatorial geodesic in K from w_{02} to w_0 of length $n-1$, $d_c(w_{02}, w_0) \le n-1$. Because the vertex w_{02} is a neighbor of w_n, $d_c(w_{02}, w_0) \ge n-1$. In conclusion $d_c(w_{02}, w_0) = n-1$.

□

5.8 Collapsing a CAT(0) 3-dimensional cubical complex

We find in this subsection metric conditions which ensure the collapsibility of a finite cubical complex of dimension 3 or less. As in [18], the result follows by applying discrete Morse theory. We start by showing that any finite, CAT(0) cubical complex of dimension 3 or less endowed with the standard piecewise Euclidean metric, admits a discrete Morse function with no critical edges and a single critical vertex.

Theorem 5.8.1. *Let K be a finite cubical complex of dimension 3 or less endowed with the standard piecewise Euclidean metric. If $|K|$ is CAT(0), then K admits a discrete Morse function with no critical edges.*

Proof. Let $W : K \to K \cup \{0\}$ be a vector field defined on K. We fix a vertex w_0 of K and we define $W(w_0) = 0$. For each vertex w different from w_0, we consider the edge $e = [w, w']$ where $[w, w', ..., w_0]$ is a combinatorial geodesic from w to w_0. Such a path exists because $|K|$ has a convex metric. We define $W(w) = e$.

For an edge e in K, if there exists a vertex $w \in K$ such that $W(w) = e$, then we define $W(e) = 0$. For an edge e in K, if there does not exist a vertex $w \in K$ such that $W(w) = e$, then we denote the endpoints of e by w_{n-1} and w_n. There exist combinatorial geodesics $[w_{n-1}, w_{n-2} ..., w_1, w_0]$ and $[w_n, \overline{w}_{n-1} ..., \overline{w}_1, w_0]$ joining the endpoints of e to w_0. We define the vector field W along these combinatorial geodesics as follows: $W(w_i) = [w_i, w_{i-1}]$ for $1 \leq i \leq n-1$, $W(\overline{w}_i) = [\overline{w}_i, \overline{w}_{i-1}]$ for $2 \leq i \leq n-1$, $W(w_n) = [w_n, \overline{w}_{n-1}]$ and $W(\overline{w}_1) = [\overline{w}_1, w_0]$.

We consider in K the closed combinatorial path $[w_0, w_1, ..., w_{n-1}, w_n, \overline{w}_{n-1}, ..., \overline{w}_1, w_0]$. Because $|K|$ is simply connected, Lemma 5.7.1 implies that there exists a cubical disk D whose exterior vertices lie on the closed combinatorial path $[v_0, v_1, ..., v_{n-1}, v_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$ and a cell map $|\phi| : |D| \to |K|$ such that $\phi(v_i) = w_i$ for all i and $\phi(\overline{v}_i) = \overline{w}_i$ for $1 \leq i \leq n-1$. We choose the disk D of minimal area. D has a unique 2-cell β such that the edge $[v_{n-1}, v_n]$ is a face of β. Because D has minimal area, $|\phi|$ preserves the dimension of cells and hence $e < |\phi|(\beta)$. We define $W(e) = |\phi|(\beta)$. For any 2-cell α of dimension 2 or 3, we define $W(\alpha) = 0$.

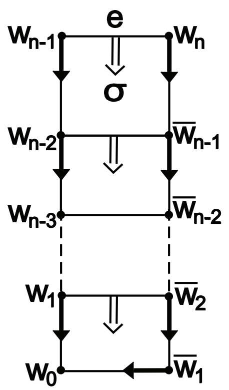
We prove next that $W : K \to K \cup \{0\}$ is the gradient vector field of a discrete Morse function with no critical edges.

According to the definition of W, if $\alpha \in Im(W)$, then $W(\alpha) = 0$ for all cells $\alpha \in K$. The definition of W also implies that either $W(\alpha) = 0$ or $W(\alpha)$ is a codimension-one face of α for all $\alpha \in K$. In order to prove that W is a discrete vector field defined on K we still have to show that for any cell $\alpha \in ImW$ there exists a unique cell γ in K such that $W(\gamma) = \alpha$. Recall that if $W(\gamma) = \alpha$ then γ must be in the boundary of α.

Let $e \in Im(W)$. Let w and w' denote the endpoints of e such that $W(w) = e$. Because W is defined along the combinatorial geodesic $[w, w' ..., w_0]$ which flows towards w_0, $d_c(w', w_0) < d_c(w, w_0)$. Hence $W(w') \neq e$. For each edge

$e \in Im(W)$, there exists therefore a unique vertex w such that $W(w) = e$.

Let e be an edge whose endpoints, denoted by w and w', are combinatorial distance n and $n-1$ from w_0. Let σ be a 2-cell in K such that $W(e) = \sigma$. Let w_{01} and w_{02} be the other two vertices spanning the 2-cell σ. Lemma 5.7.6 implies that $d_c(w_{01}, w_0) = n-2$ and $d_c(w_{02}, w_0) = n-1$. According to the definition of W, both endpoints of e map an edge of σ (different from e). So e is the only edge which can map to σ. For any 2-cell σ in $Im(W)$, there exists therefore a unique edge e with $W(e) = \sigma$. So W is a discrete vector field defined on K.



W is a discrete vector field defined on K

We prove further that W contains no non-trivial closed W-paths, neither of vertices and edges nor of edges and 2-cells.

Suppose, on the contrary, that there exists a nontrivial closed W-path of vertices and edges in K: $u_0^{(0)}, e_0^{(1)}, u_1^{(0)}, e_1^{(1)}, \ldots, u_r^{(0)}, e_r^{(1)}, u_{r+1}^{(0)} = u_0^{(0)}$. Since the W-path is non-trivial, $r \geq 0$. Because W-paths of vertices and edges in K point along geodesic paths, $d_c(u_i, w_0) = d_c(u_{i+1}, w_0) + 1$, $0 \leq i \leq r$. Hence $d_c(u_0, w_0) = d_c(u_{r+1}, w_0) + (r+1)$. Thus $d_c(u_{r+1}, w_0) = d_c(u_0, w_0) - (r+1) < d_c(u_0, w_0) = d_c(u_{r+1}, w_0)$ which is a contradiction. So W contains no nontrivial closed W-paths of vertices and edges.

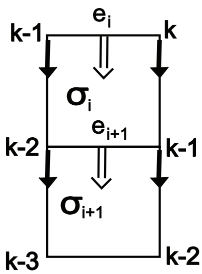
Let $e_0^{(1)}, \sigma_0^{(2)}, e_1^{(1)}, \sigma_1^{(2)}, \ldots, e_r^{(1)}, \sigma_r^{(2)}, e_{r+1}^{(1)}$ be a W-path of edges and 2-cells in K. For all $0 \leq i \leq r+1$, we denote the endpoints of the edge e_i by a_i and b_i and we denote the opposite vertices of e_i in σ_i by c_i and d_i. So $[a_i, b_i, c_i, d_i]$ is a closed combinatorial path in K. If $d_c(a_i, w_0) = k-1$ and $d_c(b_i, w_0) = k$, Lemma

5.7.6 implies that $d_c(c_i, w_0) = k - 1$ and $d_c(d_i, w_0) = k - 2$. According to the definition of W, $W(a_i) = [a_i, d_i]$, $W(b_i) = [b_i, c_i]$ and $W(e_i) = \sigma_i$. Hence $[d_i, c_i]$ is the only edge of σ_i that is neither in $Im(W)$ nor mapped by W to a 2-cell. Therefore $e_{i+1} = [d_i, c_i]$ and $W(e_{i+1}) = \sigma_{i+1}$. Lemma 5.7.6 further implies that $d_c(c_{i+1}, w_0) = k-2$ and $d_c(d_{i+1}, w_0) = k-3$. Hence $\{d_c(c_i, w_0)\}_{i=0}^{r}$ and $\{d_c(d_i, w_0)\}_{i=0}^{r}$ are non-increasing sequences. Further

$$d_c(c_i, w_0) = k - 1 = d_c(c_{i+1}, w_0) - 1$$

and

$$d_c(d_i, w_0) = k - 2 = d_c(d_{i+1}, w_0) - 1.$$



The numbers next to the vertices indicate the combinatorial distance from w_0.

Suppose that there exists a nontrivial closed W-path of edges and 2-cells in K: $e_0^{(1)}, \sigma_0^{(2)}, e_1^{(1)}, \sigma_1^{(2)}, ..., e_r^{(1)}, \sigma_r^{(2)}, e_{r+1}^{(1)} = e_0^{(1)}$. Because the W-path is nontrivial, the intersection of any two 2-cells is a face of each of them. Hence $r \geq 2$. Because $d_c(a_o, w_0) = k - 1$ and $d_c(b_o, w_0) = k$, Lemma 5.7.6 implies that $d_c(c_o, w_0) = k - 1$ and $d_c(d_o, w_0) = k - 2$. Since $e_{r+1} = e_0$, we have that $c_{r+1} = b_0$ and $d_{r+1} = a_0$. Therefore $d_c(d_{r+1}, w_0) = k - 1$ and $d_c(c_{r+1}, w_0) = k$. It follows that

$$k - 1 = d_c(c_0, w_0) \geq d_c(c_{r+1}, w_0) = k$$

and

$$k - 2 = d_c(d_0, w_0) \geq d_c(d_{r+1}, w_0) = k - 1.$$

The above relations imply a contradiction. So there exist no nontrivial closed W-paths of edges and 2-cells in K.

In conclusion W is the gradient vector field of a discrete Morse function defined on K with no critical edges and a single critical vertex.

$\square$

Lemma 5.8.2. *Let K be an n-dimensional cubical complex satisfying the property that every $(n-1)$-cell is a face of at most two n-cells. Then there exist at most two gradient paths from any critical n-cell in K to any critical $(n-1)$-cell in K.*

Proof. Let $\alpha^{(n-1)}$ and $\beta^{(n)}$ be two critical cells of K. Let $\omega = \beta^{(n)}, ..., \beta_{i-1}^{(n)}, \alpha_i^{(n-1)}$, $..., \alpha^{(n-1)}$ be a gradient path joining them. Let W be the gradient vector field associated to the path ω. Because W is a discrete vector field, there exists a unique $(n-1)$-cell α_{i-1} such that $W(\alpha_{i-1}) = \beta_{i-1}$. Since each $(n-1)$-cell is contained in at most two n-cells, there exists a unique n-cell β_{i-1} such that $\beta^{(n)}, ..., \beta_{i-1}^{(n)}, \alpha_i^{(n-1)}, \beta_i^{(n)}, ..., \alpha^{(n)}$ is a gradient path in K.

So, because the gradient path $\beta, ..., \beta_r, \alpha$ is completely determined by β_r and because there are at most two choices for β_r, there are at most two gradient paths from β to α.

$\square$

We present one of the main results of the paper.

Corollary 5.8.3. *Let K be a finite cubical complex of dimension 3 or less endowed with the standard piecewise Euclidean metric. If $|K|$ is CAT(0) and if it satisfies the property that every 2-cell of K is a face of at most two 3-cells of K, then K is collapsible.*

Proof. K admits, according to the previous theorem, a discrete Morse function with no critical edges and a single critical vertex w_0. If K is 2-dimensional, since it is contractible, the weak Morse inequalities imply that the number of critical cells of dimension 2 equals zero. So K collapses to the critical vertex w_0.

If K is 3-dimensional, according to the weak Morse inequalities we have $\chi(K) = m_0 - m_1 + m_2 - m_3 = 1 + m_2 - m_3$, where m_i denotes the number of critical cells of dimension i. So the number of critical cells of dimension 2 equals the number of critical cells of dimension 3. We will show that there exists a unique W-path from each critical 2-cell in the complex to each critical 3-cell in the complex.

We consider the Morse complex of the function f with coefficients in any field $\mathbb{F}$

$$... \to \mathfrak{M}_3 \overset{\partial_3}{\to} \mathfrak{M}_2 \overset{\partial_2}{\to} 0 \to < w_0 > \to 0.$$

Because K is contractible, $0 = H_2(K, \mathbb{F}) = \frac{Ker\partial_2}{Im\partial_3} = \frac{\mathfrak{M}_2}{Im\partial_3}$. The boundary map ∂_3 is therefore surjective. So there exists a gradient path from any critical 2-cell to any critical 3-cell.

Let $\mathbb{F} = \mathbb{Z}_2$. Because the map ∂_3 is surjective, there exists, for any critical 2-cell α, a critical 3-cell β such that $< \partial_3\beta, \alpha >= 1 mod 2$. Hence, $mod 2$, there exists a unique gradient path from β to α. Computing with coefficients in $\mathbb{Z}$, we notice that there exists an odd number of gradient paths from β to α. Because K satisfies the property that every 2-cell of K is a face of at most two 3-cells of K, the above lemma implies that there exists a unique gradient path from β to α. The critical cells of dimension 2 and 3 of K can therefore be canceled out in pairs. So K admits a new discrete Morse function with no critical cells of dimension $1, 2$ or 3 and a single critical vertex w_0. Thus K collapses to w_0.

□

Corollary 5.8.4. *Let K be a finite cubical complex of dimension 3 or less endowed with the standard piecewise Euclidean metric. If $|K|$ is CAT(k), $k \leq 0$ being a real number, and if it satisfies the property that every 2-cell of K is a face of at most two 3-cells of K, then K is collapsible.*

5.9 The geometry of CAT(0) hexagonal disks

We will show in subsection 5.10 that CAT(0) hexagonal complexes of dimension 2 are collapsible. It is hence necessary to understand the geometry of CAT(0) hexagonal disks. The aim of this subsection is to show that there exists a good direction of flow along the edges of a CAT(0) hexagonal disk.

The construction of CAT(0) hexagonal disks is similar to the one of CAT(0) cubical disks. Namely, we consider a hexagonal disk endowed with the standard piecewise Euclidean metric such that each of its interior vertices has degree at least 3. The standard piecewise Euclidean metric structure on the hexagonal disk becomes then CAT(0).

The following lemma gives an important property which holds for any hexagonal disk whose interior vertices have degree at least 3. The proof was given in section 4.2.

Lemma 5.9.1. *Let D be a hexagonal disk whose interior vertices have degree at least 3. Then*

$$\sum_{v\in\partial D}\left(\tfrac{5}{2}-\mathrm{degv}\right)\geq 3,$$

summing over the boundary vertices of D.

Definition 5.9.2. *A* geodesic disk *is a hexagonal disk D that satisfies* deg $v \geq 3$ *for all interior vertices v, and whose exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[v_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$, $n \geq 4$.*

Lemma 5.9.3. *Let D be a geodesic disk whose exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[v_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$, $n \geq 4$. Then D has an exterior vertex v such that* deg $v = 3$.

Proof. For $1 \leq k \leq n-1$, the degree of v_k must be at least 3. Otherwise the vertices $v_{k-2}, v_{k-1}, v_k, v_{k+1}, v_{k+2}, v_{k+3}$ would span a 2-cell in D, contradicting the fact that $[v_n, ..., v_{k-2}, v_{k-1}, v_k, v_{k+1}, v_{k+2}, v_{k+3}, ..., v_0]$ is a combinatorial geodesic. Similarly, deg $\overline{v}_k \geq 3$ for $1 \leq k \leq n-1$.

We consider the boundary vertices of D, $v_0, v_1, \overline{v}_1, \overline{v}_{n-2}, v_{n-1}, \overline{v}_{n-1}$ and v_n. Because $(\frac{5}{2} - \deg v_i) \leq \frac{1}{2}$ for $i \in \{0, 1, n-1, n\}$, while $(\frac{5}{2} - \deg \overline{v}_i) \leq \frac{1}{2}$ for $i \in \{1, n-2, n-1\}$, Lemma 5.7.1 implies

$$3 \leq (\tfrac{5}{2} - \deg v_0) + (\tfrac{5}{2} - \deg v_1) + (\tfrac{5}{2} - \deg \overline{v}_1)+$$

$$(\tfrac{5}{2} - \deg \overline{v}_{n-2}) + (\tfrac{5}{2} - \deg v_{n-1}) + (\tfrac{5}{2} - \deg \overline{v}_{n-1}) + (\tfrac{5}{2} - \deg v_n)+$$

$$+ \sum_{v \in \partial D, v \notin \{v_0, v_1, \overline{v}_1, \overline{v}_{n-2}, v_{n-1}, \overline{v}_{n-1}, v_n\}} (\tfrac{5}{2} - \deg v) \leq$$

$$\leq \tfrac{7}{2} + \sum_{v \in \partial D, v \notin \{v_0, v_1, \overline{v}_1, \overline{v}_{n-2}, v_{n-1}, \overline{v}_{n-1}, v_n\}} (\tfrac{5}{2} - \deg v).$$

Thus there exists an exterior vertex v of D, $v \notin \{v_0, v_1, \overline{v}_1, \overline{v}_{n-2}, v_{n-1}, \overline{v}_{n-1}, v_n\}$ such that $\deg v \leq 3$. So D has an exterior vertex $v \notin \{v_0, v_1, \overline{v}_1, \overline{v}_{n-2}, v_{n-1}, \overline{v}_{n-1}, v_n\}$ such that $\deg v = 3$.

□

The following definition generalizes the notion of geodesic disk.

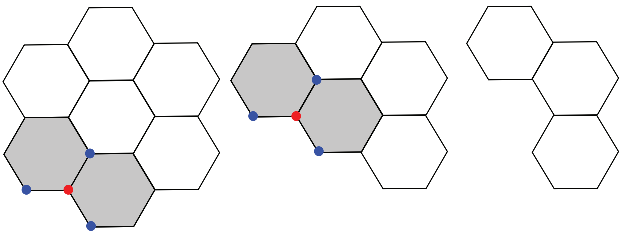
Definition 5.9.4. *Let J be a hexagonal complex whose underlying space is homeomorphic to $\mathbb{R}^2$ such that $\deg v \geq 3$ for all interior vertices v. A connected, finite subcomplex S of J is called a* string of pearls *if it is simply connected and if its exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[v_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$, $n \geq 4$ in J.*

Let S be a string of pearls whose exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[v_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$, $n \geq 4$. The following theorem proves that each vertex in S lies on a combinatorial geodesic from v_n to v_0.

Theorem 5.9.5. *Let S be a string of pearls whose exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[v_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$, $n \geq 4$. Then each vertex of S lies on a combinatorial geodesic from v_n to v_0.*

Proof. Lemma 5.9.3 implies that S has an exterior vertex v_k, $1 \leq k \leq n-1$ such that $\deg v_k = 3$. So the closed star of v_k in S contains two 2-cells and their faces. We consider the subcomplex $S_1 = S - \overline{St} v_k$ of S obtained by deleting the closed star of v_k in S. We notice that S_1 is simply connected and that each of its interior vertices has degree at least 3. So the subcomplex S_1 remains a string of pearls. Each exterior vertex of S_1 lies therefore on a combinatorial geodesic from v_n to v_0.

We retract further and obtain each time a string of pearls. Because S is finite, we reach, after a finite number of steps, a string of pearls S' with no



Deleting 2-cells in a string of pearls

interior vertices. Because every exterior vertex of S' lies on a combinatorial geodesic from v_n to v_0, the theorem follows.

□

The following corollary states that the combinatorial distance function measured along the edges of a string of pearls, is maximized on its boundary.

Corollary 5.9.6. *Let S be a string of pearls whose exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[v_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$, $n \geq 4$. Then, for all interior vertices v of S, we have $d_c(v, v_0) < n$.*

The following lemma provides information regarding the structure of the star of an exterior vertex in a string of pearls.

Lemma 5.9.7. *Let S be a string of pearls whose exterior vertices lie on the combinatorial geodesics $[v_n, v_{n-1}, ..., v_1, v_0]$ and $[v_n, \overline{v}_{n-1}, ..., \overline{v}_1, v_0]$, $n \geq 4$. If $v_{n-1} \neq \overline{v}_{n-1}$ and $v_{n-2} \neq \overline{v}_{n-2}$, then the vertices $v_n, v_{n-1}, \overline{v}_{n-1}$, $v_{n-2}, \overline{v}_{n-2}$ and v_{n-3} span a 2-cell in S.*

Proof. For $n = 2$, the exterior vertices of S lie on the combinatorial geodesics $[v_3, v_2, v_1, v_0]$ and $[v_3, \overline{v}_2, \overline{v}_1, v_0]$. Thus deg $v_3 \geq 2$, deg$\overline{v}_2 \geq 2$, deg $v_2 \geq 2$, deg $v_1 \geq 2$, deg $\overline{v}_1 \geq 2$ and deg $v_0 \geq 2$. Lemma 5.9.1 further implies

$$(\tfrac{5}{2} - \deg v_3) + (\tfrac{5}{2} - \deg v_2) + (\tfrac{5}{2} - \deg \overline{v}_2) +$$

$$+(\tfrac{5}{2} - \deg v_1) + (\tfrac{5}{2} - \deg \overline{v}_1) + (\tfrac{5}{2} - \deg v_0) \geq 3.$$

Hence the following inequalities hold:

$$\deg v_3 \leq 2, \deg v_2 \leq 2, \deg \overline{v}_2 \leq 2,$$

$$\deg v_1 \leq 2, \deg \overline{v}_1 \leq 2, \deg v_0 \leq 2.$$

So, since the vertices $v_3, v_2, v_1, v_0, \overline{v}_1$ and $\overline{v}_2$ have each degree exactly 2 in the subcomplex bounded by the closed combinatorial path $[v_3, v_2, v_1, v_0, \overline{v}_1, \overline{v}_2, v_3]$, they span a 2-cell in S.

In general, let $[v_{k+1}, v_k, ..., v_1, v_0]$ and $[v_{k+1}, \overline{v}_k, ..., \overline{v}_1, v_0]$ be the combinatorial geodesics the exterior vertices of S lie on. Suppose that the vertices $v_{k+1}, v_k, v_{k-1}, v_{k-2}, \overline{v}_{k-1}$ and $\overline{v}_k$ do not span a 2-cell in S. The vertex v_{k+1} has therefore $r > 2$ neighbors. Assume without loss of generality that $\deg v_{k+1} = 3$. We denote by v'_k the third neighbor of v_{k+1}, besides v_k and $\overline{v}_k$. Theorem 5.9.5 implies that v_{k+1} is the only neighbor of v'_k combinatorial distance $k+1$ from v_0. Theorem 5.9.5 further implies that $d_c(v_k, v_0) = d_c(\overline{v}_k, v_0) = k$, while $d_c(v_{k-1}, v_0) = d_c(\overline{v}_{k-1}, v_0) = k-1$. Because D is a hexagonal disk, v'_k has no neighbors combinatorial distance k from v_0 and no neighbors combinatorial distance $k-1$ from v_0. It follows by induction that v'_k is the only neighbor of v_{k-2}, combinatorial distance $k-1$ from v_0. Hence v_{k-2} is the only neighbor of v'_k, combinatorial distance $k-2$ from v_0. Thus $\deg v'_k = 2$. Since v'_k is an interior vertex of a string of pearls, this is a contradiction. The vertices v_{k+1}, v_k, $v_{k-1}, v_{k-2}, \overline{v}_{k-1}$ and $\overline{v}_k$ span therefore a 2-cell in S.

□

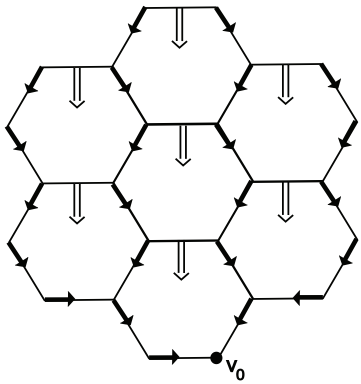
5.10 Collapsing a CAT(0) 2-dimensional hexagonal complex

We consider in this subsection 2-dimensional, CAT(0) hexagonal complexes and prove that they are collapsible by applying discrete Morse theory.

Theorem 5.10.1. *Let K be a 2-dimensional hexagonal complex endowed with the standard piecewise Euclidean metric. If each interior vertex of K has degree at least 3, then K is collapsible.*

Proof. Notice that $|K|$ is a CAT(0) space, and hence a CAT(k) space, $k \leq 0$ being a real number.

We define on K the vector field $V : K \to K \bigcup \{0\}$. We fix a vertex v_0 of K and we define $V(v_0) = 0$. For each vertex v of K different from v_0, we define $V(v) = e = [v, u]$, where $[v, u, ..., v_0]$ is any combinatorial geodesic from v to v_0. For each edge e in the image of V, we define $V(e) = 0$. For each edge e not in the image of V, we consider the string of pearls S bounded by e and the V-paths from the endpoints of e to v_0. Because e belongs to the boundary of S, there exists a unique 2-cell σ such that $e < \sigma$. We define $V(e) = \sigma$.



The gradient vector field of a discrete Morse function defined on K

To show that V is a discrete vector field defined on K, we must verify whether for each 2-cell $\sigma \in ImV$, there exists a unique edge e such that $V(e) = \sigma$. Let e be an edge in K whose endpoints are combinatorial distance n and $n-1$ from v_0. Corollary 5.9.6 and Lemma 5.9.7 imply that e is the only edge mapped by V to σ. V is therefore a discrete vector field defined on K. By its definition, the arrows of V always point closer to v_0. So V contains no nontrivial closed V-paths. V is therefore a gradient vector field defined on K. Thus we can associate it a discrete Morse function with no critical edges and a single critical vertex.

The weak Morse inequalities imply that K has no critical cells of dimension 2. So the discrete Morse function defined on K has no critical cells of dimension 1 or 2 and a single critical vertex. The collapsibility of K follows.

□

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