Oana Ruxandra Bode (Tuns)

Particular optimization problems with application in Economy

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This book is dedicated to my family, i.e. my parents Ioan and Maria Tuns, former teachers of Mathematics at Secondary School Mărișelu, Bistrița-Năsăud county, as well as my brother Ciprian, and his family, Simona, Maria and Iulia Tuns, who must know that they will always be present in my thoughts and in my heart. Throughout the elaboration of the book they demonstrated, at all times, their full support and most understanding. For this and many more, my sincere "Thank you!" from the bottom of my heart.

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Contents

Introduction

"Life is about decisions. Decisions, no matter if made by a group or an individual, usually involve several conflicting objectives. The observation that real world problems have to be solved optimally according to criteria, which prohibit an "ideal" solution - optimal for each decision-maker under each of the criteria considered - has led to the development of multicriteria optimization."
(EHRGOTT M. [26])

Operations research, often considered to be a sub-field of mathematics, is a discipline that deals with the application of advanced analytical methods in helping to make better decisions. It leads at the optimal solutions or near optimal solutions to complex decision-making problems. It is often concerned with determining the maximum (of profit, performance or yield) or minimum (of loss, risk or cost) of some real-world objective.

It is well known that the operations research, as they are nowadays, originate during the second world war, from the practical necessity to find the more efficient military sources. We recall that the operations research give a generic answer to the following question: if we suppose that if the implications of the choice of the elements of a given set is known, then one should determine the element (better said all the elements) of the set which satisfies some given conditions such that the result of the above implications is optimal.

Very useful results generated by the studies on such problems can be found in the daily life. Some of these results were rewarded with the Nobel Prize for Economical Sciences, among which we recall the following:

• in 1969, RAGNAR FRISCH and JAN TINBERGEN get this prize for their scientific work regarding the application of mathematics and statistical methods to economical theories and problems;

• in 1972, John R. Hicks and Kenneth Arrow get this prize for theories that help assessing business risk and government economic as well as welfare policies;

• in 1975, Leonid Kantorovich and Tjalling Koopmans get this prize for their scientific work regarding the theory of optimum allocation of resources;

• in 1982, George Stigler gets this prize for the scientific work on government regulation in the economic problem of how prices operate to balance the procedure supply with the buyers desire;

• in 1988, Maurice Allais gets this prize for his development of theories for better understanding market behavior and the efficient use of resources;

• in 1994, Reinhard Selten, John F. Nash Jr. and John C. Harsanyi get this prize for their work in game theory;

• in 2005, Robert J. Aumann and Thomas Schelling get this prize for having enhanced our understanding of conflict and cooperation through game theory analysis;

• in 2008, Paul Krugman gets this prize for his analysis of trade patterns and location of economic activity;

• in 2010, Peter Diamond, Dale T. Mortensen and Christopher A. Pissarides get this prize for their analysis of markets with search frictions.

Over time, operations research knew a very quick development. Hence, nowadays it has a lot of branches among which we point out the optimization theory (it is the modern term of operations research). Optimization theory includes the calculus of variations, control theory, convex optimization theory, decision theory, game theory, linear optimization, etc. The study of optimization problems was closely connected to practical problems.

The scientific results within the present book introduce some particular types of optimization problems generated by concrete economic problems. In this way we highlight the well-deserved high rank of optimization theory among mathematical areas due to its countless applications in practical areas. For each such a problem we give a method or an algorithm that can be used for solving it and some necessary and sufficient conditions for the optimal solutions of each problem.

As the title of the book suggests, we study different types of particular multicriteria and multilevel optimization problems, the bond being represented by the discrete point of view. All types of the studied problems are based on concrete applications that can be found in real life situations.

The opportunity of studying such an exciting research field represents a real privilege. The present book contains the author's own results, obtained alone or in joint works, addressing concrete economic problems from different economic fields, such as: costs management area, portfolio theory area, technology transfer area and assignment of the unemployed persons to professional training programs area.

In what follows, we give a description of how is the present book organized: the entire content is split into six chapters, followed by the Bibliography.

Chapter 1, entitled Preliminaries, contains a brief background concerning multicriteria, lexicographic and multilevel optimization problems. Also, we point out the case in which in a multilevel optimization problem the coefficients depend on one or more parameters. A problem of this kind, obtained by mathematically modeling a practical problem, is given in Chapter 6 of the present book.

Chapter 2 contains original results of scientific research belonging to the author of this book and can be found in papers [75], [115], [113] and [**?**].

We begin our exposure explaining what we understand by lexicographic multicriteria bottleneck problems with p bottleneck objective functions (LpBP). Furthermore, in Lemma 2.1.2 and Theorem 2.1.3 we study the structure of the set of optimal solutions of this type of problem.

In Section 2.2 we introduce the notion of optimal solution with pipeline property for the lexicographic bottleneck problems and we discuss some aspects about the set of all optimal solutions with pipeline property. We introduce the notion of minimum point with pipeline property of a function on a set (Definition 2.2.1). We note that within this Definition the function f does not appear, but Theorem 2.2.2 justifies the use of the term "minimum". An example of minimum points which does not have the pipeline property is given in Example 2.2.3. In Proposition 2.2.4 and Proposition 2.2.5 we give two different methods that can be used to verify that a point has the pipeline property. Some properties of the set of all minimum points with pipeline property are given in Theorem 2.2.2, Proposition 2.2.7, Proposition 2.2.8 and Proposition 2.2.9.

In Section 2.3 we consider the lexicographic bottleneck problems in which the set of all feasible solutions is discrete. The structure of the set of optimal solutions (Theorem 2.3.6 and Theorem 2.3.7) and the structure of the set of optimal solutions with pipeline property for this particular type of problems are discussed (Proposition 2.3.8, Corollary 2.3.9, Proposition 2.3.10 and Proposition 2.3.11).

Section 2.4 contains a method (based on Theorem 2.4.2) to determine an optimal solution with pipeline property for lexicographic discrete bottleneck problems. This method is a type of weighted methods. The novelty is that the weighted type introduced by us allows the direct getting of the optimal points with pipeline property. It generalizes both the method used by Bandopadhyaya L. [6] and the method given by Zarepisheh M. and Khorram E. [131].

In **Chapter 3**, based on a concrete problem concerning the planning of how to collect and transport the milk by a dairy products manufacturing company under the restrictions of minimizing the quantity of stored milk and the transport costs, and taking into account some other given requirements, we study a special type of bilevel optimization problem in which the set of feasible solutions is the set of subgraphs of a given graph. These subgraphs fulfill some given restrictions. We note that the studied problem mathematically models a concrete costs management problem and it can be considered a type of traveling salesman problem.

The novelty of this problem is given, on one hand, by the mathematical model which we introduce and, on the other hand, by the fact that we use the splitting technique which allows us to reduce the solving of this problem (see Lemma 3.2.1, Lemma 3.2.2, Lemma 3.2.3, Theorem 3.2.4 and Theorem 3.2.5) to solving three problems: denoted by (PP), $(P_1(H_0))$, respectively $(P_2(H_0))$, H_0 being an optimal solution of the first problem. The problem $(P_1(H_0))$ is a classic problem of determining the minimum of a given function on a given set of subgraphs of a graph. The problem $(P_2(H_0))$ is a lexicographic bicriteria optimization problem. By introducing Theorem 3.2.6 and by using the method presented in Section 2.4 we reduce the solving of this last problem to solving again a classic problem of determining the minimum of a given function on a given set of subgraphs of a graph. The authors achievements within this chapter can be found in paper authored by Goina D. and Tuns (Bode) O.R. [42]. It completes the results obtained by Ruuska S., Miettinen K. and Wiecek M.M. [100].

Chapter 4 deals with the study, from the optimization point of view, of some concrete economic problems involving assigning unemployed persons to professional training programs. We begin by justifying the importance of the professional training programs for the unemployed persons.

In Section 4.2 we formulate the two studied economic problems. We note that both problems represent new types of generalizations of classic assignment problems. Therefore, these problems complete the results obtained by Pentico D.W. [91].

We continue by analysing in Section 4.3 the first economic problem, denoted by (AEP_1). We mathematically model it in Subsection 4.3.1 and then, in the next subsection, we give some necessary and sufficient optimality conditions (Proposition 4.3.3, Proposition 4.3.5, Proposition 4.3.6, Proposition 4.3.7 and Theorem 4.3.14). Based on Proposition 4.3.3 and Proposition 4.3.5, in Subsection 4.3.3 we give a polynomial technique for solving the problem (PM).

In Section 4.4 we study the second problem, denoted by (AEP_2). In Proposition 4.4.1 it is given a necessary and sufficient condition such that a feasible solution is optimal. Then, we present the way in which the above proposed technique can be used to solve this problem. The technique we introduce is more efficient than the one given by Della Croce F., Paschos V.Th. and Tsoukias A. [23]. The author's achievements within this area of research can be found in papers [**?**] and [116].

In **Chapter 5** we turn our attention to the economic-financial problems related to portfolio theory area.

In Section 5.1, after giving a brief background concerning modern portfolio theory in Subsection 5.1.1, we emphasize in Subsection 5.1.2 the most known portfolio selection models, i.e. the Markowitz's portfolio selection models. Furthermore, in Subsection 5.1.3 we introduce a relation between the portfolio selection models of Markowitz's type and the bicriteria optimization. By means of this relation, we give a new approach for the portfolio selection problem (Proposition 5.1.2 and Proposition 5.1.4). Using the results within this paragraph, in Subsection 5.1.4 we analyse a particular type of portfolio selection problems. The mathematical model attached to this type of problem is a fractional pseudo boolean optimization problem. The scientific results within Subsection 5.1.3 and Subsection 5.1.4 belong to the author and can be found in paper [109].

In Section 5.2 we begin by presenting two different problems which we want to study. We formulate in Subsection 5.2.1 the particular type of portfolio selection problem which we mathematically model and solve in Subsection 5.2.2. This mathematical model represents a bilevel assignment optimization problem of cost type. Based on the restrictions of this problem we use the splitting technique in order to solve it (see Theorem 5.2.1, Theorem 5.2.2 and Corollary 5.2.5). In Section 5.2.3 we extend the economic problem. The mathematical model attached to this economic problem is a bilevel optimization problem for which the lower level function is bicriteria of cost-bottleneck type. As far as we know, this kind of bilevel optimization problem is not discussed in literature. For solving this problem we use both the splitting technique and the technique introduced in Section 2.4 (see Theorem 5.2.9, Theorem 5.2.10, Theorem 5.2.12, Theorem 5.2.13 and Theorem 5.2.14). An algorithm that can be used to solve this problem and an example to emphasize how this algorithm works are given.

Chapter 6 is devoted to the study of the economic problems related to technology transfer area.

We begin our exposure by giving in Section 6.1 a brief background concerning technology transfer, then we continue with the formulation in Section 6.2 of our concrete economic problem: we consider a n differentiated Stackelberg model, when the leader firm engages in an research and development process that gives an endogenous cost-reducing innovation. Our goal is to study, in Sections that follow, the licensing of the cost-reduction innovation, i.e. the patent licensing contracts cases (no-licensing case, per-unit royalty licensing case, fixed-fee licensing case and two-part tariff licensing case) when the patentee is an insider.

In Section 6.3 we attach the mathematical model to the concrete economic problem in the benchmark case. The novelty consists in the fact that this mathematical model is a three-level parametric optimization problem with two parameters within the objective functions. This allows us to solve the mathematical problem by using the variables of the upper level problems as parameters in the lower level problems (Proposition 6.3.2, Proposition 6.3.4, Proposition 6.3.6 and Proposition 6.3.8). We determine the feasibility domain of the parameter which represents the degree of the differentiation of goods in Proposition 6.3.6, Remark 6.3.7 and Remark 6.3.9. We note that for the particular case when $n = 1$ and both parameters belong to interval $]0, 1[$, the optimal solution of the mathematical problem coincides with the optimal solution of the economic problem that can be found in the next subsection. More, the result that the absolute value of the parameter which represents the degree of the differentiation of goods cannot exceed 1 has an important economic significance. The result justifies the condition that this parameter belongs to interval $]0, 1[$, which is frequently used in the economic literature.

In Subsection 6.3.2 we recall, for the particular case when $n = 1$, the economic problem formulated in Section 6.2. Within this paragraph we determine the optimal value in this case of study for some other variables which have an important economic significance, such as: the profits of both firms (leader and follower), the consumer surplus and the social welfare. We remark that we denote all these variables by using the specific economic notations. We evaluate the effects of the degree of the differentiation of goods over all these variables and also over the optimal innovation size and optimal outputs of both firms (Theorem 6.3.11 and Remark 6.3.12). As the main novelty of these results we note the fact that the mathematical solutions for the values of the optimal innovation size and optimal outputs for both firms coincide with the values that we get strictly from the economic point of view when $n = 1$.

In Section 6.4 we attach the mathematical model to the concrete economic problem in the per-unit royalty licensing case. The novelty consists in the fact that this mathematical model is a four-level parametric optimization problem with two parameters within the objective functions. Again we solve the mathematical problem by using the variables of the upper level problems as parameters in the lower level problems (Proposition 6.4.2). In Subsection 6.4.2 we recall, for the particular case when $n = 1$, the economic problem formulated in Section 6.2. As in Subsection 6.3.2, we determine the optimal value for this case of study for the innovation size, the output and the profit of both firms, the consumer surplus and the social welfare. Again, we evaluate the effects of the degree of the differentiation of goods over all these variables (Theorem 6.4.3 and Remark 6.4.4). Also, we note that for this case of study the mathematical solutions for the values of the optimal innovation size and optimal outputs for both firms coincide with the values that we get strictly from the economic point of view when $n = 1$.

In Section 6.5, respectively 6.6, we solve the economic problem formulated in Section 6.2 for the particular case when $n = 1$ and when the technology license occurs by means of a fixed-fee, respectively by means of a two-part tariff. By means of Remark 6.5.1 and Theorem 6.5.2, respectively Theorem 6.6.2 and Theorem 6.6.3, we give the economic interpretation of the mathematical results that follows in this case of study.

The personal contribution of the author in this area may be described by means of the following classification:

- the analysis and comparison of the possible cases of the licensing contract (pre-licensing, licensing by means of a per-unit royalty, licensing by means of a fixed-fee and licensing by means of a two-part tariff) in a differentiated-good Stackelberg duopoly, where one of the firms invests in research and development in order to get a cost-reducing innovation. Based on the identity of the patentee, we analyse the case when the leader firm is the innovator (i.e. only the leader firm engages in process innovation) [31], [32], [112] and [114];

- the analysis of the licensing by means of a per-unit royalty and the licensing by means of a fixed-fee, in the Cournot and Bertrand models; the comparison between these results and the ones obtained by Li C. and Ji X. [67], for Cournot and Bertrand duopolies [**?**].

- the mathematical modeling by using the multilevel parametric optimization problems of the benchmark case and per-unit royalty case in the n differentiated Stackelberg duopoly. In this way, we get some mathematical explanations to the economic restrictions.

We note that all the studied problems within the present book points out some new types of discrete optimization problems which have not been studied so far.

Chapter 1

Preliminaries

Real life situations impose the necessity of studying from the mathematical point of view some concrete economic problems. As already mentioned in the Introduction, this study is realized by analyzing four different types of economic problems: costs management problems, portfolio selection problems, technology transfer problems and problems concerning the assignment of the unemployed persons to professional training programs.

For the mathematical modeling of the above types of economic problems and their study we use different mathematical tools and notions. Therefore, for the easier lecturing of the present book, we start by presenting basic notions and results with respect to multicriteria optimization problems, lexicographic optimization problems and multilevel optimization problems.

1.1 Brief Background Concerning Multicriteria Optimization Problems

There exists many practical problems in which the aim is to realize concomitantly more objectives. These problems are named multi-objective or multicriteria optimization problems and are generated by many concrete problems which are based on real life situations. The study of this type of problems was made in different ways, depending, on one hand, on the practical necessities (if the study was generated by a real practical problem) or, on the other hand, on the conjuncture in which it was wished to make the study.

Let n and p be natural non-zero numbers.

Let $\Omega \subseteq \mathbb{R}^n$, $S \subseteq \Omega$ and let $f = (f_1, f_2, ..., f_p) : \Omega \to \mathbb{R}^p$ be a given function.

The *maximization problem* of the function f on S is generically denoted by

$$\text{(PVMax)} \qquad \begin{cases} (f(x)) = \begin{pmatrix} f_1(x) \\ \cdots \\ f_p(x) \end{pmatrix} \to \text{v} - \max, \\ x \in S. \end{cases} \tag{1.1}$$

and the *minimization problem* of the function f on S by

$$\text{(PVMin)} \qquad \begin{cases} (f(x)) = \begin{pmatrix} f_1(x) \\ \cdots \\ f_p(x) \end{pmatrix} \to \text{v} - \min, \\ x \in S. \end{cases} \tag{1.2}$$

A point $x^0 \in S$ is called *global maximum point* of the function f on S or *ideal point* of the problem (PVMax) if x^0 is a maximum point of f_i with respect to the set S, for all $i \in \{1, \dots, p\}$. A point $x^0 \in S$ is called *global minimum point* of the function f on S or *ideal point* of the problem (PVMin) if x^0 is a minimum point of f_i with respect to the set S, for all $i \in \{1, \dots, p\}$.

Generally, the ideal point doesn't exist. Hence, the following question is raised: what can replace the global maximum (global minimum) point in case of the vector functions? Researches answering to this question can be classified in several categories, among which we cite the followings:

• Building a synthesis function $F : \Omega \to \mathbb{R}$ and identifying the set of optimal solutions of the problem (PVMax), respectively (PVMin), with the set of maximum points, respectively with the set of minimum points, of F with respect to Ω. It is the most common way that is used in the economic approach of the problems with more objective functions. Different ways of choosing the function F are given by Andraşiu M., Baciu A., Pascu A., Puşcaş E. and Taşnadi Al. [3], May K.O. [85], Sawaragi Y., Nakayama H. and Tanino T. [102], Stancu-Minasian I.M. [107], Yu P.L. [130], Zeleny M. [132] and Zionts S. [133]. Using the synthesis function is the base for the weighted method. Also, in this category we can frame the ε-restrictive method and adaptive method.

• The second category includes the global approaches of the problem, by considering an ideal solution (see Ehrgott M. [26]) and then selecting from the feasible solutions the one which is the closest to the ideal solution, or by introducing the r-balanced point and set (see Galperin E.A. [38]), or through other procedures.

• Defining a preference relation $\prec$ (i.e. a transitive binary relation) (see Yu P.L. [130]) on the set $f(\Omega)$ and identifying the optimal solutions of the problem (PV) as those points $x^0 \in \Omega$ whose image by the function f is a non-dominant element of the set $f(\Omega)$

in view of the preference relation defined. We mention that a point $x^0 \in \Omega$ is called a non-dominated point of f on Ω if there exists no $x \in \Omega$ such that $f(x) \prec f(x^0)$. Preference relations generated by a cone were introduced in 1974 by Yu P.L. [129]. By choosing proper cones it can be obtained classes of preference relations and their corresponding types of non-dominated points. We just recall the efficient points and the lexicographic optimal points because they are often used in the present book.

The notion of *efficient point* was introduced in 1951 by Koopmans (see Koopmans T.C., *Acticity Analysis of Production and Allocation*, Cowles Commision Monograph 13, New York, Wiley, 1951), but the idea underlying it belongs to Vittorio Pareto and was presented in the book *Cours d'Economie Politique*, edited in 1896 in Switzerland at the Lausanne Rouge printing house. Unanimously, it is accepted that this work is the first attempt to solve a problem of multicriteria decision.

Definition 1.1.1 *A point $x \in S$ is called max-efficient point of the function f on S (or Pareto maximum point of the problem PVMax) if there exists no point $y \in S$ such that the following two conditions to be fulfilled:*

$$(i)\ f_i(y) \geqq f_i(x),\ \textit{for all}\ i \in \{1, \dots, p\} \tag{1.3}$$

and

$$(ii)\ \textit{there exists at least one}\ \ j \in \{1, \dots, p\}\ \textit{such that}\ f_j(y) > f_j(x). \tag{1.4}$$

A point $x \in S$ is called min-efficient point of the function f on S (or Pareto minimum point of the problem PVMin) if there exists no point $y \in S$ such that the following two conditions to be fulfilled:

$$(i)\ f_i(y) \leqq f_i(x),\ \textit{for all}\ i \in \{1, \dots, p\} \tag{1.5}$$

and

$$(ii)\ \textit{there exists at least one}\ \ j \in \{1, \dots, p\}\ \textit{such that}\ f_j(y) < f_j(x). \tag{1.6}$$

Remark 1.1.2 *Let $\Omega \subseteq \mathrm{I\!R}^n$ and $S \subseteq \Omega$.*

A point $x^0 \in S$ is a max-efficient point of the function $f = (f_1, \dots, f_p) : \Omega \to \mathrm{I\!R}^p$ if and only if the system

$$\begin{cases} f_1(x) \geqq f_1(x^0), \\ \dots \\ f_p(x) \geqq f_p(x^0), \\ \sum\limits_{j=1}^{p} f_k(x) > \sum\limits_{j=1}^{p} f_k(x^0), \\ x \in S, \end{cases} \tag{1.7}$$

is incompatible.

A point $x^0 \in S$ is a min-efficient point of the function f on S if and only if the system

$$\begin{cases} f_1(x) \leqq f_1(x^0), \\ \ldots \\ f_p(x) \leqq f_p(x^0), \\ \sum\limits_{j=1}^{p} f_k(x) < \sum\limits_{j=1}^{p} f_k(x^0), \\ x \in S, \end{cases} \tag{1.8}$$

is incompatible.

Remark 1.1.3 *We note that* (*see* Popovici N. [*95*], *Remark* 1.2.6) *if it is considered the cone $C = \mathbb{R}^p_+ \setminus \{0_p\}$ and the binary relation denoted by $\geq$ and defined, for all $u, v \in \mathbb{R}^p$, by*

$$u \geq v \ \text{ if and only if } u - v \in \mathbb{R}^p_+ \setminus \{0_p\}, \tag{1.9}$$

then this is a preference relation. The inverse of the relation $\geq$, denoted by $\leq$, is also a preference relation.

A point $x^0 \in S$ is called *non-dominated point of f on S with respect to the preference relation* $\geq$ if there exists no $x \in S$ such that $f(x) \geq f(x^0)$. Immediately we can see that a point $x^0 \in S$ is a max-efficient point of f on S if and only if it is a non-dominated point of f on S with respect to the preference relation $\geq$. Analogously, a point $x^0 \in S$ is called *non-dominated point of f on S with respect to the preference relation* $\leq$ if there exists no $x \in S$ such that $f(x) \leq f(x^0)$. Obviously, a point $x^0 \in S$ is a min-efficient point of f on S if and only if it is a non-dominated point of f on S with respect to the preference relation $\leq$.

Among the numerous papers in which the multicriteria optimization problems are studied, in order to get some concrete methods to solve it, we cite the following papers authored by: Ehrgott M. [26], Jahn J. [56], Luc D.T. [71], Miettinen K. [87], Yu P.L. [130], Popovici N. [95], Roy B. [99], Stancu-Minasian I.M. [107] and Steuer R.E. [108]. Also, we recall some romanian papers very useful from the practical point of view, such as the papers authored by Andraşiu M., Baciu A., Pascu A., Puşcaş E. and Taşnadi Al. [3], Baciu A., Puşcaş A. and Puşcaş E. [5], Neamţiu L. [89], Stancu-Minasian I.M. [107] and Ţigan Şt., Achimaş A., Coman I., Drugan T. and Iacob E. [118]. As well, another application very useful from the practical point of view in the portfolio theory area can be found in the paper authored by Tuns (Bode) O.R. [109].

1.2 Brief Background Concerning Lexicographic Optimization Problems

In real-world optimization problems often appear situations when conflicting objectives exist in a decision problem, but for reasons outside the control of the decision maker the objectives have to be considered in a hierarchical manner, i.e. each objective, except the first one, is required to be realized with respect to the set of optimal solutions of the previous objective. As it is pointed out by Marques-Silva J., Argelich J., Graça A. and Lynce I. [83], given a sequence of cost functions, an optimization criterion is said to be lexicographic whenever there exists a preference in the order in which cost functions are optimized. Lexicographic optimization plays an important role in goal optimization. More often, it is used in linear and combinatorial optimization than in nonlinear optimization.

Emelichev V.A. and Gurevsky E.E. [27] use the lexicographic optimization in order to investigate the stability of the optimal solution set for a multicriteria boolean problem. Their results represent a prosecution of the research made by Berdysheva R.A. and Emelichev V.A. [7] concerning the stability of the optimal solution for a discrete optimization problem against perturbations of coefficient of objective functions or of functions which give the restrictions. Brown A., Gedlaman A. and Martinez S. in [13] study the general linear piecewise lexicographic optimization problem and obtain a few extensions of some important theorem from linear optimization.

There are many other examples where optimization is expected to be lexicographic, such as Soizic A., Bonnans J.F., Paraisy R. and Veyrat S. [103] or Marques-Silva J., Argelich J., Graça A. and Lynce I. [83]. Relevant results concerning lexicographic optimization are given by Ehrgott M. [26] or Storengy S.A., Bonnans J.F. and Paraisy R. [103].

Lexicographic goal optimization represents the initial goal optimization and, as can be seen in [**?**], its formulations "ordered the unwanted deviations into a number of priority levels, with the minimization of a deviation in a higher priority level being infinitely more important than any deviations in lower priority levels". This type of optimization have to be used when there exists a clear priority ordering amongst the goals to be achieved. An algorithm showing how a lexicographic goal programme can be solved as a series of linear programmes is given by Ignizio J.P. [54].

Lexicographic multiobjective optimization problems (LMOP) appear when conflicting objectives exist in a decision problem and, additional, those objectives have to be considered in a hierarchical order which is outside the control of the decision maker.

Among the multiple examples of this type of problems that can be found in the literature we recall the papers authored by Bhushan M. and Rengaswamy R. [10], Hernndez Lerma O. and Hoyos-Reyes L.F. [52], and Weber E., Rizzoli A., Soncini-Sessa R. and Castelletti A. [128]. In [10] we can find formulations which incorporate some robustness enhancing criteria while designing cost-optimal sensor network for reliable fault diagnosis. These formulations are presented in a lexicographic optimization framework. The application of those formulations is demonstrated on the Tennessee Eastman case study. In [52] the authors introduced a lexicographic multiobjective control formulation of the priority assignment problem for a discrete-time single-server queueing system with q competing classes of customers and a discounted cost criterion. In [128] the authors described the optimization of water resources planning for Lake Verbano (Lago Maggiore) in northern Italy. The main objective was to determine an optimal policy for the management of the water supply over some planning horizon. This problem has a lexicographic nature since the order to achieve all the objectives (to maximize flood protection, minimize supply shortage for irrigation and maximize electricity generation) was prescribed by law.

Formulation of the lexicographic optimization problem

Let n and p be natural non-zero numbers.

Let $\Omega \subseteq \mathbb{R}^n$, $S \subseteq \Omega$ and let $f = (f_1, f_2, ..., f_p) : \Omega \to \mathbb{R}^p$ be a given function.

The set (see Popovici N. [95], Definition 1.5.1)

$$C_{\text{lex}} := \{0_p\} \cup \{x = (x_1, \ldots, x_p) \in \mathbb{R}^p | \exists i \in \{1, \ldots, p\} \text{ s.t.}$$

$$x_i > 0, \; \nexists j \in \{1, \ldots, p\}, \; j < i \text{ s.t. } x_j \neq 0\}$$

is called lexicographic cone from $\mathbb{R}^p$.

Remark 1.2.1 (*see* Popovici N. [*95*], *Corollary* 1.5.1) *The binary relation denoted by $\geq_{lex}$ and defined, for each $u, v \in \mathbb{R}^p$, by*

$$u \geq_{lex} v \textit{ if and only if } u - v \in C_{lex}, \tag{1.10}$$

is a total order relation (called lexicographic order relation), that is compatible with the operations of sum of vectors and multiplication of vectors by scalars. The inverse of the relation $\geq_{lex}$, denoted by $\leq_{lex}$, has the same properties.

Definition 1.2.2 *A point $x^0 \in S$ is called lexicographic maximum point of f on S if there exists no $x \in S$ such that $f(x) \geq_{lex} f(x^0)$. Immediately we can see that a point is a lexicographic maximum point of f on S if and only if is a non-dominated point of f on S with respect to the relation $\geq_{lex}$.*

Analogously, a point $x^0 \in S$ is called lexicographic minimum point of f on S if there exists no $x \in S$ such that $f(x) \leq_{lex} f(x^0)$. Immediately we can see that a point is a lexicographic minimum point of f on S if and only if is a non-dominated point of f on S with respect to the relation $\leq_{lex}$.

Generally, we can write the lexicographic optimization problem as follows:

$$\text{(LP)} \quad \begin{cases} f(x) = (f_1(x), \ldots, f_m(x)) \rightarrow \text{lex} - \max \quad (\text{or lex} - \min), \\ x \in S. \end{cases}$$

We have to point out the fact that if we compare the efficient solutions with the lexicographic solutions, then the essential characteristic of efficiency is the existence of tradeoff between objectives, while lexicographic optimality implies a ranking of the objectives in the sense that optimization of function f_m is only considered once optimality for objectives $\{1, \ldots, m-1\}$ has been established. As pointed out by Ehrgott M. [26], that means objective 1 has the highest priority, and only in the case of multiple optimal solutions objectives f_2 and further objectives are considered. This priority ranking implies the absence of tradeoffs between criteria. An improvement in an objective f_k, $k \in \{1, \ldots, m\}$, can never compensate the deterioration of any $f_i, i < k$. The hierarchy among criteria allows us to solve lexicographic optimization problems sequentially, minimizing one objective f_k at a time and using optimal objective values of $f_i, i < k$, as constraints (see Algorithm 5.1 from [26]).

Remark 1.2.3 *We recall that any lexicographic maximum (respectively, minimum) point of f on S or any optimal point of the problem (LP) is a max-efficient (respectively, min-efficient) point. The converse is not true.*

We point out the above Remark by the following example.

Example 1.2.4 *Let $f = (f_1, f_2) : [0, 2] \rightarrow \mathbb{R}^2$ given by*

$$f_1(x) = x, \; f_2(x) = (x-1)^2, \; \forall\, x \in [0, 2].$$

It is obvious that the system (1.8) is incompatible. Applying Remark 1.1.2, it results that the point $x^0 = 1$ is a min-efficient point. The point $x^0 = 1$ is not a lexicographic minimum point, because if we take $x = 0$ we get that $f(0) = (0, 1) <_{lex} (1, 0) = f(1)$.

There exist many algorithms that can be used to solve the problem (LP). Zarepisheh M. and Khorram E. [131] notice that there exist two different methods: the sequential method and the weighted method.

I. *The sequential method*, given for the first time by IGNIZIO J.P. in [54], consists in determining successive the set of optimal solutions of the function f_i with respect to the set of optimal solutions of the function f_{i-1}. If we consider $S_0 := S$, then, taking i from the set $I = \{1, \ldots, m\}$, at each iteration we have to determine the set S_i of all minimum points of the function f_i on S_{i-1}. It is obvious that the set of optimal solutions of the problem (LP) it is equal to the set S_m. If the functions f_i, $i \in I$, are linear and S is a convex polyhedron, then this method is similar with the method given in 1970 by MARUŞCIAC I. and RĂDULESCU M. [84], and by ISERMANN H. in 1982. For solving the problem in this case, the above authors used a specific simplex method.

II. *The weighted method* is based on the existence of a real number M and on identification of the set of optimal solutions of the problem (LP) with the set of optimal solutions of the problem

$$(PU) \quad \begin{cases} F(x) \to \min, \\ x \in S, \end{cases}$$

where $F(x) = \sum_{i=1}^{m} M^{m-i} f_i(x), \ \forall\, x \in S$.

This method can be used only if the set S is discrete and finite or $f(S)$ is discrete and finite. Therefore, this method is considered only to solve linear LMOPs and LMOPs with a finite discrete feasible region. In this method we should minimize one objective function over the original feasible region, while in the sequential method we should usually solve r optimization problems, and in each iteration one constraint is added to the feasible region.

As pointed out by ZAREPISHEH M. and KHORRAM E. [131], some other applications of LMOP can be found in the papers authored by DESAULNIERS G. (2007), ERDOĞAN G. ET AL. (2010), GASCON V. ET AL. (2000).

We remark that DELLA CROCE F., PASCHOS V. TH. and TSOUKIAS A. [23] consider a generalization of the bottleneck type problem, in the sense that require the determination of that solution wherefore the biggest time is as small as possible and it is fulfill as few times as possible, then the biggest time to be the smallest and to be fulfill as few times as possible etc. For that, an algorithm is proposed. This algorithm reduces the solving of the problem by solving a lexicographic optimization problem whose objective function has all the components of bottleneck type. In this paper are given two examples of applying the algorithm for the lexicographic bottleneck shortest path problem and lexicographic bottleneck assignment problem. The algorithm complexity depends directly on complexity of the algorithm used to solve the classic bottleneck problem appropriate for the type of problem considered.

1.3 Brief Background Concerning Multilevel Optimization Problems

Multilevel optimization and subsequently bilevel optimization have lately become important areas in optimization. The investigations of such types of problems are strongly motivated by their actual real-life applications in areas such as economics, medicine, engineering etc. The increasing number of these applications have led mathematicians to develop new theories and mathematical models. In mathematical terms, the bilevel optimization problem is an optimization problem where a subset of the variables is constrained to be an optimal solution of a given optimization problem parameterized by the remaining variables.

From the brief presentation made by Vicente L.N. [119] we recall the followings:

The original formulation of the bilevel optimization problem appeared in 1973, in a paper authored by Bracken J. and McGill J. [11]. But, the bilevel optimization problem has its origins in the work of H.F. von Stackelberg [105] or [106], which, by studying the real market situations, introduces so-called Stackelberg game. He used, for the first time, an hierarchical model to describe real market situations. This model reflects that there exist market situations when different decision makers are not able to make their decisions independently, being forced to act according to a certain hierarchy. The simplest case of such a situation is that one when there are only two decision makers on the market: the leader (upper level) and the follower (lower level) and, more over, the lower level actions depend on upper level decisions. Although the bilevel optimization problem was, for the first time, introduced by Bracken J. and McGill J. [11], it was Candler W. and Norton R. [16] who used, for the first time in 1977, the term of optimization problem on more levels and, subsequently, bilevel optimization problem, for this type of optimization problems.

Among major works in this area of research we recall the ones authored by Dempe St. [24], Colson B., Marcotte P. and Savard G. [21], Vicente L., Savard G. and Júdice J. [121] (where the linear bilevel optimization problems are studied), Duca D. and Lupşa L. [25] (where transport bilevel problems are discussed), and Kosuch St., Le Bodic P., Leung J. and Lisser A. [63] (where the stochastic aspect is treated).

Multilevel optimization problems are used for modeling many types of concrete problems, such as: the network design problem [12], optimal pricing problem [70], the optimal signal setting problem [66], transportation problem [25], train set organization [39] and allocation problem [88].

A detailed bibliography of works in the field of bilevel and multilevel optimization problems is given by Vicente L.N. and Calamai P.H. [120].

Hansen P., Jaumard B. and Savard G. [51] have shown, in 1992, that linear bilevel optimization problems are NP-complete. Therefore, besides classical methods used in solving bilevel optimization problems, approximate methods are used, like methods based on the penalty technique (see [20] or [77]) and genetic algorithms (see [68]).

A study of the bilevel multiobjective optimization problems, in a general sense, is given by Liu Y. and Mei J. [69]. A global optimization strategy for the solution of hierarchical multilevel and decentralized multilevel programs based on multiparametric optimization is given by Faisca N.P., Saraiva P.M., Rustem B. and Pistikopoulos E.N. [28]. The case of both (upper and lower level) linear functions is discussed by Gaur A. and Arora S.R. [41] and Wang Z.W., Nagasawa H. and Nishiyama N. [123] (in this case, functions being of integer variables). A generalization of bilevel optimization problems is given in [1], where the real objective functions are replaced with multiobjective functions.

Now, we present the bilevel optimization problem as it is given in [24].

Let us consider the sets $D \subseteq \mathbf{R}^n \times \mathbf{R}^m$, $X \subseteq \mathbf{R}^n$, $Y \subseteq \mathbf{R}^m$, and let $F : D \to \mathbf{R}$, $f : D \to \mathbf{R}$, $G : D \to \mathbf{R}^p$ and $g : D \to \mathbf{R}^q$ be some given functions.

We introduce the set

$$S = \{x \in X,\ y \in Y \mid (x, y) \in D,\ G(x, y) \leqq 0_p,\ g(x, y) \leqq 0_q\}.$$

For each $x \in X$, we denote by

$$S_x = \{y \in Y \mid (x, y) \in D,\ G(x, y) \leqq 0_p\}$$

and, under the hypothesis that $S_x \neq \emptyset$, we denote by

$$S_x^* \ = \ \{\arg\min(\arg\max)\, \{f(x, y) | y \in S_x\}\}.$$

In mathematical terms, using the above notations, a bilevel optimization problem can be formulated as follows:

$$(BP) \quad \begin{cases} F(x, y) \ \to \ \min(\max), \\ \text{such that} \\ G(x, y) \leqq 0_p, \\ x \in X, \\ y \in S_x^*. \end{cases} \tag{1.11}$$

Furthermore, we give the terminology used in the literature for the case of bilevel optimization problems. We agree that:

• the set S is called *relaxed feasible set* or *constrained region* attached to the feasible solutions set of the problem (BP);

• for each $x \in X$, the set S_x is called *lower level feasible solutions set* determined by $x \in X$;

• for each $x \in X$, the set S_x^* is called *follower's rational reaction set* determined by $x \in X$;

• the set $RI = \{(x,y) \in S \,|\, y \in S_x^*\}$ is called *the set of feasible solutions* or *inducible region*;

• for each $x \in X$, the number $f(x,y)$, where $y \in S_x^*$, is called *lower level optimal value* corresponding to x;

• the function F is called *upper level objective function*;

• the function f is called *lower level objective function*;

• for $x \in X$, the problem

$$(P_l(x)) \quad \begin{cases} f(x,y) \;\rightarrow\; \min(\max), \\ \text{such that} \\ g(x,y) \leqq 0_q, \\ y \in S_x, \end{cases}$$

is called *lower level problem* determined by x;

• the problem

$$(P_u) \quad \begin{cases} F(x,y) \;\rightarrow\; \min(\max), \\ \text{such that} \\ (x,y) \in RI, \end{cases}$$

is called *upper level problem*.

The generalization of bilevel optimization problems was made in many directions, such as: increasing the number of levels (on vertical), increasing the number of lower-level functions corresponding to the existence of several lower-level decision makers independent between them, replacing the functions F (see ALVES M.J., DEMPE ST. and JÚDICE J.J. [1]) and/or f by vector functions. The case in which F and f are multifunction is discussed by LIU Y. and MEI J. [69].

Let us consider the natural non-zero numbers n_i and p_i, $i \in \{1, \ldots, m\}$.

Let $f_i : \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m} \;\rightarrow\; \mathbb{R}$, $g_i : \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m} \;\rightarrow\; \mathbb{R}^{p_i}$, $i \in \{0, 1, \ldots, m\}$ be some given functions and let $S_i \subseteq \mathbb{R}^{n_i}$, $i \in \{0, 1, \ldots, m\}$.

By using the formulation given in [28], the multilevel decentralized optimization problem (MLDP) can be defined as follows: to determine the minimum of the function

$$(MLDP) \quad \begin{cases} f_0(x, y^1, \ldots, y^m) \ \to \ \min, \quad \text{(the upper level - level 0)} \\ \text{such that} \\ g_0(x, y^1, \ldots, y^m) \leq 0_{m_0}, \\ x \in S_0, \\ \quad \text{if} \\ \qquad [y^1, \ldots, y^m] \text{ is an optimal solution of the problem} \\ \qquad f_1(x, y^1, \ldots, y^m) \ \to \ \min, \quad \text{(lower level 1 - level 1)} \\ \qquad \text{such that} \\ \qquad g_1(x, y^1, \ldots, y^m) \leq 0_{p_1}, \\ \qquad y^1 \in S_1, \\ \qquad \quad \text{if} \\ ... \\ \qquad\quad [y^i, \ldots, y^m] \text{ is an optimal solution of the problem} \\ \qquad\quad f_1(x, y^1, \ldots, y^i, \ldots, y^m) \ \to \ \min, \quad \text{(lower level i - level i)} \\ \qquad\quad \text{such that} \\ \qquad\quad g_i(x, y^1, \ldots y^i, \ldots, y^m) \leq 0_{p_i}, \\ \qquad\quad y^i \in S_i, \\ \qquad\qquad \text{if} \\ ... \\ \qquad\qquad y^m \text{ is an optimal solution of the problem} \\ \qquad\qquad f_m(x, y^1, \ldots, y^i, \ldots, y^m) \ \to \ \min, \quad \text{(lower level m - level m)} \\ \qquad\quad \text{such that} \\ \qquad\quad g_i(x, y^1, \ldots y^i, \ldots, y^m) \leq 0_{p_m}, \\ \qquad\quad y^m \in S_m. \end{cases}$$

Regarding the methods used for solving these problems, they are very different, depending on the type of functions that appear in the problem definition. Different methods for solving the bilevel optimization problems are given in [24]. A very used technique appeal to parametric optimization (see, for example, [28]), in the sense that variables that appear in the lower level problems are considered as parameters in the main problem, respectively each variable from the lower level k is considered as parameter in the problem of level $k + 1$.

We note that there exist some cases (see [34]) in which the solving of the bilevel optimization problem can be reduced to solving a multicriteria optimization problem. Then, the new problem can be solved by using the weighted method.

Within the present book we study some types of bilevel optimization problems which are solved by reducing the initial problem to a multicriteria lexicographic optimization problem.

Bilevel parametric optimization

Another aspect, which is new and very used in practical applications, is the one which implies to consider the bi and multilevel optimization problem in which the coefficients of the objective functions or of the restrictions depend on one or more parameters. Such kind of problems are named bilevel/multilevel parametric optimization problems. It is very important to remark the fact that this type of problems differ from those in which the variables of the lower levels are parameters for the upper levels.

In what follows, we introduce a bilevel optimization problem in which the coefficients of the objective functions and of the restrictions depend on one or more parameters.

Let m, n, p, q, r be natural non-zero numbers, Ω, A, $B \subseteq \mathbb{R}^n \times \mathbb{R}^m$ and $T \subseteq \mathbb{R}^r$. Let $F : \Omega \times T \to \mathbb{R}$, $f : \Omega \times T \to \mathbb{R}$, $G : \Omega \times T \to \mathbb{R}^p$ and $g : \Omega \times T \to \mathbb{R}^q$ be some given functions.

Let us consider the parametric optimization problem

$$(BPP) \quad \begin{cases} F(x, y; t) \ \to \max \\ \text{such that} \\ (x, y) \in \Omega \bigcap A \qquad , t \in T, \\ G(x, y; t) \leq 0_p \\ y \in S^*(x; t) \end{cases} \tag{1.12}$$

where

$$S^*(x; t) \ = \ \text{argmax}\, \{g(x, y; t) \,|\, y \in \Omega_{x;t}\}, \tag{1.13}$$

$$\Omega_{x;t} = \{y \in \mathbb{R}^n \,|\, (x, y) \in \Omega \bigcap B, \ g(x, y; t) \leq 0_q\}. \tag{1.14}$$

In fact, to solve a parametric optimization problem means to specify, for each value of the parameter $t \in T$, which is the solution of the bilevel optimization problem obtained if the parameter is fixed to this value. It is well known that in case of the bilevel optimization we work under the hypothesis that both lower level objective functions and upper level objective function are bounded on the set of optimal solutions. Therefore, it is very important that, for each $t \in T$, to know if the set of feasible solutions is non-void and, also, if the objective function is bounded on the corresponding set of feasible solutions. Hence, we consider the set

$$T^* \ = \ \{t \in T \,|\, S^*(x; t) \neq \emptyset, \ \forall\, x \in \Omega\} \tag{1.15}$$

named the characteristic variation set (domain) of the parameter.

In Chapter 6 we study some multilevel optimization problems in which the coefficients depend on two parameters. Therefore, we agree that in case of a multilevel parametric optimization problem in which the coefficients depend on one or more parameters we introduce for each parameter the characteristic variation set as being composed from each values of the parameter for which the set of all lower level feasible solutions is non-void and the corresponding objective functions are bounded and reach its bound.

Chapter 2

Lexicographic Multicriteria Bottleneck Problems and Optimal Points with Pipeline Property

In the present chapter we augment the existent results regarding the lexicographic optimization problems with the special case of the lexicographic optimization problems with p objective functions of bottleneck type (or time type), generic denoted by us with LpBP. For this, we introduce a new type of optimal solutions named by us *optimal solution with pipeline property*. This type of points are introduced by using the idea of the papers authored by BANDOPADHYAYA L. [6], LUPŞA L. and BLAGA L.R. [72] or LUPŞA L. and BLAGA L.R. [73]. The study of these points can be reached when solving bilevel optimization problems in which the lower level function is bicriteria, of bottleneck type. For example, in GOINA D. and TUNS (BODE) O.R. [42] or in TUNS (BODE) O.R. [110].

The present chapter is organized as follows: in Section 2.1, after we introduce the lexicographic multicriteria bottleneck optimization problem (LBP), we study the structure of the set of optimal solutions of it. Such kind of study has not been done so far in the literature from this area of research. In Section 2.2 we introduce the notion of optimal solution with pipeline property for LBP and we discuss some aspects about the set of all optimal solutions with pipeline property for LBP. In the second part of this chapter we consider the LBPs in which the set of all feasible solutions is discrete. We denote this type of LBP by LDBP (lexicographic discrete optimization problem with bottleneck objective functions). The structure of the set of optimal solutions for LDBP and the structure of the set of optimal solutions with pipeline property for LDBP are discussed in Section 2.3. As we consider discrete problems, in order to study the convexity properties, in the first part of this section we recall some notions of strongly 2-convexity with respect to $\mathbb{N}^n$

given by CRISTESCU G. and LUPŞA L. in [22].

In Section 2.4 a new technique to determine an optimal solution with pipeline property for LDBP is given. We note that Theorem 2.4.2 represents a generalization of the technique used in [6], as well as of the technique given in [91] as the weighted method. In addition, we point out that the results within this chapter are used in the entire book.

We note that the scientific results within this chapter belong to the author and a part of these results can be found in the papers authored by TUNS (BODE) O.R. and LUPŞA L. [115] and [75]. In [115] the properties of the optimal solutions for a lexicographic discrete optimization problem with $p-$bottleneck objective functions are studied. A special type of these optimal solutions, named by us optimal solution with pipeline property, is pointed out. In [75] the structure of the set of optimal solutions with pipeline property for a lexicographic discrete optimization problem with $p-$bottleneck objective functions is studied.

2.1 Lexicographic Multicriteria Bottleneck Problems

Let m, n, p be natural non-zero numbers such that $1 \leq p \leq m$.

Let $I := \{1, \ldots, m\},\ J := \{1, \ldots, n\},\ H := \{1, \ldots, p\}$.

Let $\Omega \subseteq \mathbb{R}_+^n$ and $f = (f_1, \ldots, f_m) : \Omega \to \mathbb{R}^m$. We say that the vector function $f = (f_1, \ldots, f_m)$ has p components of *bottleneck type* if there exist p natural numbers $i_1, \ldots, i_p$ and a real matrix $T = [t_{hj}]$ with p lines and n columns such that, for each $h \in H$,

$$f_{i_h}(x) \ = \ \max\{t_{hj} \cdot \operatorname{sgn}(x_j) \,|\, j \in J\},\ \forall\, x \in \Omega \tag{2.1}$$

or

$$f_{i_h}(x) \ = \ \min\{t_{hj} \cdot \operatorname{sgn}(x_j) \,|\, j \in J\},\ \forall\, x \in \Omega. \tag{2.2}$$

Let $I^c := I \setminus \{i_h | h \in H\}$.

In the present chapter we consider the case (2.1). For each $h \in H$ we set

$$Z_h = \{t_{hj} | j \in J\},\ q_h := \operatorname{card}(Z_h) \text{ and } K_h := \{1, \ldots, q_h\}.$$

We renumber, for each $h \in H$, the elements of set Z_h by $z_{h1}, \ldots, z_{hq_h}$, such that to have

$$z_{h1} \ > \ z_{h2} \ > \ldots > \ z_{hq_h}. \tag{2.3}$$

For each $h \in H$ and $k \in K_h$ we set $L_{hk} := \{j \in J | t_{hj} = z_{hk}\}$.

Remark 2.1.1 (TUNS (BODE) O.R. [*113*]). *Using the above notations, we conclude that for each $h \in H$ we have $f_{i_h}(x) = z_{hr}$ if and only if*

$$\{j \in L_{hk} | x_j > 0\} = \emptyset,\ \forall\, k \in K_h,\ k < r,\ \text{ and } \{j \in L_{hr} | x_j > 0\} \neq \emptyset.$$

Let S be a non-void subset of Ω. We consider the following optimization problem

$$(\text{LpBP}) \quad \begin{cases} f(x) \rightarrow \text{lex} - \text{min} \quad (\text{or lex} - \text{max}), \\ x \in S, \end{cases}$$

named by us *lexicographic optimization problem with p bottleneck objective functions.*

We note that in case S is a finite subset of $\mathbb{N}^n$ the lexicographic bottleneck problem can be seen as a generalization of the classic bottleneck assignment problems, as we can see in Pentico D.W. [91]. The lexicographic bottleneck problem have been studied for example by Burkard R.E. and Rendl F. [15], or by Sokkalingam P.T. and Aneja Y.P. [104]. In [15] the authors study combinatorial optimization problems with bottleneck objective function, where any feasible solution has the same number of elements. In addition to minimizing the largest element of a feasible solution they are interested in minimizing also the second largest element, the third largest element, and so on. For this version of the bottleneck problem two generic solution procedures are developed and analyzed by the authors. In [91] it can be find some of the variety of models of the bottleneck assignment problems, such as the lexicographic bottleneck problem, the assignment problem with side constraints and r-lexicographic multi-objective problem. In [104] are studied combinatorial problems with lexicographic bottleneck objective function, where in addition to minimizing the largest element of a feasible solution, the authors are also interested in minimizing the second largest element, the third largest element, and so on. The authors consider assignment and general matching problems. They propose an approach for solving the lexicographic bottleneck optimization problem by solving bottleneck and zero$-$one sum optimizations for at most t iterations and by reducing the problem size in each iteration.

Pastor R. [90] considers a new important kind of lexicographic bottleneck problem named Lexicographic Bottleneck Assembly Line Balancing Problem, obtained from classic assembly line balancing problem. Later Pastor R., Chuca I. and Carcía-Villoria A. [90] give three heuristic algorithms for solving this kind of problems.

We note that we can find very often bottleneck type functions in the literature from different areas of research. We point out this fact by our studies from the following Chapters of the present book.

In what follows, we study the properties of the set of optimal solutions of the (LpBP) problem. If, additional, S is a non-void finite subset of $\Omega \bigcap \mathbb{N}^n$, then we use the notion of lexicographic discrete optimization problem with p bottleneck objective functions, denoted by (LDpBP). Some properties of the optimal solutions for this type of problems are studied in the second part of the present chapter. As far as we know, so far it has not been studied neither the structure of the set of optimal solutions for LBPs, nor our proposed method to determine an optimal solution with pipeline property for a LBP.

Let $\hat{S}_0 := S$ and, for each $i \in I$, let $\hat{S}_i$ be the set of all minimum points of f_i on set $\Lambda := \hat{S}_{i-1}$. Obviously, *a point x^0 of S is an optimal solution for problem* (LpBP) *or a minimum (maximum) lexicographic point of f on S if x^0 is a minimum (maximum) point of function f_i with respect to set $\hat{S}_{i-1}$, for each $i \in I$.*

Let us denote by $\hat{S}$ the set of all optimal solutions for (LpBP), i.e. the set of all minimum lexicographic points of f on S. Let us remark that $\hat{S}_i \subseteq \hat{S}_{i-1}$, for each $i \in I$. Therefore, a point $x^0 \in S$ is an optimal solution for problem (LpBP) if and only if $x^0 \in \hat{S}_m$. Then, $\hat{S} = \hat{S_m}$.

Lemma 2.1.2 (TUNS (BODE) O.R. [*113*]). *If $\Lambda \subseteq S$ is a convex set and $h \in H$, then the set $\hat{\Lambda}_{i_h}$ of all minimum points of f_{i_h} on Λ is convex.*

Proof. If $\Lambda = \emptyset$, then $\hat{\Lambda}_k = \emptyset$. Therefore, the Lemma is true.

Let now $\Lambda \neq \emptyset$.

As $f_{i_h}(\Lambda)$ is a non-void finite set, there exists $\underline{m}_{i_h} := \min\{f_{i_h}(x)|x \in \Lambda\}$.

Let $x', x'' \in \hat{\Lambda}_{i_h}$. Then,

$$f_{i_h}(x') = f_{i_h}(x'') = \max\{t_{hj}\mathrm{sgn}(x'_j)|j \in J\} = \underline{m}_{i_h}.$$

Let $v \in]0,\ 1[$. Since Λ is convex, we have $(1-v)x' + vx'' \in \Lambda$. We have to prove that $(1-v)x' + vx'' \in \hat{\Lambda}_{i_h}$, i.e. $\max\{t_{hj}\mathrm{sgn}((1-v)x'_j + vx''_j)|j \in J\} = \underline{m}_{i_h}$. As $\Lambda \subseteq S \subseteq \mathrm{I\!R}^n_+$, we get that, for each $j \in J$, $\mathrm{sgn}((1-v)x'_j + vx''_j) = 1$ if and only if $x'_j \neq 0$ or $x''_j \neq 0$. It follows that $\max\{t_{hj}\mathrm{sgn}((1-v)x'_j + vx''_j)|j \in J\} = \max\{t_{hj}\mathrm{sgn}(x'_j),\ t_{hj}\mathrm{sgn}(x''_j))|j \in J\} = \underline{m}_{i_h}$. This implies that $f_{i_h}((1-v)x' + vx'') = \underline{m}_{i_h}$. Then, $(1-v)x' + vx'' \in \hat{\Lambda}_{i_h}$. Therefore, $\hat{\Lambda}_{i_h}$ is convex. ■

As a consequence of Lemma 2.1.2 we obtain a very relevant result for the practical applications.

Theorem 2.1.3 (TUNS (BODE) O.R. [*113*]). *If S is a non-void and convex set and functions f_k, $k \in I \setminus \{i_h|h \in H\}$, are convex, then set $\hat{S}$ is convex, too.*

Proof. Let us consider $k = 1$ and $\Lambda := S$. As S is a non-void and convex set, applying Lemma 2.1.2 we get that the set $\hat{S}_1$, which is equal to set $\hat{\Lambda}$, is convex. Let us take by turns $k = 2, \ldots, k = m$. Taking, each time, $\Lambda := \hat{S}_{k-1}$ and applying Lemma 2.1.2, we deduce that $\hat{S}_k$, the set of minimum points of f_k on $\hat{S}_{k-1}$, is convex because is equal to $\hat{\Lambda}$. Therefore, as $\hat{S} = \hat{S}_m$, we deduce that $\hat{S}$ is convex. ■

2.2 Optimal Solutions with Pipeline Property

Let $\Lambda \subseteq S$, $h \in H$.

Definition 2.2.1 (Tuns (Bode) O.R. *and* Lupşa L. [*115*]). *A point $x^0 \in \Lambda$ is said to be a minimum point with pipeline property of f_{i_h} on Λ if, for all $x \in \Lambda$, we have*
a) $\sum\limits_{j \in L_{hk}} sgn(x_j^0) = \sum\limits_{j \in L_{hk}} sgn(x_j)$, $\forall\, k \in K_h$,
or
b) there exists a natural number $r \in K_h$ such that $\sum\limits_{j \in L_{hr}} sgn(x_j^0) < \sum\limits_{j \in L_{hr}} sgn(x_j)$, *and, if* $r \geq 2$, *then* $\sum\limits_{j \in L_{hk}} sgn(x_j^0) = \sum\limits_{j \in L_{hk}} sgn(x_j)$, $\forall\, k \in \{1, ..., r-1\}$.

In what follows, for each $h \in H$, we denote by $\tilde{\Lambda}_{i_h}$ the set of all minimum points with pipeline property of f_{i_h} on Λ.

Theorem 2.2.2 (Tuns (Bode) O.R. [*113*]). *If $h \in H$ and $x^0 \in \tilde{\Lambda}_{i_h}$, then x^0 is a minimum point of f_{i_h} on Λ.*

Proof. Let us suppose that x^0 is not a minimum point of f_{i_h} on Λ. Then, there exists $x \in \Lambda$ such that $f_{i_h}(x) < f_{i_h}(x^0)$. Based on (2.3) we deduce that there exist $r, s \in K_h$, $r < s$, such that $f_{i_h}(x) = z_{hs}$ and $f_{i_h}(x^0) = z_{hr}$. Then, since $S \subseteq \mathbb{R}^n_+$, we have

$$\sum_{j \in L_{hk}} \operatorname{sgn}(x_j) = 0,\ \forall k \in K_h,\ k < s \text{ and } \sum_{j \in L_{hs}} \operatorname{sgn}(x_j) > 0,$$

and

$$\sum_{j \in L_{hk}} \operatorname{sgn}(x_j^0) = 0,\ \forall k \in K_h, k < r \text{ and } \sum_{j \in L_{hr}} \operatorname{sgn}(x_j^0) > 0.$$

As $r < s$, we deduce that $\sum\limits_{j \in L_{hr}} \operatorname{sgn}(x_j^0) > \sum\limits_{j \in L_{hr}} \operatorname{sgn}(x_j)$, which contradicts the fact that x^0 has the pipeline property. ■

Recalling that we denoted by $\hat{\Lambda}_{i_h}$ the set of all minimum points of function f_{i_h} on Λ, from Theorem 2.2.2 we get that

$$\tilde{\Lambda}_{i_h} \subseteq \hat{\Lambda}_{i_h}. \tag{2.4}$$

But, based on the below example, there exist minimum points of f_{i_h} with respect to Λ which does not have the pipeline property.

Example 2.2.3 (Tuns (Bode) O.R. [*113*]). *Let us consider* $\Lambda = [1, 2] \times [0, 2] \times [0, 1] \times [0, 1]$.
Let $f = (f_1, f_2) : \Lambda \to \mathbb{R}^2$ *be the function given by*

$f_1(x_1,x_2,x_3,x_4)=\max\{2sgn(x_1),\ 3sgn(x_2), 2sgn(x_3)\}$, *for all* $(x_1,x_2,x_3,x_4)\in\Lambda$,
$f_2(x_1,x_2,x_3,x_4)=\max\{3sgn(x_1),\ 2sgn(x_2), sgn(x_3)\}$, *for all* $(x_1,x_2,x_3,x_4)\in\Lambda$.

We have:

$\min\{f_1(x)|x\in\Lambda\} = 2$, $\hat{\Lambda}_1=\{(x_1,0,x_3,x_4)|x_1\in[1,\ 2],\ x_3\in[0,\ 1],\ x_4\in[0,\ 1]\}$,
$Z_1=\{3,2,0\}$, $z_{11}=3$, $z_{12}=2$, $z_{13}=0$, $L_{11}=\{2\}$, $L_{12}=\{1,3\}$, $L_{13}=\{4\}$.

For each $x\in\{(x_1,0,x_3,x_4)|x_1\in[1,\ 2],\ x_3\in[0,1],\ x_4\in[0,1]\}$ *we have*

$$\sum_{j\in L_{11}} sgn(x_j)=0,$$

$$\sum_{j\in L_{12}} sgn(x_j)=\begin{cases}1, & \textit{if}\ x_1\in[1,\ 2],\ x_3=0,\ x_4\in[0,\ 1],\\ 2, & \textit{if}\ x_1\in[1,\ 2],\ x_3\in]0,\ 1],\ x_4\in[0,\ 1],\end{cases}$$

$$\sum_{j\in L_{13}} sgn(x_j)=\begin{cases}0, & \textit{if}\ x_1\in[1,\ 2],\ x_3=0,\ x_4=0,\\ 1, & \textit{if}\ x_1\in[1,\ 2],\ x_3\in]0,\ 1],\ x_4\in]0,\ 1].\end{cases}$$

Therefore, $\tilde{\Lambda}_1=\{(x_1,0,0,0)|x_1\in[1,\ 2]\} = [(1,0,0,0),\ (2,0,0,0)]$.

For each $x\in\tilde{\Lambda}_1$ *we have* $f_1(x)=2$, $\sum_{j\in L_{11}} sgn(x_j)=0$, $\sum_{j\in L_{12}} sgn(x_j)=1$, $\sum_{j\in L_{13}} sgn(x_j)=0$. *Any point of the set* $\hat{\lambda}\setminus\tilde{\lambda}$ *is a minimum point of function* f_1, *point which does not have the pipeline property. For example, the point* $(1,0,1,1)$, *because* $f(1,0,1,1)=\max\{2,0,2\}=2$, $\sum_{j\in L_{11}} sgn(x_j)=0$, $\sum_{j\in L_{12}} sgn(x_j)=2$, $\sum_{j\in L_{13}} sgn(x_j)=1$.

Under the hypothesis that $\Lambda\subseteq S\subseteq \mathbb{R}^n_+$ and $h\in H$, in the following two Propositions we give two different methods that can be used to verify that a point has the pipeline property.

Proposition 2.2.4 (Tuns (Bode) O.R. [*113*]). *If* $x\in\tilde{\Lambda}_{i_h}$, $f_{i_h}(x)=z_{hr}$ *and* $y\in\Lambda$, *then* $y\in\tilde{\Lambda}_{i_h}$ *if and only if*

$$f_{i_h}(x)=f_{i_h}(y)\ \textit{and}\ \sum_{j\in L_{hk}} sgn(x_j)=\sum_{j\in L_{hk}} sgn(y_j),\ \forall\, k\in\{r,\dots,q_h\}. \tag{2.5}$$

Proof. *Necessity:* Let $y\in\tilde{\Lambda}_{i_h}$. In view of Theorem 2.2.2 both x and y are minimum points of f_{i_h} on Λ. Therefore, $f_{i_h}(y)=f_{i_h}(x)=z_{hr}$.

Let us suppose that (2.5) does not hold and let s be the smallest value of $k\in\{r,\dots,q_h\}$ such that $\sum\limits_{j\in L_{hs}}\mathrm{sgn}(x_j)\neq\sum\limits_{j\in L_{hs}}\mathrm{sgn}(y_j)$.

Two cases can occur:

Case 1. $\sum\limits_{j\in L_{hs}}\mathrm{sgn}(x_j)>\sum\limits_{j\in L_{hs}}\mathrm{sgn}(y_j)$; then x does not have the pipeline property.

Case 2. $\sum_{j\in L_{hs}} \mathrm{sgn}(x_j) < \sum_{j\in L_{hs}} \mathrm{sgn}(y_j)$; then y does not have the pipeline property.
As in both cases we get a contradiction, it follows that

$$\sum_{j\in L_{hk}} \mathrm{sgn}(x_j) = \sum_{j\in L_{hk}} \mathrm{sgn}(y_j),\ \forall\, k \in \{r,\ldots,q_h\}.$$

Sufficiency: As $x \in \tilde{\Lambda}_{i_h}$, from Theorem 2.2.2 one gets that x is a minimum point of f_{i_h} on Λ. From $f_{i_h}(x) = f_{i_h}(y)$ we deduce that $f_{i_h}(y) \le f_{i_h}(z),\ \forall\, z \in \Lambda$.

If $r > 1$, then (2.3) implies that

$$\sum_{j\in L_{hk}} \mathrm{sgn}(y_j) = \sum_{j\in L_{hk}} \mathrm{sgn}(x_j) = 0,\ \forall\, k \in \{1,\ldots,r-1\}. \tag{2.6}$$

Therefore, if $r \in K_h$ and $r > 1$, then for all $k \in \{1,\ldots,r-1\}$ we have

$$\sum_{j\in L_{hk}} \mathrm{sgn}(y_j) \le \sum_{j\in L_{hk}} \mathrm{sgn}(z_j),\ \text{for all } z \in \Lambda. \tag{2.7}$$

Let us suppose, by indirect proof, that $y \notin \tilde{\Lambda}_{i_h}$. Then, there exist $z \in \Lambda$ and $k \in \{r,\ldots,q_h\}$ such that

$$\sum_{j\in L_{hk}} \mathrm{sgn}(z_j) \ < \sum_{j\in L_{hk}} \mathrm{sgn}(y_j). \tag{2.8}$$

Therefore, supposing that k is the smallest value wherefore the inequality fulfill, from (2.5), (2.6) and (2.7) we deduce that $\sum_{j\in L_{hk}} \mathrm{sgn}(z_j) < \sum_{j\in L_{hk}} \mathrm{sgn}(x_j)$, which contradicts the fact that $x \in \tilde{\Lambda}_{i_h}$. Hence, $y \in \tilde{\Lambda}_{i_h}$. ■

Proposition 2.2.5 (Tuns (Bode) O.R. [*113*]). *If $x \in \tilde{\Lambda}_{i_h}$ and $y \in \Lambda$, then $y \in \tilde{\Lambda}_{i_h}$ if and only if*

$$\sum_{j\in L_{hk}} sgn(x_j) \ = \sum_{j\in L_{hk}} sgn(y_j),\ \forall\, k \in K_h. \tag{2.9}$$

Proof. *Necessity:* Let $y \in \tilde{\Lambda}_{i_h}$ and $z_r = f_{i_h}(x)$. In view of Proposition 2.2.4 and of Remark 2.1.1, we just need to prove that if $r > 1$, then $\sum_{j\in L_{hk}} \mathrm{sgn}(y_j) = 0,\ \forall\, k \in \{1,\ldots r-1\}$. Let us suppose that there exists $k \in \{1,\ldots r-1\}$ such that $\sum_{j\in L_{hk}} \mathrm{sgn}(y_j) \ne 0$. As $y \in \mathbb{R}^n_+$ we deduce that $\sum_{j\in L_{hk}} \mathrm{sgn}(y_j) \ge 1$. Now, let us suppose that we choose the smallest value of k wherefore the inequality fulfill. Then, $f_{i_h}(y) = z_k > z_r$, which, based on Proposition 2.2.4, contradicts the fact that $y \in \tilde{\Lambda}_{i_h}$.

Sufficiency: Let $z_r = f_{i_h}(x)$. Based on Remark 2.1.1 we have $\sum_{j\in L_{hk}} \mathrm{sgn}(x_j) = 0$, for all $k \in \{1,\ldots,r-1\}$, and $\sum_{j\in L_{hr}} \mathrm{sgn}(x_j) \ne 0$. From (2.9) we deduce that $\sum_{j\in L_{hk}} \mathrm{sgn}(y_j) = 0$, for all $k \in \{1, \ldots, r-1\}$, and $\sum\limits_{j \in L_{hr}} \operatorname{sgn}(y_j) \neq 0$. Applying again Remark 2.1.1, we get that $f_{i_h}(y) = z_r = f_{i_h}(x)$. Based on the above and on (2.9), we obtain that the hypothesis of the Proposition 2.2.4 occurs. Hence, $y \in \tilde{\Lambda}_{i_h}$. ■

Let $h \in H$. In what follows, we study the properties of the set $\tilde{\Lambda}_{i_h}$.

Let $\Lambda \subseteq S \subseteq \mathbb{R}^n_+$, $h \in H$ and $x, y \in \Lambda$.

For each $k \in K_h$, we set

$$L^x_{hk} := \{j \in L_{hk} | \operatorname{sgn}(x_j) = 1, \operatorname{sgn}(y_j) = 0\},$$
$$L^y_{hk} := \{j \in L_{hk} | \operatorname{sgn}(x_j) = 0, \operatorname{sgn}(y_j) = 1\},$$
$$L^{xy}_{hk} := \{j \in L_{hk} | \operatorname{sgn}(x_j) = 1, \operatorname{sgn}(y_j) = 1\}.$$

Proposition 2.2.6 (Lupşa L. *and* Tuns (Bode) O.R. [75]). *Let $x, y \in \Lambda$. It is not difficult to see that:*

i) $\sum\limits_{j \in L_{hk}} sgn(x_j) = card(L^x_{hk}) + card(L^{xy}_{hk})$, $\sum\limits_{j \in L_{hk}} sgn(y_j) = card(L^y_{hk}) + card(L^{xy}_{hk})$, $\forall\, k \in K_h$;

ii) If $\lambda \in]0, 1[$, then

$$\sum_{j \in L_{hk}} sgn((1-\lambda)x_j + \lambda y_j) = card(L^x_{hk}) + card(L^y_{hk}) + card(L^{xy}_{hk}),\ \forall\, k \in K_h; \tag{2.10}$$

iii) If $x, y \in \tilde{\Lambda}_{i_h}$, then $card(L^x_{hk}) = card(L^y_{hk})$, $\forall\, k \in K_h$;

iv) If $y \in \hat{\Lambda}_{i_h} \setminus \tilde{\Lambda}_{i_h}$, then there exist $x \in \tilde{\Lambda}_{i_h}$ and $r \in K_h$ such that $card(L^x_{hr}) < card(L^y_{hr})$, and, if $r \geq 2$, then $card(L^x_{hk}) = card(L^y_{hk})$, $\forall\, k \in \{1, \ldots, r-1\}$.

Proof. i) Let $k \in K_h$ and $L_{hk} = L^x_{hk} \cup L^y_{hk} \cup L^{xy}_{hk}$.

As $L^x_{hk} \cap L^y_{hk} = \emptyset$, $L^x_{hk} \cap L^{xy}_{hk} = \emptyset$, $L^y_{hk} \cap L^{xy}_{hk} = \emptyset$ and $\operatorname{sgn}(x_j) = 0$, for all $j \in L^y_{hr}$, we get that $\sum\limits_{j \in L_{hk}} \operatorname{sgn}(x_j) = \sum\limits_{j \in L^x_{hk}} \operatorname{sgn}(x_j) + \sum\limits_{j \in L^{xy}_{hk}} \operatorname{sgn}(x_j) = \operatorname{card}(L^x_{hk}) + \operatorname{card}(L^{xy}_{hk})$. Analogously, we have $\sum\limits_{j \in L_{hk}} \operatorname{sgn}(y_j) = \sum\limits_{j \in L^y_{hk}} \operatorname{sgn}(y_j) + \sum\limits_{j \in L^{xy}_{hk}} \operatorname{sgn}(y_j) = \operatorname{card}(L^y_{hk}) + \operatorname{card}(L^{xy}_{hk})$.

ii) Let $k \in K_h$. As the sets L^x_{hk}, L^y_{hk} and L^{xy}_{hk} are two by two disjoints, we have

$$\sum_{j \in L_{hk}} \operatorname{sgn}((1-\lambda)x_j + \lambda y_j) = \sum_{j \in L^x_{hk}} \operatorname{sgn}((1-\lambda)x_j) + \sum_{j \in L^y_{hk}} \operatorname{sgn}(\lambda y_j) +$$
$$+ \sum_{j \in L^{xy}_{hk}} \operatorname{sgn}((1-\lambda)x_j + \lambda y_j) = \operatorname{card}(L^x_{hk}) + \operatorname{card}(L^y_{hk}) + \operatorname{card}(L^{xy}_{hk}).$$

iii) From i) and (2.5) it follows that $\operatorname{card}(L^x_{hk}) = \operatorname{card}(L^y_{hk})$, for all $k \in K_h$.

iv) As $y \in (\hat{\Lambda}_{i_h} \setminus \tilde{\Lambda}_{i_h})$, from Definition 2.2.1 we deduce that there exists a point $x \in \hat{\Lambda}_{i_h}$ and there exists $r \in K_h$ such that $\sum\limits_{j \in L_{hr}} \operatorname{sgn}(x_j) < \sum\limits_{j \in L_{hr}} \operatorname{sgn}(y_j)$ and, if $r \geq 2$, then $\sum\limits_{j \in L_{hk}} \operatorname{sgn}(x_j) = \sum\limits_{j \in L_{hk}} \operatorname{sgn}(y_j)$, for all $k \in \{1, \ldots, r-1\}$. Based on i), we obtain that $\operatorname{card}(L^x_{hr}) < \operatorname{card}(L^y_{hr})$, and, if $r \geq 2$, then $\operatorname{card}(L^x_{hk}) = \operatorname{card}(L^y_{hk})$, $\forall\, k \in \{1, \ldots, r-1\}$. ■

Proposition 2.2.7 (Tuns (Bode) O.R. [*113*]). *If set $\Lambda \subseteq S$ is convex, $h \in K_h$ and $y \in (\hat{\Lambda}_{i_h} \setminus \tilde{\Lambda}_{i_h})$, then there exists $x \in \tilde{\Lambda}_{i_h}$ such that $]x,\, y] \subseteq (\hat{\Lambda}_{i_h} \setminus \tilde{\Lambda}_{i_h})$.*

Proof. Based on iv) from Proposition 2.2.6 it results that there exists x, minimum point of f_{i_h}, and there exists $r \in K_h$ such that $\mathrm{card}(L^x_{hr}) < \mathrm{card}(L^y_{hr})$, and, if $r \geq 2$, then $\mathrm{card}(L^x_{hk}) = \mathrm{card}(L^y_{hk})$, $\forall\, k \in \{1, \ldots, r-1\}$. Therefore, $\mathrm{card}(L^y_{hr}) > 0$. Let $\lambda \in]0,\, 1]$ and $z := (1-\lambda)x + \lambda y$. If z is not a minimum point of f_{i_h} on Λ, then it does not have the pipeline property.

Let us suppose that z is a minimum point of f_{i_h} on Λ. By using (2.10) we obtain that

$$\sum_{j \in L_{hr}} \mathrm{sgn}(z_j) = \mathrm{card}(L^x_{hr}) + \mathrm{card}(L^y_{hr}) + \mathrm{card}(L^{xy}_{hr}) > \mathrm{card}(L^x_{hr}) + \mathrm{card}(L^{xy}_{hr}) = \sum_{j \in L_{hr}} \mathrm{sgn}(x_j).$$

In view of Proposition 2.2.4 it follows that z does not have the pipeline property. As λ was chosen arbitrary from the interval $]0,\, 1]$, we deduce that $]x,\, y] \subseteq (\Lambda \setminus \tilde{\Lambda}_{i_h})$. ■

Proposition 2.2.8 (Tuns (Bode) O.R. [*113*]). *If set $\Lambda \subseteq S$ is convex, $h \in H$, $x, y \in \hat{\Lambda}_{i_h}$ and $]x,\, y[\cap \tilde{\Lambda}_{i_h} \neq \emptyset$, then $]x,\, y[\subseteq \tilde{\Lambda}_{i_h}$.*

Proof. Let $z \in]x,\, y[\cap \tilde{\Lambda}_{i_h}$. Let us suppose that there exists $u \in]x,\, y[$ such that $u \notin \tilde{\Lambda}_{i_h}$. Two cases can occur: $z \in]x,\, u[$ or $z \in]u,\, y[$. Applying Proposition 2.2.7, in both cases we obtain that $z \notin \tilde{\Lambda}_{i_h}$. This contradicts the hypothesis. Therefore, $]x,\, y[\subseteq \tilde{\Lambda}_{i_h}$. ■

Proposition 2.2.9 (Tuns (Bode) O.R. [*113*]). *If set $\Lambda \subseteq S$ is convex, $h \in H$, $x, y \in \tilde{\Lambda}_{i_h}$, $x \neq y$, then $[x,\, y] \subseteq \tilde{\Lambda}_{i_h}$ if and only if $L^x_{hk} = \emptyset$ (or equivalent $L^y_{hk} = \emptyset$), for all $k \in K_h$.*

Proof. *Necessity:* Let $\lambda \in]0,\, 1[$ and $z := (1-\lambda)x + \lambda y$. From (2.10) one gets that

$$\sum_{j \in L_{hk}} \mathrm{sgn}((1-\lambda)x_j + \lambda y_j) = \mathrm{card}(L^x_{hk}) + \mathrm{card}(L^y_{hk}) + \mathrm{card}(L^{xy}_{hk}),\ \forall\, k \in K_h.$$

If, by indirect proof, there exists $r \in K_h$ such that $L^x_{hr} \neq \emptyset$, then Proposition 2.2.6 implies that $\sum_{j \in L_{hr}} \mathrm{sgn}((1-\lambda)x_j + \lambda y_j) > \mathrm{card}(L^y_{hr}) + \mathrm{card}(L^{xy}_{hr}) = \sum_{j \in L_{hr}} \mathrm{sgn}(y_j) = \sum_{j \in L_{hr}} \mathrm{sgn}(x_j)$. Therefore, z has not the pipeline property. This contradicts the hypothesis.

Sufficiency: Let $x, y \in \tilde{\Lambda}_{i_h}$, $\lambda \in]0,\, 1[$ and $z := (1-\lambda)x + \lambda y$. In view of Lemma 2.1.2 we get that z is a minimum point of f_{i_h} on Λ. As $L^x_{hk} = \emptyset$, for each $k \in K_h$, based on Proposition 2.2.6 iii), we get that $L^y_{hk} = \emptyset$, for each $k \in K_h$.

Then, $\sum_{j \in L_{hk}} \mathrm{sgn}(x_j) = \mathrm{card}(L^{xy}_{hk})$ and $\sum_{j \in L_{hk}} \mathrm{sgn}(y_j) = \mathrm{card}(L^{xy}_{hk})$. From (2.10) we get that $\sum_{j \in L_{hk}} \mathrm{sgn}(z_j) = \sum_{j \in L_{hk}} \mathrm{sgn}((1-\lambda)x_j + \lambda y_j) = \mathrm{card}(L^{xy}_{hk})$, $\forall\, k \in K_h$. As z is a

minimum point of f_{i_h} on Λ, applying Proposition 2.2.4 we deduce that z has the pipeline property. As $\lambda \in]0,\, 1[$ was arbitrary chosen and $x, y \in \tilde{\Lambda}_{i_h}$, we deduce that $[x,\, y] \subseteq \tilde{\Lambda}_{i_h}$.
■

We remark the fact that the condition $L^x_{hk} = \emptyset$, $\forall\, k \in K_h$, which is equivalent with the condition $L^y_{hk} = \emptyset$, $\forall\, k \in K_h$, is essential, as can be seen also from the following example.

Example 2.2.10 (Tuns (Bode) O.R. [*113*]). *Let* $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ *be the function given by*

$$f_1(x_1, x_2) = \max\{2sgn(x_1), 2sgn(x_2)\} \text{ and } f_2(x_1, x_2) = x_1 + x_2,\ \forall\, x = (x_1, x_2) \in \mathbb{R}^2.$$

Let $\Lambda = [(2, 0),\, (0, 2)] = \{(2(1-\lambda),\, 2\lambda) | \lambda \in [0,\, 1]\}$. *One gets that* $f_1(x) = 2$, $\forall\, x \in \Lambda$. *Therefore,* $\hat{\Lambda}_1 = [(2, 0),\, (0, 2)]$, $Z_1 = \{2\}$, $q_1 = 1$, $z_{11} = 2$, $L_{11} = \{1, 2\}$ *and for* $x \in \Lambda$ *we have*

$$\sum_{j \in L_{11}} = \begin{cases} 1, \text{ if } x = (2,0) \text{ or } x = (0,2), \\ 2, \text{ if } x \in \Lambda \setminus \{(2,0),\, (0,2)\}. \end{cases}$$

Hence, $\tilde{\Lambda} = \{(2, 0),\, (0, 2)\}$. *We note that, if we have* $x = (2, 0)$ *and* $y = (0, 2)$, *then* $L^x_{11} = \{1\} \neq \emptyset$, *which implies that the hypothesis of the Proposition 2.2.9 are not fulfilled.*

We note that in Example 2.2.3, taking $x = (1, 0, 0, 0)$ and $y = (2, 0, 0, 0)$, we have $L^x_{11} = \emptyset$, $L^x_{12} = \emptyset$, $L^x_{13} = \emptyset$. As $x, y \in \tilde{\Lambda}_1$, the hypothesis of the Proposition 2.2.9 are fulfilled, so it results that $[(1, 0, 0, 0),\, (2, 0, 0, 0)] \subseteq \tilde{\Lambda}_1$. In Example 2.2.3, by using the definition we showed that in fact $\tilde{\Lambda}_1 = [(1, 0, 0, 0),\, (2, 0, 0, 0)]$.

In what follows, we introduce the notion of optimal solution with pipeline property for (LpBP) problem.

Definition 2.2.11 (Tuns (Bode) O.R. *and* Lupşa L. [*115*]). *A point* $x^0 \in S$ *is called an optimal solution with pipeline property for (LpBP) or minimum lexicographic point with pipeline property of* f *on* S *if* x^0 *is a minimum point of function* f_k *on set* S_{k-1} *for each* $k \in I$ *and, additionally, if* $k \in \{i_h | h \in H\}$ *then it has the pipeline property, where* $S_0 = S$ *and* S_k *denotes:*
i) the set of all minimum points of function f_k *with respect to the set* S_{k-1}, *i.e. the set* $\hat{S}_{k-1}$, *if* $k \in I \setminus \{i_h | h \in H\}$; *or*
ii) the set of all minimum points with pipeline property of function f_k *with respect to the set* S_{k-1}, *i.e. the set* $\tilde{S}_{k-1}$, *if* $k \in \{i_h | h \in H\}$.

Let us denote by $\tilde{S}$ the set of all minimum lexicographic points with pipeline property of f on S.

Example 2.2.12 (TUNS (BODE) O.R. [*113*]). *Let us recall Example 2.2.3. If* $\Lambda = S = [1, 2] \times [0, 2] \times [0, 1] \times [0,1]$*, then we have* $\tilde{\Lambda}_1 = \{(x_1, 0, 0, 0)|x_1 \in [1, 2]\}$*. Hence,* $\tilde{S}_1 = \{(x_1, 0, 0, 0)|x_1 \in [1, 2]\}$*. Considering now* $\Lambda := \tilde{S}_1$*, we have* $\min\{f_2(x)|x \in \Lambda\} = \min\{3sgn(x_1)|x \in \tilde{S}_1\} = 3$*. As* $f_2(x) = 3$*, for all* $x \in \tilde{S}_1$*, we deduce that* $\hat{\Lambda} = \tilde{S}_1$*. Since* $Z_2 = \{3, 2, 1, 0\}$*, we have* $z_{21} = 3$*,* $z_{22} = 2$*,* $z_{23} = 1$*,* $z_{24} = 0$*,* $L_{21} = \{1\}$*,* $L_{22} = \{2\}$*,* $L_{23} = \{3\}$*,* $L_{24} = \{4\}$*. For each* $x \in \hat{\Lambda}$ *we have* $\sum\limits_{j \in L_{21}} sgn(x_j) = 1$*,* $\sum\limits_{j \in L_{22}} sgn(x_j) = 0$*,* $\sum\limits_{j \in L_{23}} sgn(x_j) = 0$*,* $\sum\limits_{j \in L_{24}} sgn(x_j) = 0$*. We deduce that* $\tilde{\Lambda} = \hat{\Lambda}$*. Therefore,* $\tilde{S}_2 := \tilde{\Lambda} = \{(x_1, 0, 0, 0)|x_1 \in [1, 2]\}$*.*

A method to determine the elements of the set $\tilde{S}$ is given in the last paragraph of the present chapter.

2.3 Lexicographic Discrete Bottleneck Problems

In this section the lexicographic discrete optimization problems with p bottleneck objective functions (LDpBP) are studied. Although, exteriorly, such kind of problems are very restrictive, they appear in real life situations. A first example it can be found in the paper authored by BANDOPADHYAYA L. [6], whence we adopt the term *pipeline property.* Then, we recall the papers authored by LUPŞA L. and BLAGA L.R. [72] and LUPŞA L. and BLAGA L.R. [73]. Another two examples, derived from the portfolio theory area, respectively from the vehicle routing problems area, can be found in the papers authored by GOINA D. and TUNS (BODE) O.R. [42], TUNS (BODE) O.R. [110] and TUNS (BODE) O.R. [111]. These last examples are presented in the following chapters of the book.

In what follows, we consider the (LpBP) problem under the additional hypothesis that $S \subseteq (\Omega \cap \mathbb{N}^n)$. We denote this problem by (LDpBP). We begin by giving two examples.

Example 2.3.1 (TUNS (BODE) O.R. *and* LUPŞA L. [*115*]).
Let $S = \{(-1, -1), (-1/2, -1)\}$ *and* $f : S \to \mathbb{R}$*,* $f(x_1, x_2) = \max\{2x_1, x_2\}$*, for all* $(x_1, x_2) \in S$*. Obviously,* $f(-1, -1) = f(1/2, -1) = -1$*. Both points* $x^0 = (-1, -1)$ *and* $x = (-1/2, -1)$ *are minimum points of* f *with respect to* S*, but only* $(-1, -1)$ *has the pipeline property because* $\sum\limits_{j \in L_{-1}} sgn(x_j^0) < \sum\limits_{j \in L_{-1}} sgn(x_j)$*.*

Example 2.3.2 (TUNS (BODE) O.R. *and* LUPŞA L. [*115*]).
Let $\Omega = \mathbb{R}^3$ *and* $S = ([0, 2] \times [1, 3] \times [0, 1]) \cap \mathbb{N}^3 = \{0, 1, 2\} \times \{1, 2, 3\} \times \{0, 1\}$.
Let $f = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ *be the function given by:*

$f_1(x_1,x_2,x_3) = -5 + x_3^2 - x_3,$
$f_2(x_1,x_2,x_3) = \max\{2sgn(x_1), 2sgn(x_2), 3sgn(x_3)\},$
$f_3(x_1,x_2,x_3) = -8 + x_1^2 + x_3^2$, *for all* $x = (x_1,x_2,x_3) \in \mathbb{R}^3$.
We have: $\min\{f_1(x)|x \in S\} = -5$, $S_1 = S$, $T = \{z_1 = 3,\ z_2 = 2\}$,
$L_1 = \{3\}$, $L_2 = \{1,\ 2\}$,
$\min\{f_2(x)|x \in S_1\} = 2$, $S_2 = \{(x_1,x_2,0)|x_1 \in \{0,1,2\},\ x_2 \in \{1,2,3\}\}$,
$\sum\limits_{j\in L_2} x_j = 1$, *if* $x \in \{(0,1,0),(0,2,0),(0,3,0)\}$,
$\sum\limits_{j\in L_2} x_j = 2$, *if* $x \in \{(1,1,0),(1,2,0),(1,3,0),(2,1,0),(2,2,0),(2,3,0)\}$,
$\tilde{S}_2 = \{(0,1,0),(0,2,0),(0,3,0)\}$, $\min\{f_3(x)|x \in \tilde{S}_2\} = -8$, $S_3 = \tilde{S}_2$.
Hence, $\tilde{S} = \{(0,1,0),(0,2,0),(0,3,0)\}$.

In what follows, some properties concerning the structure of the set of optimal solutions of (LDpBP) problem are studied by Lupşa L. and Tuns (Bode) O.R. in [75] and [115]. As this problem is discrete, in order to study the convexity of the set of all optimal solutions and of the set of all optimal solutions with pipeline property, we recall the notions of strongly 2-convexity with respect to $\mathbb{N}^n$ given by Cristescu G. and Lupşa L. in [22].

Definition 2.3.3 (Cristescu G. *and* Lupşa L. [*22*]). *A subset* U *of* $\mathbb{R}^n$ *is called strongly 2-convex with respect to* $\mathbb{N}^n$*, if for each two points* $x', x'' \in U$ *and any real number* $\alpha \in [0,1]$ *such that* $(1-\alpha)x' + \alpha x'' \in \mathbb{N}^n$*, we have* $(1-\alpha)x' + \alpha x'' \in U$.

Obviously, if $U \subseteq \mathbb{R}^n$ is a convex set, then $U \cap \mathbb{N}^n$ is strongly 2-convex. Generally, the converse is not true. For example, the set

$$U = \Big\{(x,y) \in \mathbb{R}^2 | x \in [1/2, 3/2[,\ y \geq 0\Big\} \cup \Big\{(x,y) \in \mathbb{R}^2 | x \in [3/2, 2[,\ y \geq 1\Big\}$$

is not convex, but it is strongly 2-convex.

Now, let $U \subseteq \mathbb{R}^n$ be a strongly 2-convex set with respect to $\mathbb{N}^n$. Particularizing the Definition 11.1.1 from Cristescu G. and Lupşa L. [22], we say that:

Definition 2.3.4 *A mapping* $\varphi : U \to \mathbb{R}$ *is called strongly 2-convex with respect to* $\mathbb{N}^n \times \mathbb{R}$ *if the set* $epi\varphi = \{(x,y) \in U \times \mathbb{R} \mid y \geq \varphi(x)\}$ *is strongly 2-convex with respect to* $\mathbb{N}^n \times \mathbb{R}$.

Regarding the strongly 2-convex sets and strongly 2-convex functions, we recall the following important result: Using Theorem 11.4.4 from [22] we get that:

Theorem 2.3.5 (Cristescu G. *and* Lupşa L. [*22*]). *If* $U \subseteq \mathbb{R}^n$ *is strongly 2-convex with respect to* $\mathbb{N}^n$ *and mapping* $\varphi : U \to \mathbb{R}$ *is strongly 2-convex with respect to* $\mathbb{N}^n \times \mathbb{R}$*, then the set of all minimum points of* φ *on* U *is strongly 2-convex with respect to* $\mathbb{N}^n$.

Let $\Omega \subseteq \mathbb{R}_+^n$, $S \subseteq (\Omega \cap \mathbb{N}^n)$ and $f = (f_1, \ldots, f_p) : \Omega \to \mathbb{R}^p$.

Theorem 2.3.6 (Tuns (Bode) O.R. *and* Lupşa L. [*115*]). *If set $\Lambda \subseteq S$ is strongly 2-convex with respect to $\mathbb{N}^n$ and $h \in H$, then set $\hat{\Lambda}_{i_h}$ of all minimum points of f_{i_h} on Λ is strongly 2-convex with respect to $\mathbb{N}^n$.*

Proof. As $f_{i_h}(\Lambda)$ is a non-void and finite set, there exists $\underline{m}_{i_h} := \min\{f_{i_h}(x) | x \in \Lambda\}$.

Let $x', x'' \in \hat{\Lambda}_{i_h}$. Then, there exists $j_1, j_2 \in J$ such that

$$f_{i_h}(x') = \max\{t_{hj}\mathrm{sgn}(x'_j) | j \in J\} = \underline{m}_{i_h} \text{ and } f_2(x'') = \max\{t_{hj}\mathrm{sgn}(x''_j) | j \in J\} = \underline{m}_{i_h}.$$

Let $\lambda \in]0, 1[$ be such that $z := (1-v)x' + vx'' \in \mathbb{N}^n$. Since Λ is strongly 2-convex with respect to $\mathbb{N}^n$, it follows that $(1-v)x' + vx'' \in \Lambda$. Based on the proof of Lemma 2.1.2 it results that $f_{i_h}((1-v)x' + vx'') = \underline{m}_{i_h}$. Hence, $(1-v)x' + vx'' \in \hat{\Lambda}_{i_h}$. Therefore, $\hat{\Lambda}_{i_h}$ is strongly 2-convex with respect to $\mathbb{N}^n$. ■

Theorems 2.3.5 and 2.3.6 implies that:

Corollary 2.3.7 (Tuns (Bode) O.R. *and* Lupşa L. [*115*]). *If set S is non-void and strongly 2-convex with respect to $\mathbb{N}^n$ and functions f_k, $k \in I \setminus \{i_h | h \in H\}$, are strongly 2-convex with respect to $\mathbb{N}^n$, then set $\hat{S}$ is strongly 2-convex with respect to $\mathbb{N}^n$, too.*

Similar properties with the ones presented above can be obtained in case it is studied the set of optimal points with pipeline property of a bottleneck type function on a strongly 2-convex subset of the domain of definition.

Proposition 2.3.8 (Lupşa L. *and* Tuns (Bode) O.R. [*75*]). *If set $\Lambda \subseteq S$ is strongly 2-convex with respect to $\mathbb{N}^n$, $h \in H$, y is a minimum point of f_{i_h} on Λ, but $y \notin \tilde{\Lambda}_{i_h}$, then there exists $x \in \tilde{\Lambda}_{i_h}$ such that if $\lambda \in]0, 1[$ and $z := (1-\lambda)x + \lambda y \in \mathbb{N}^n$, then $z \notin \tilde{\Lambda}_{i_h}$.*

Proof. From Proposition 2.2.6 iv), it follows that there exist x minimum point of f_{i_h} and $r \in K_h$ such that $\mathrm{card}(L^x_{hr}) < \mathrm{card}(L^y_{hr})$, and if $r \geq 2$ then $\mathrm{card}(L^x_{hk}) = \mathrm{card}(L^y_{hk})$, $\forall\, k \in \{1, \ldots, r-1\}$. Therefore, $\mathrm{card}(L^y_{hr}) > 0$.

If z is not a minimum point of f_{i_h} on Λ, then it does not have the pipeline property. Let us suppose that z is a minimum point of f_{i_h} on Λ. By using (2.10) we obtain that $\sum\limits_{j \in L_{hr}} \mathrm{sgn}(z_j) = \mathrm{card}(L^x_{hr}) + \mathrm{card}(L^y_{hr}) + \mathrm{card}(L^{xy}_{hr}) > \mathrm{card}(L^x_{hr}) + \mathrm{card}(L^{xy}_{hr}) = \sum\limits_{j \in L_{hr}} \mathrm{sgn}(x_j)$. In view of Proposition 2.2.4 it follows that z does not have the pipeline property. ■

It is not difficult to see that:

Corollary 2.3.9 (Lupşa L. *and* Tuns (Bode) O.R. [*75*]). *If set $\Lambda \subseteq S$ is strongly 2-convex with respect to $\mathbb{N}^n$, $h \in H$, x and y are minimum points of f_{i_h} on Λ, but $x, y \notin \tilde{\Lambda}_{i_h}$, and $\lambda \in [0, 1]$ is such that $z := (1-\lambda)x + \lambda y \in \mathbb{N}^n$, then $z \notin \tilde{\Lambda}_{i_h}$.*

Proof. If $\lambda = 0$, respectively $\lambda = 1$, then $z = x$, respectively $z = y$. Therefore, $z \notin \tilde{\Lambda}_{i_h}$. If $\lambda \in]0,\, 1[$, then applying Proposition 2.3.8, we deduce that $z \notin \tilde{\Lambda}_{i_h}$. ■

Let $x, y \in \Lambda$ be two given points. We denote by $[x,\, y]$ the straight-line segment determined by the points x and y and by $]x,\, y[$ the open straight-line segment determined by the points x and y, i.e.

$$[x,\, y] = \{(1-\lambda)x + \lambda y \,|\, \lambda \in [0,\, 1]\},$$

respectively

$$]x,\, y[= \{(1-\lambda)x + \lambda y \,|\, \lambda \in]0,\, 1[\}.$$

Proposition 2.3.10 (Lupşa L. *and* Tuns (Bode) O.R. [75]). *If set $\Lambda \subseteq S$ is strongly 2-convex with respect to $\mathbb{N}^n$, $h \in H$, $x, y \in \Lambda$ and there exists $z \in]x,\, y[\cap \mathbb{N}^n$ such that $z \in \tilde{\Lambda}_{i_h}$, then $]x,\, y[\cap \mathbb{N}^n \subseteq \tilde{\Lambda}_{i_h}$.*

Proof. Let us suppose that there exists $u \in]x,\, y[\cap \mathbb{N}^n$ such that $u \notin \tilde{\Lambda}_{i_h}$. Two cases can occur: $z \in]x,\, u[$ or $z \in]u,\, y[$. Applying Proposition 2.3.8, in both cases we get that $z \notin \tilde{\Lambda}_{i_h}$. This contradicts the hypothesis. Therefore, $]x,\, y[\cap \mathbb{N}^n \subseteq \tilde{\Lambda}_{i_h}$. ■

Proposition 2.3.11 (Lupşa L. *and* Tuns (Bode) O.R. [75]). *If set $\Lambda \subseteq S$ is strongly 2-convex with respect to $\mathbb{N}^n$, $h \in H$, $x, y \in \tilde{\Lambda}_{i_h}$ and $]x,\, y[\cap \mathbb{N}^n \neq \emptyset$, then the set $[x,\, y]$ is strongly 2-convex with respect to $\mathbb{N}^n$ if and only if $L^x_{hk} = \emptyset$ (or equivalent $L^y_{hr} = \emptyset$), for all $k \in K_h$.*

Proof. *Necessity:* Let $\lambda \in]0,\, 1[$ be such that $z := (1-\lambda)x + \lambda y \in \mathbb{N}^n$. From (2.10) it follows that $\sum\limits_{j \in L_{hk}} \operatorname{sgn}((1-\lambda)x_j + \lambda y_j) = \operatorname{card}(L^x_{hk}) + \operatorname{card}(L^y_{hk}) + \operatorname{card}(L^{xy}_{hk})$, $\forall\, k \in K_h$. If, by indirect proof, there exists $r \in K_h$ such that $L^x_{hr} \neq \emptyset$, then Proposition 2.2.6 implies that $\sum\limits_{j \in L_{hr}} \operatorname{sgn}((1-\lambda)x_j + \lambda y_j) > \operatorname{card}(L^y_{hr}) + \operatorname{card}(L^{xy}_{hr}) = \sum\limits_{j \in L_{hr}} \operatorname{sgn}(\lambda y_j) = \sum\limits_{j \in L_{hr}} \operatorname{sgn}(y_j) = \sum\limits_{j \in L_{hr}} \operatorname{sgn}(x_j)$. Therefore, z does not have the pipeline property. This contradicts the hypothesis.

Sufficiency: Let $x, y \in \tilde{\Lambda}_{i_h}$ and $\lambda \in [0,\, 1]$ be such that $z := (1-\lambda)x + \lambda y \in \mathbb{N}^n$. Theorem 2.3.6 implies that z is a minimum point of f_{i_h} on Λ. As $L^x_{hk} = \emptyset$, for each $k \in K_h$, from Proposition 2.2.6 iii), one gets that $L^y_{hk} = \emptyset$, for each $k \in K_h$. Hence, $\sum\limits_{j \in L_{hr}} \operatorname{sgn}(x_j) = \operatorname{card}(L^{xy}_{hr})$ and $\sum\limits_{j \in L_{hr}} \operatorname{sgn}(y_j) = \operatorname{card}(L^{xy}_{hr})$. In view of (2.10) it follows that $\sum\limits_{j \in L_{hk}} \operatorname{sgn}(z_j) = \sum\limits_{j \in L_{hk}} \operatorname{sgn}((1-\lambda)x_j + \lambda y_j) = \operatorname{card}(L^{xy}_{hk})$. As z is a minimum point of f_{i_h} on Λ, applying Proposition 2.2.4, we deduce that z has the pipeline property. As $\lambda \in [0,\, 1]$ was arbitrary chosen, we deduce that any point $z \in [0,\, 1] \cap \mathbb{N}^n$ has the pipeline property.
■

We note that the condition $L^x_{hk} = \emptyset, \forall\, k \in K_h$, equivalent with the condition $L^y_{hr} = \emptyset, \forall\, k \in K_h$, is essential, as can be seen also from the following example.

Example 2.3.12 (Lupşa L. *and* Tuns (Bode) O.R. [75]).

Let $S = \{(2,0), (1,1), (0,2)\}$. *Let* $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ *be the function given by:* $f_1(x_1,x_2) = x_1 + x_2$ *and* $f_2(x_1,x_2) = \max\{2sgn(x_1), 2sgn(x_2)\}$, *for all* $x = (x_1,x_2) \in \mathbb{R}^2$. *One gets that* $\min\{f_1(x)|x \in S\} = 2$, $S_1 = S$, $T = \{z_1 = 2\}$, $L_2 = \{1,2\}$, $\min\{f_2(x)|x \in S_1\} = \min\{\max\{2,0\}, \max\{2,2\}, \max\{0,2\}\} = 2$, $\hat{S}_2 = \{(2,0), (1,1), (0,2)\}$. *As* $sgn(2) + sgn(0) = 1$, $sgn(1) + sgn(1) = 2$, $sgn(0) + sgn(2) = 1$, *it follows that* $\tilde{S} = \{(2,0), (0,2)\}$. *It is easy to see that* $(1,1) \in \hat{S}_2 \cap \mathbb{N}^2$, *but* $(1,1) \notin \tilde{S}$.

2.4 A Method to Determine an Optimal Solution with Pipeline Property for (LDpBP) Problem

In the present paragraph a method to determine an optimal solution with pipeline property for (LDpBP) problem is given. This method is a type of weighted methods. The novelty is that the weighted type introduced by us allows the direct getting of the optimal points with pipeline property.

Let m, n, p be natural non-zero numbers such that $1 \le p \le m$.

Let $I := \{1,\ldots,m\}$, $J := \{1,\ldots,n\}$, $H := \{1,\ldots,p\}$ and let i_h, $h \in H$, be p distinct natural numbers such that $i_1 < \ldots < i_p$.

Let $S \subseteq \mathbb{N}^n$. We consider $f = (f_1,\ldots,f_m) : S \to \mathbb{N}^m$ the vector function with p components of bottleneck type, given by (2.1), where the numbers t_{hj}, $h \in H$, $j \in J$, are integers. We consider the problem:

$$\text{(LDpBP)} \quad \begin{cases} f(x) = (f_1(x),\ldots,f_m(x)) \to \text{lex} - \min, \\ x \in S. \end{cases}$$

In what follows, we give an algorithm that can be used to determine the optimal solutions with pipeline property for the problem (LDpBP). This algorithm is based on the weighted method.

For each $h \in H$ we set $q_h + 1$ numbers, denoted by M_{hk}, $k \in \{q_h, q_{h-1},\ldots,0\}$, such that

$$M_{hq_h} = 1 \tag{2.11}$$

and

$$M_{hk} = 1 + \sum_{j=k+1}^{q_h} M_{hj} \cdot \text{card}(L_{hj}),\ \forall\, k \in \{q_{h-1},...,0\}. \tag{2.12}$$

It is easy to see that

$$M_{h0} \ = \ 1 + \sum_{j=1}^{q_h} M_{hj} \cdot \mathrm{card}(L_{hj}). \tag{2.13}$$

Let

$$M_0 := 1 + \max\{M_{h0} | h \in H\}. \tag{2.14}$$

Obviously, for each $r \in K_h$ we have

$$\sum_{k=r}^{q_h} M_{hk} \cdot \mathrm{card}(L_{hk}) = \begin{cases} M_{h,r-1} - 1 \leq M_{h0} - 1 \leq M_0 - 1 < M_0, \text{ if } r > 1, \\ M_{h0} - 1 \leq M_0 - 1 < M_0, \text{ if } r = 1. \end{cases} \tag{2.15}$$

For each $i \in I$ let us consider a real number $\bar{f}_i$, $\bar{f}_i \geq 2$, such that

$$f_i(x) \leq \bar{f}_i, \; \forall\, x \in S. \tag{2.16}$$

Let

$$\lambda = 1 + \max\{1 + M_0, \{\bar{f}_i | i \in I\}\}. \tag{2.17}$$

It is easy to see that

$$\lambda^{m+1-i} - 2\lambda^{m-i} > 0, \forall\, i \in I, \text{ and } \lambda^{m+2-i_{h_0}} - (M - 0 + 1)\lambda^{m-i_{h_0}} > 0, \forall\, h \in H. \tag{2.18}$$

Remark 2.4.1 (Tuns (Bode) O.R. [?]). *Let $u, v \in S$, $u \neq v$, $i_0 \in I^c$, $h \in H$ and $k \in K_h$. The following inequalities fulfill:*

(i) *if* $f_{i_0}(u) > f_{i_0}(v)$, *then* $f_{i_0}(u) - f_{i_0}(v) \geq 1$;

(ii) $\sum\limits_{i \in I^c, i \geq i_0} \lambda^{m-i}(f_i(u) - f_i(v)) \geq - \sum\limits_{i \in I^c, i \geq i_0} \lambda^{m-i} f_i(v) > \sum\limits_{i \in I^c, i \geq i_0} \lambda^{m+1-i}$;

(iii) *if* $f_{i_h}(u) = z_{hs} \in Z_h$, *then* $\sum\limits_{j \in L_{hs}} sgn(u_j) \geq 1$;

(iv) $\sum\limits_{j \in L_{hk}} (sgn(u_j) - sgn(v_j)) \geq - \sum\limits_{j \in L_{hk}} sgn(v_j) \geq - card(L_{hk})$;

(v) $\sum\limits_{k \in K_h} M_{hk} \sum\limits_{j \in L_{hk}} (sgn(u_j) - sgn(v_j)) \geq - \sum\limits_{k \in K_h} M_{hk} card(L_{hk}) = 1 - M_{h0} > 1 - M_0$.

Now, let us consider the problem:

$$(\mathrm{PUP}) \begin{cases} F(x) = M_0 \sum\limits_{i \in I^c} \lambda^{m-i} f_i(x) + \sum\limits_{h \in H} \lambda^{m+1-i_h} \sum\limits_{k \in K_h} (M_{hk} \sum\limits_{j \in L_{hk}} \mathrm{sgn}(x_j)) \to \min, \\ x \in S. \end{cases}$$

Theorem 2.4.2 (Tuns (Bode) O.R. [?]). *A point $x^0 \in S$ is an optimal solution with pipeline property of the problem (LDpBP) if and only if it is an optimal solution of the problem (PUP).*

Proof. *Necessity:* Let x^0 be an optimal solution with pipeline property for (LDpBP). We prove that this point is a minimum point of F. Let $x \in S$. Since x^0 is an optimal solution with pipeline property for (LDpBP), three cases can occur.

Case 1. There exists $i_0 \in I^c$ such that:
i) $f_l(x^0) = f_l(x)$, for all $l \in I$, $l < i_0$;
ii) $f_{i_0}(x^0) < f_{i_0}(x)$; and
iii) $\sum_{j\in L_{hk}} \mathrm{sgn}(x_j^0) = \sum_{j\in L_{hk}} \mathrm{sgn}(x_j)$, for all $h \in H$ with $i_h < i_0$ and for all $k \in K_h$.

Two cases can occur: $i_0 = m$ or $i_0 < m$.

If $i_0 = m$, then

$$F(x) - F(x^0) = M_0(f_m(x) - f_m(x^0)).$$

Remark 2.4.1 (i), implies that $F(x) - F(x^0) \geq M_0 \cdot 1 = M_0 \geq 1 > 0$.

Let $i_0 < m$. In this case one gets that

$$F(x) - F(x^0) = M_0\lambda^{m-i_0}(f_{i_0}(x) - f_{i_0}(x^0)) + M_0\Big(\sum_{i\in I^c, i>i_0} \lambda^{m-i}\Big(f_i(x) - f_i(x^0)\Big)\Big)$$

$$+ \sum_{h\in H, i_h>i_0} \lambda^{m-i_h+1} \sum_{k\in K_h} \Big(M_{hk} \sum_{j\in L_{hk}} \Big(\mathrm{sgn}(x_j) - \mathrm{sgn}(x_j^0)\Big)\Big).$$

In view of Remark 2.4.1 (i), (ii) and (v), successive we obtain that

$$\begin{aligned} F(x) - F(x^0) &> M_0\lambda^{m-i_0} - M_0\Big(\sum_{i\in I^c, i>i_0} \lambda^{m-i} f_i(x^0)\Big) - \sum_{h\in H, i_h>i_0} \lambda^{m-i_h+1}(M_0 - 1) \\ &\geq M_0\lambda^{m-i_0} - M_0 \sum_{i\in I, i>i_0} \lambda^{m-i+1} + \sum_{h\in H, i_h>i_0} \lambda^{m-i_h+1} \\ &\geq M_0\frac{\lambda^{m+1-i_0} - 2\lambda^{m-i_0} + \lambda}{\lambda - 1} + \sum_{h\in H, i_h>i_0} \lambda^{m-i_h+1} > 0. \end{aligned}$$

Case 2. There exist $h_0 \in H$ and $s \in K_{h_0}$ such that:
i) $f_l(x^0) = f_l(x)$, $\forall\, l \in I^c$, $l < i_{h_0}$;
ii) $\sum_{j\in L_{hk}} \mathrm{sgn}(x_j^0) = \sum_{j\in L_{hk}} \mathrm{sgn}(x_j)$, for each $h \in H$, with $h < h_0$, and for each $k \in K_h$;
iii) $\sum_{j\in L_{h_0 s}} \mathrm{sgn}(x_j^0) < \sum_{j\in L_{h_0 s}} \mathrm{sgn}(x_j)$, and, if $s > 1$, then
iv) $\sum_{j\in L_{h_0 k}} \mathrm{sgn}(x_j^0) = \sum_{j\in L_{h_0 k}} \mathrm{sgn}(x_j)$, $\forall\, k \in K_{h_0}$, $k < s$.

Again, two cases can occur: $i_{h_0} = m$ or $i_{h_0} < m$.

If $i_{h_0} = m$, then

$$F(x) - F(x^0) = \lambda\Big(M_{h_0 s} \sum_{j\in L_{h_0 s}} \mathrm{sgn}(x_j) + \sum_{k\in K_{i_{h_0}}, k>s} M_{h_0 k} \sum_{j\in L_{h_0 k}} \Big(\mathrm{sgn}(x_j) - \mathrm{sgn}(x_j^0)\Big)\Big).$$

In view of Remark 2.4.1 (iv) and of (2.12), it follows that

$$F(x) - F(x^0) \geq \lambda\Big(M_{h_0 s} - \sum_{k \in K_{i_{h_0}}, k>s} M_{h_0 k}\mathrm{card}(L_{h_0 k})\Big) = \lambda(M_{h_0 s} - (M_{h_0,s} - 1)) = \lambda > 0.$$

If $i_{h_0} < m$, then

$$F(x) - F(x^0) = M_0\Big(\sum_{i \in I^c, i > i_{h_0}} \lambda^{m-i}\Big(f_i(x) - f_i(x^0)\Big)\Big) + \lambda^{m+1-i_{h_0}}\Big(M_{h_0 s}\sum_{j \in L_{h_0 s}} \mathrm{sgn}(x_j)\Big)$$
$$+ \sum_{h \in H} \lambda^{m+1-i_h}\Big(\sum_{k \in K_h} M_{hk}\Big(\sum_{j \in L_{hk}} (\mathrm{sgn}(x_j) - \mathrm{sgn}(x_j^0))\Big)\Big).$$

Applying Remark 2.4.1 (ii), (iii) and (v), one gets that

$$\begin{aligned}
F(x) - F(x^0) \;&\geq\; -M_0\Big(\sum_{i \in I^c, i > i_{h_0}} \lambda^{m-i} f_i(x^0)\Big) + \lambda^{m+1-i_{h_0}} M_{h_0 s}\Big(\sum_{j \in K_{h_0} s} \mathrm{sgn}(x_j)\Big) \\
&\quad -\lambda^{m+1-i_{h_0}} \sum_{k \in K_{h_0}, k>s} M_{h_0 k}\Big(\sum_{j \in L_{h_0 k}} \mathrm{sgn}(x_j^0)\Big) \\
&\quad - \sum_{h \in H, h > h_0} \lambda^{m-i_h+1}\Big(\sum_{k \in K_h} M_{hk}\sum_{j \in L_{hk}} \mathrm{sgn}(x_j^0)\Big) \\
&> -M_0 \sum_{i \in I^c, i > i_{h_0}} \lambda^{m+1-i} + \lambda^{m+1-i_{h_0}} M_{h_0 s} \\
&\quad -\lambda^{m+1-i_{h_0}} \sum_{k \in K_{h_0}, k>s} M_{h_0 k}\mathrm{card}(L_{h_0 k}) \\
&\quad - \sum_{h \in H, h > h_0} \lambda^{m-i_h+1}\Big(\sum_{k \in K_h} M_{hk}\mathrm{card}(L_{hk})\Big) \\
&= -M_0 \sum_{i \in I^c, i > i_{h_0}} \lambda^{m+1-i} + \lambda^{m+1-i_{h_0}}\Big(M_{h_0 s} - \sum_{k \in K_{h_0}, k>s} M_{h_0 k}\mathrm{card}(L_{h_0 k})\Big) \\
&\quad - \sum_{h \in H, h > h_0} \lambda^{m-i_h+1}\Big(\sum_{k \in K_h} M_{hk}\mathrm{card}(L_{hk})\Big) \\
&\geq -M_0 \sum_{i \in I^c, i > i_{h_0}} \lambda^{m+1-i} + \lambda^{m+1-i_{h_0}} - M_0 \sum_{h \in H, h > h_0} \lambda^{m-i_h+1} \\
&= \lambda^{m+1-i_{h_0}} - M_0 \sum_{i \in I, i > i_{h_0}} \lambda^{m-i_h+1} \\
&= \lambda^{m+1-i_{h_0}} - M_0 \frac{\lambda^{m+1-i_{h_0}} - \lambda}{\lambda - 1} \\
&= \frac{\lambda^{m+2-i_{h_0}} - \lambda^{m+1-i_{h_0}} - M_0\lambda^{m+1-i_{h_0}} + M_0\lambda}{\lambda - 1} \\
&= \frac{\lambda^{m+1-i_{h_0}}(\lambda - 1 - M_0) + M_0\lambda}{\lambda - 1} \;>\; 0.
\end{aligned}$$

Case 3. We have that

i) $f_i(x^0) = f_i(x)$, for all $i \in I^c$;

ii) $\sum_{j \in L_{hk}} \mathrm{sgn}(x_j^0) = \sum_{j \in L_{hk}} \mathrm{sgn}(x_j)$, for all $h \in H$ and for all $k \in K_h$.

It follows that $F(x) - F(x^0) = 0$.

Therefore, in each of the above three cases, we get that

$$F(x) - F(x^0) \geq 0.$$

As $x \in S$ was arbitrary chosen, we can conclude that x^0 is a minimum point of F on S.

Sufficiency: Let x^0 be a minimum point of F on S. If x^0 is not a minimum point with pipeline property of f on S, then, based on Definitions 1.2.2, 2.2.1 and 2.2.11, there exists $l \in I$ such that x^0 is not a minimum point of f_l on S_{l-1} if $l \in I^c$, or x^0 is not a minimum point with pipeline property of f_l on S_{l-1} if $l \in I \setminus I^c$.

Let i be the smallest value of $l \in I$ wherefore the optimality condition does not take place.

Two cases can occur:

Case 1: $i \in I^c$. Then, for each $l \in I$, $l < i$, we have $f_l(x^0) = f_l(x)$, for all $x \in S_l$, and there exists $x \in S_{i-1}$ such that $f_i(x^0) > f_i(x)$.

Analogously as in *Case 1* from *Necessity*, it follows that $F(x^0) - F(x) > 0$.

This contradicts that x^0 is a minimum point of F on S.

Case 2: $i \in I \setminus I^c$. Then, there exists an unique $h \in H$ such that $i = i_h$. We note that since x^0 is not a minimum point with pipeline property of f_{i_h} on S_{i_h-1}, based on Proposition 2.2.5, there exist $x \in S_{i_h-1}$ and $k_0 \in K_{i_h}$ such that

$$\sum_{j \in L_{hk_0}} \operatorname{sgn}(x_j^0) > \sum_{j \in L_{hk_0}} \operatorname{sgn}(x_j)$$

and, for each $k \in K_h$ with $k < k_0$, we have that

$$\sum_{j \in L_{hk}} \operatorname{sgn}(x_j^0) = \sum_{j \in L_{hk_0}} \operatorname{sgn}(x_j).$$

Analogously as in *Case 2* from *Necessity*, it follows that $F(x^0) - F(x) > 0$.

This contradicts that x^0 is a minimum point of F on S. ■

Based on Theorem 2.4.2, the solving of the problem (LDpBP) can be replaced by solving the problem (PUP). Therefore, the method is based on setting the numbers M_{hk}, $k \in \{q_h, q_{h-1}, \ldots, 0\}$, given by (2.11) and (2.12), and considering values for M_0 given by (2.14), respectively for λ given by (2.17), with which we consider problem (PUP).

Chapter 3

An Application Related to Firm's Costs Management

In the present chapter we present an application generated by a concrete costs management problem. This problem points out a relation that can exists, in some cases, between the bilevel optimization and the lexicographic optimization.

We note that the problem studied in the present chapter can be seen, on one hand, as a traveling salesman problem (TSP). It is well known that the TSP is one of the oldest and most studied combinatorial problem. From the mathematical point of view, researches concerning the TSP have an important role in development of the graph theory. Applegate D.L., Bixby R.E., Chvátal V. and Cook W.J. [4] present various aspects of the TSP, especially the ones related to the methods and algorithms of solving it. Gutin G. and Punnen A. [44], as well as Toth P. and Vigo D. [117], provide some of the problems known under the generic name of *The Vehicle Routing Problem*, which represent a generalization of the TSP.

On the other hand, the studied problem represents a problem of generating new types of routes. We study this problem by using the bilevel optimization as a mathematical tool.

The present chapter is organized as follows: in Section 3.1 we formulate the problem, named by us *The Milk Collection Problem*. We note that this problem is based on a concrete practical problem. In Section 3.2 we give a generalization of this problem, pointing out a method for solving it.

The results within this chapter belong to the author and can be found in the paper authored by Goina D. and Tuns (Bode) O.R. [42].

3.1 The Milk Collection Problem

A dairy products manufacturing company collects twice a day the milk from a certain area. Collection points are located only on roads linking villages in the area. The milk is brought to the collection points by the owners. The quantity of milk delivered depends on the time when the collection is scheduled. Some providers can bring the milk to the collection points only in the morning. Others only in the evening, and some of them both in the morning and in the evening. There exists a possibility for some providers who deliver milk in the morning to store it (in conditions that do not impair the milk quality) and to offer it for delivery only in the evening. The others do not have this possibility. The providers impose that either the entire quantity of milk offered is collected by the dairy products manufacturing company or nothing. The milk is collected by the dairy products manufacturing company in the morning and in the evening using a collector tank, which has a capacity denoted by $\overline{Q}$.

The problem that arises is that of planning the providers:
- those who bring milk to the collection points in the morning, and the milk is collected by the collector tank in the morning;
- those who bring milk to the collection points in the morning, but it is necessary to store it until evening, when it will be collected by the collector tank;
- those who bring milk to the collection points in the evening, and the milk is collected by the collector tank in the evening,

such that the total cost required for milk collection in a day to be minimum and a collection point to be visited by the collector tank at most once in the morning and at most once in the evening.

The providers planning must satisfy the following requirements:
a) the quantity of milk collected in the morning not exceed the capacity $\overline{Q}$ of the collector tank;
b) the quantity of milk collected in the evening (which may be from the evening milk or from the stored one) not exceed the capacity $\overline{Q}$ of the collector tank;
c) the quantity of milk collected in the morning and the quantity collected in the evening must be greater than a specified quantity, denoted by $\underline{Q}$, in order to ensure the continuity in the production process;
d) the quantity of stored milk to be minimum and in the same time fulfilling the conditions a)-c).

The Mathematical Model of the Milk Collection Problem

Let n be the number of the providers and $N = \{1, ..., n\}$.

Let us denote by L_i, $i \in \{0, 1, ..., n+1\}$, the collection point where the provider i brings the milk. Let L_0 be the location where the collector tank starts and L_{n+1} be the location where the collector tank must return. We agree that they coincide, i.e. $L_0 = L_{n+1}$. The collector tank transportation cost between each two locations $i, j \in \{0, 1, \ldots, n+1\}$ is known. Let us denote this cost by c_{ij}. By q_i^1, respectively q_i^2, we denote the quantity of milk that can be delivered by the provider $i \in N$ in the morning, respectively in the evening.

Let I_1 be the set of indices corresponding to the collection points where providers can deliver milk only in the morning, but can not store it. Let I_2 be the set of indices corresponding to the collection points where providers can deliver milk both in the morning and in the evening, but in case they deliver milk in the morning do not accept to store it. Let I_3 be the set of indices corresponding to the collection points where providers can deliver milk both in the morning and in the evening, and accept to store the morning milk in case it is required. Let I_4 be the set of indices corresponding to the collection points where providers can deliver milk only in the evening.

It is obvious that

$$I_1 \cap I_2 = \emptyset,\ I_1 \cap I_3 = \emptyset,\ I_1 \cap I_4 = \emptyset,\ I_2 \cap I_3 = \emptyset,\ I_2 \cap I_4 = \emptyset,\ I_3 \cap I_4 = \emptyset,$$

$$I_1 \cup I_2 \cup I_3 \cup I_4 = N.$$

Now, let us consider the complete undirected graph $G = (\tilde{N}; E)$, where

$$\tilde{N} = N \cup \{0\} \cup \{n+1\}\ =\ \{0, 1, ..., n, n+1\} \text{ and } E = \{\{i, j\} \,|\, i \in \tilde{N},\ j \in \tilde{N},\ j \neq i\}.$$

The graph vertices correspond to the locations L_i, $i \in \{0, 1, \ldots, n+1\}$.

Let us denote by Λ the set of subgraphs $\Gamma = (N_\Gamma, E_\Gamma)$ of $G = (\tilde{N}; E)$. We weight the graph G using the cost matrix $C = [c_{ij}]_{i,j \in \tilde{N}}$, where c_{ij}, with $i \neq j$, it is the minimum transport cost of the collector tank from the location i to location j, and $c_{ii} = +\infty$, for each $i \in \tilde{N}$. As well, we attach to each node $i \in N$ two positive weights, q_i^1 and q_i^2.

In order to elaborate the mathematical model for this problem we consider two subgraphs $G_1 = (N_1, E_1)$ and $G_2 = (N_2, E_2)$ of G as variables. The set of nodes of the first subgraph, N_1, corresponds to the indices of the collection points where the milk is collected in the morning. The set of nodes of the second subgraph, N_2, corresponds to the indices of the collection points where the milk is collected in the evening.

The main objective is to determine the sets N_1 and N_2 such that the total transport cost in one day to be minimum and the problem restrictions to be fulfilled.

Let us note that for a fixed set N_1 it is obtained a minimum cost of the morning collection if it is followed a Hamiltonian circuit of minimum value in G_1. Analogous, for a fixed set N_2 it is obtained a minimum cost of the evening collection (stored milk or delivered to the collection points only in the evening) if it is followed a Hamiltonian circuit of minimum value in G_2. Therefore, if for a subgraph Γ of G we denote by $h(\Gamma)$ the value of a Hamiltonian circuit of minimum value corresponding to the subgraph Γ, then the minimum cost of milk collection in one day it is equal to $h(G_1) + h(G_2)$.

The graphs G_1 and G_2 can not be chosen randomly; they must fulfill the problem restrictions. Thus, in the morning the collector tank can collect only from the nodes in which the providers deliver milk only in the morning. Therefore, $N_1 \subseteq I_1 \cup I_2 \cup I_3$. In the evening the collector tank can collect milk only from the nodes in which the providers deliver milk only in the evening or in which the morning milk was stored until evening. Therefore, $N_2 \subseteq I_2 \cup I_3 \cup I_4$.

In each collection point where the providers can deliver milk both in the morning and in the evening and where there exists the possibility to store the morning milk until evening, the quantity of milk delivered in the morning is collected just once: in the morning or in the evening. Therefore, the following condition occurs: $N_1 \cap N_2 \cap I_3 = \emptyset$.

The quantity of milk collected in the morning, equal to $\sum_{i \in N_1} q_i^1$, must be greater than or equal to $\underline{Q}$, and can not exceed the collector tank capacity $\overline{Q}$. Also, the quantity of milk collected in the evening, equal to $\sum_{i \in N_2 \cap I_3} q_i^1 + \sum_{i \in N_2} q_i^2$, can not exceed the collector tank capacity $\overline{Q}$ and must be greater than or equal to $\underline{Q}$.

Let S be the set of all pairs (G_1, G_2) of subgraphs of the weighted graph G, $G_1 = (N_1, E_1)$ and $G_2 = (N_2, E_2)$, satisfying the conditions (3.1)-(3.5):

$$N_1 \subseteq I_1 \cup I_2 \cup I_3, \tag{3.1}$$

$$N_2 \subseteq I_2 \cup I_3 \cup I_4, \tag{3.2}$$

$$N_1 \cap N_2 \cap I_3 = \emptyset, \tag{3.3}$$

$$\underline{Q} \leq \sum_{i \in N_1} q_i^1 \leq \overline{Q}, \tag{3.4}$$

$$\underline{Q} \leq \sum_{i \in N_2 \cap I_3} q_i^1 + \sum_{i \in N_2} q_i^2 \leq \overline{Q}. \tag{3.5}$$

In order to plan the evening collection it is necessary that the planning of the morning collection and of the stored milk to be done with respect to the capacity type restrictions (3.4) and (3.5). For a given N_2, the quantity of stored milk is equal to $\sum_{i \in N_2 \cap I_3} q_i^1$.

Let g be the real function defined on the set Λ of subgraphs of G, given by

$$g(\Gamma) = \sum_{i \in N_\Gamma \cap I_3} q_i^1, \ \forall\, \Gamma = (N_\Gamma, E_\Gamma) \in \Lambda. \tag{3.6}$$

For each $G_1 \in \Lambda$ let us denote by

$$S(G_1) = \{G_2 \in \Lambda \,|\, (G_1, G_2) \in S\}.$$

If $S(G_1) \neq \emptyset$, then the minimum quantity of stored milk (which can be determined taking into account the morning planning, i.e. knowing N_1) is obtained solving the following problem:

$$\begin{cases} g(G_2) \;\rightarrow\; \min, \\ G_2 \in S(G_1). \end{cases}$$

Let us denote by $S^*(G_1)$ the set of optimal solutions of this problem, i.e.

$$S^*(G_1) \;=\; \operatorname{argmin}\Big\{g(G_2) \,|\, G_2 \in S(G_1)\Big\}.$$

Under these circumstances, the milk collection problem is reduced to solving the following bilevel optimization problem:

$$(BP) \qquad \begin{cases} h(G_1) \,+\, h(G_2) \;\rightarrow\; \min, \\ \text{such that} \\ (G_1, G_2) \in S, \\ G_2 \in S^*(G_1). \end{cases}$$

In what follows, we give a method for solving the problem (BP), in a little more general context.

3.2 Generalization of the Mathematical Model for the Milk Collection Problem

Let N be a finite set, $G = (N, E)$ be a weighted graph and let $I \subset N$.

Let Λ be the set of subgraphs $\Gamma = (N_\Gamma, E_\Gamma)$ of G with $N_\Gamma \neq \emptyset$ and $E_\Gamma \neq \emptyset$.

Let $\mathcal{C}_1$ be the set of those elements $G_1 = (N_1, E_1)$ of Λ which fulfill some given conditions. Also, let $\mathcal{C}_2$ be the set of those elements $G_2 = (N_2, E_2)$ of Λ which fulfill other given conditions. In both cases, the conditions are some restrictions imposed to be fulfilled by the set of nodes N_1, respectively N_2. It can be defined, for instance, by inequalities or equalities generated by some given functions or by some inclusions. For example, recalling the milk collection problem, $\mathcal{C}_1$ it is the set of those subgraphs which verify the conditions (3.1) and (3.4), while $\mathcal{C}_2$ it is the set of those subgraphs which verify the conditions (3.2) and (3.5).

Furthermore, for a subgraph $\Gamma \in \Lambda$, we denote by $h(\Gamma)$ the value of a Hamiltonian circuit of minimum value corresponding to it.

Now, let a and b be positive numbers and let $F : \Lambda \times \Lambda \to \mathbb{R}_+$ be the function given by

$$F(G_1, G_2) \;=\; a \cdot h(G_1) + b \cdot h(G_2),\; \forall\; (G_1, G_2) \in \Lambda \times \Lambda. \tag{3.7}$$

Also, let $g : \Lambda \to \mathbb{N}$ be a given function. For the milk collection problem, the function g returns the quantity of stored morning milk which it is collected in the evening.

The bilevel problem proposed to be solved is:

$$(PBG) \quad \begin{cases} F(G_1, G_2) \;\to\; \min, \\ G_1 \in \mathcal{C}_1, \\ G_2 \in S^*(G_1), \end{cases}$$

where $S^*(G_1)$ it is the set of optimal solutions of the problem

$$(P(G_1)) \quad \begin{cases} g(G_2) \;\to\; \min, \\ G_2 \in \mathcal{C}_2, \\ N_1 \cap N_2 \cap I = \emptyset. \end{cases}$$

Let us denote by

$$S = \Big\{(G_1, G_2) \in \Lambda \times \Lambda \,|\, G_1 \in \mathcal{C}_1,\; G_2 \in \mathcal{C}_2,\; N_1 \cap N_2 \cap I = \emptyset\Big\}$$

and

$$S_1 = \Big\{G_1 \in \Lambda \mid \exists\, G_2 \in \Lambda \text{ s. t. } (G_1, G_2) \in S\Big\},$$

i.e.

$$S_1 = \Big\{G_1 \in \mathcal{C}_1 \mid \exists\, G_2 \in \mathcal{C}_2 \text{ s. t. } N_1 \cap N_2 \cap I = \emptyset\Big\}.$$

For each $G_1 \in S_1$ we consider the set

$$S(G_1) \;=\; \Big\{G_2 \in \mathcal{C}_2 \mid (G_1, G_2) \in S\Big\} \;=\; \Big\{G_2 \in \mathcal{C}_2 \mid N_1 \cap N_2 \cap I = \emptyset\Big\}.$$

It is easy to see that $S(G_1)$ it is the set of feasible solutions of the problem $(P(G_1))$.

Let $H \in 2^I$. We consider the problems:

$$(P_1(H)) \quad \begin{cases} h(G_1) \;\to\; \min, \\ G_1 \in \mathcal{C}_1, \\ N_1 \cap I = H \end{cases}$$

and

$$(P_2(H)) \quad \begin{cases} \begin{pmatrix} g(G_2) \\ h(G_2) \end{pmatrix} \;\to\; \text{lex} - \min, \\ G_2 \in \mathcal{C}_2, \\ N_2 \cap H = \emptyset. \end{cases}$$

Let us denote by h_1^H the optimal value of the problem $(P_1(H))$ and by (g_2^H, h_2^H) the optimal value of the problem $(P_2(H))$.

Also, we define the function $\tilde{F} : 2^I \to \mathrm{I\!R}$,

$$\tilde{F}(H) = a \cdot h_1^H + b \cdot h_2^H, \ \forall\ H \in 2^I, \tag{3.8}$$

and we consider the problem

$$(PP) \quad \begin{cases} \tilde{F}(H) \ \to \ \min, \\ H \in 2^I. \end{cases}$$

Furthermore, we establish relations between the feasible solutions, respectively between the optimal solutions, of the problems (PBG) and (PP).

Lemma 3.2.1 (Goina D. *and* Tuns (Bode) O.R. [*42*]). *If $G_1^0 \in \mathcal{C}_1$, G_2^0 it is a feasible solution of the problem $(P(G_1^0))$ and $H^0 = N_1^0 \cap I$, then G_2^0 it is a feasible solution of the problem $(P_2(H^0))$.*

Proof. As $G_2^0 \in S(G_1^0)$ we deduce that $G_2^0 \in \mathcal{C}_2$ and $N_1^0 \cap N_2^0 \cap I = \emptyset$. From the last equality, considering that $H^0 = N_1^0 \cap I$, we get that $N_2^0 \cap H^0 = N_2^0 \cap N_1^0 \cap I = \emptyset$. Hence, G_2^0 it is a feasible solution of the problem $(P_2(H^0))$. ■

Lemma 3.2.2 (Goina D. *and* Tuns (Bode) O.R. [*42*]). *If (G_1^0, G_2^0) it is a feasible solution of the problem (PBG) and $H^0 = N_1^0 \cap I$, then G_1^0 it is a feasible solution of the problem $(P_1(H^0))$ and G_2^0 it is a feasible solution of the problem $(P_2(H^0))$.*

Proof. Since (G_1^0, G_2^0) it is a feasible solution of the problem (PBG) and $H^0 = N_1^0 \cap I$, immediately results that G_1^0 it is a feasible solution of the problem $(P_1(H^0))$ and G_2^0 it is an optimal solution of the problem $(P(G_1^0))$. Because any optimal solution it is a feasible one, applying Lemma 3.2.1, we deduce that G_2^0 it is a feasible solution of the problem $(P_2(H^0))$. ■

Lemma 3.2.3 (Goina D. *and* Tuns (Bode) O.R. [*42*]). *If $H^0 \in 2^I$, G_1^0 it is a feasible solution of the problem $(P_1(H^0))$ and G_2^0 it is an optimal solution of the problem $(P_2(H^0))$, then (G_1^0, G_2^0) it is a feasible solution of the problem (PBG).*

Proof. Since G_1^0 it is a feasible solution of the problem $(P_1(H^0))$, we have

$$G_1^0 \in \mathcal{C}_1, \tag{3.9}$$

$$N_1^0 \cap I = H^0. \tag{3.10}$$

Because any optimal solution it is a feasible one, we have

$$G_2^0 \in \mathcal{C}_2, \tag{3.11}$$

$$N_2^0 \cap H^0 = \emptyset. \tag{3.12}$$

From (3.10) and (3.12) it follows that

$$N_1^0 \cap N_2^0 \cap I \ = \ N_2^0 \cap H^0 = \emptyset. \tag{3.13}$$

Based on (3.11) and (3.13) we deduce that G_2^0 it is a feasible solution of the problem $(P(G_1^0))$, i.e. $G_2^0 \in S(G_1^0)$. We must prove that G_2^0 it is an optimal solution of this problem. Let us suppose the opposite. Then, there exists $G_2 \in S(G_1^0)$ such that $g(G_2) < g(G_2^0)$, which implies

$$\begin{pmatrix} g(G_2) \\ h(G_2) \end{pmatrix} <_{\text{lex}} \begin{pmatrix} g(G_2^0) \\ h(G_2^0) \end{pmatrix}. \tag{3.14}$$

Recalling Lemma 3.2.1, since $G_2 \in S(G_1^0)$, we deduce that G_2 it is a feasible solution of the problem $(P_2(H^0))$; the inequality (3.14) contradicts the fact that G_2^0 it is an optimal solution of the problem $(P_2(H^0))$. Therefore, $G_2^0 \in S^*(G_1^0)$. Considering now the fact that $G_1^0 \in \mathcal{C}_1$, it results that (G_1^0, G_2^0) it is a feasible solution of the problem (PBG). ■

Theorem 3.2.4 (Goina D. *and* Tuns (Bode) O.R. [42]). *If (G_1^0, G_2^0) it is an optimal solution of the problem (PBG), then taking $H^0 = N_1^0 \cap I$ the following sentences are true:*
i) G_1^0 it is an optimal solution of the problem $(P_1(H^0))$;
ii) G_2^0 it is an optimal solution of the problem $(P_2(H^0))$;
iii) H^0 it is an optimal solution of the problem (PP).

Proof. i) Lemma 3.2.2 implies that G_1^0 it is a feasible solution of the problem $(P_1(H^0))$. Let us suppose that G_1^0 it is not an optimal solution of the problem $(P_1(H^0))$. Then, there exists a feasible solution G_1 of the problem $(P_1(H^0))$ such that

$$h(G_1) < h(G_1^0). \tag{3.15}$$

As G_1 it is a feasible solution of the problem $(P_1(H^0))$, we have $G_1 \in \mathcal{C}_1$ and $N_1 \cap I = H^0$. It follows that $N_1 \cap N_2^0 \cap I = H^0 \cap N_2^0 = N_1^0 \cap I \cap N_2^0 \ = \ \emptyset$. Therefore, $G_2^0 \in S(G_1)$. Two cases can occur:
1) $G_2^0 \in S^*(G_1)$; or 2) $G_2^0 \notin S^*(G_1)$.

If $G_2^0 \in S^*(G_1)$, then (G_1, G_2^0) it is a feasible solution of the problem (PBG). Since $a > 0$, from (3.15) we deduce that $ah(G_1)+bh(G_2^0) \ < \ ah(G_1^0)+bh(G_2^0)$, which contradicts the optimality of (G_1^0, G_2^0).

Now, let us suppose that $G_2^0 \notin S^*(G_1)$. Then, there exists $G_2 \in S(G_1)$ such that

$$g(G_2) < g(G_2^0). \tag{3.16}$$

Because $G_2 \in S(G_1)$, we have $G_2 \in \mathcal{C}_2$ and $N_2 \cap N_1 \cap I = \emptyset$. On the other hand, G_1 being a feasible solution of $(P_1(H^0))$, we have $N_1 \cap I = H^0$. It results that

$$N_1^0 \cap N_2 \cap I \;=\; H^0 \cap N_2 \;=\; N_1 \cap I \cap N_2 \;=\; \emptyset.$$

So, $G_2 \in S(G_1^0)$. Therefore, (3.16) contradicts the fact that $G_2^0 \in S^*(G_1^0)$.

Hence, G_1^0 it is an optimal solution of the problem $(P_1(H^0))$.

ii) From Lemma 3.2.2 ii), it results that G_2^0 it is a feasible solution of $(P_2(H^0))$. Let us suppose that G_2^0 it is not an optimal solution of $(P_2(H^0))$. Since the set of solutions of the problem $(P_2(H^0))$ is a finite and nonempty set, the problem will have optimal solutions. Let $\tilde{G}_2 = (\tilde{N}_2, \tilde{E}_2)$ be an optimal solution of this problem. Based on Lemma 3.2.3, $(G_1^0, \tilde{G}_2)$ it is a feasible solution of the problem (PBG). As we supposed the contrary, we have

$$\begin{pmatrix} g(\tilde{G}_2) \\ h(\tilde{G}_2) \end{pmatrix} <_{\text{lex}} \begin{pmatrix} g(G_2^0) \\ h(G_2^0) \end{pmatrix}. \tag{3.17}$$

Two cases can occur:

1) $g(\tilde{G}_2) < g(G_2^0)$; or 2) $g(\tilde{G}_2) = g(G_2^0)$ and $h(\tilde{G}_2) < h(G_2^0)$.

In the first case, we deduce that $G_2^0 \notin S^*(G_1^0)$, which contradicts the fact that (G_1^0, G_2^0) it is a feasible solution of (PBG).

In the second case, we have

$$F(G_1^0, \tilde{G}_2) = ah(G_1^0) + bh(\tilde{G}_2) \;<\; ah(G_1^0) + bh(G_2^0) = F(G_1^0, G_2^0),$$

which contradicts the optimality of (G_1^0, G_2^0).

iii) Since the set 2^I is a nonempty and finite set, the problem (PP) has an optimal solution. Let us suppose that H^0 it is not an optimal solution of the problem (PP), and let H^* be the optimal solution of the problem (PP). Under these circumstances, we have

$$\tilde{F}(H^*) < \tilde{F}(H^0). \tag{3.18}$$

Let us notice that, from i) and ii), taking into account the way in which the functions F and $\tilde{F}$ are defined (see (3.7) and (3.8)), one gets that

$$\tilde{F}(H^0) \;=\; ah_1^{H^0} + bh_2^{H^0} \;=\; ah(G_1^0) + bh(G_2^0) = F(G_1^0, G_2^0). \tag{3.19}$$

Now, let G_1^* be an optimal solution of the problem $(P_1(H^*))$ and G_2^* be an optimal solution of the problem $(P_2(H^*))$. Then,

$$\tilde{F}(H^*) \;=\; ah_1^{H^*} + bh_2^{H^*} \;=\; ah(G_1^*) + bh(G_2^*) \;=\; F(G_1^*, G_2^*). \tag{3.20}$$

From (3.18)-(3.20) it results that

$$F(G_1^*, G_2^*) \; < \; F(G_1^0, G_2^0). \tag{3.21}$$

On the other hand, applying Lemma 3.2.3, we deduce that (G_1^*, G_2^*) it is a feasible solution of the problem (PBG). Hence, (3.21) contradicts the optimality of (G_1^0, G_2^0). ■

Theorem 3.2.5 (Goina D. *and* Tuns (Bode) O.R. [*42*]). *If H^0 it is an optimal solution of the problem (PP) and G_1^0, respectively G_2^0, it is an optimal solution of the problem $(P_1(H^0))$, respectively $(P_2(H^0))$, then (G_1^0, G_2^0) it is an optimal solution of the problem (PBG).*

Proof. As G_1^0, respectively G_2^0, it is an optimal solution of $(P_1(H^0))$, respectively $(P_2(H^0))$, we get that $h_1^{H^0} = h(G_1^0)$ and $h_2^{H^0} = h(G_2^0)$. Therefore,

$$\tilde{F}(H^0) = ah(G_1^0) + bh(G_2^0) = F(G_1^0, G_2^0). \tag{3.22}$$

On the other hand, applying Lemma 3.2.3, we get that (G_1^0, G_2^0) it is a feasible solution of (PBG). If (G_1^0, G_2^0) it is not an optimal solution of the problem (PBG), then there exists $(\tilde{G}_1 = (\tilde{N}_1, \tilde{E}_1), \tilde{G}_2 = (\tilde{N}_2, \tilde{E}_2))$ feasible solution of (PBG), such that

$$ah(\tilde{G}_1) + bh(\tilde{G}_2) \; = \; F(\tilde{G}_1, \tilde{G}_2) < F(G_1^0, G_2^0) \; = \; ah(G_1^0) + bh(G_2^0). \tag{3.23}$$

From (3.23) we deduce that $h(\tilde{G}_1) < h(G_1^0)$ or $h(\tilde{G}_2) < h(G_2^0)$.

Let $\tilde{H} = \tilde{N}_1 \cap I$. Since $(\tilde{G}_1, \tilde{G}_2)$ it is a feasible solution of (PBG) it follows that $\tilde{G}_2 \in S^*(\tilde{G}_1)$. It implies that $h_2^{\tilde{H}} = F(\tilde{G}_1, \tilde{G}_2)$. Hence, (3.23) implies $\tilde{F}(\tilde{H}) < \tilde{F}(H^0)$. This contradicts the optimality of H^0. ■

Let

$$\lambda \; \geq \; 1 + \max\{F(G_1, G_2), \; \forall \; (G_1, G_2) \in \Lambda\}. \tag{3.24}$$

Let $G_1 \in S_1$ and $H \in 2^I$, fulfilling the following condition:

$$N_1 \cap I = H. \tag{3.25}$$

Let us consider the problem:

$$(PL_2(H)) \quad \begin{cases} \lambda \cdot g(G_2) + F(G_1, G_2) \; \rightarrow \; \min, \\ G_2 \in \mathcal{C}_2, \\ H \cap N_2 = \emptyset. \end{cases}$$

Theorem 3.2.6 (Goina D. *and* Tuns (Bode) O.R. [*42*]). *If $G_1 \in S_1$ and $H \in 2^I$ such that the condition (3.25) is fulfilled, then an element G_2 it is an optimal solution of the problem $(PL_2(H))$ if and only if it is an optimal solution of the problem $(P_2(H))$.*

Proof. First, let us remark that both problems have the same set of feasible solutions.

Necessity: Let G_2 be an optimal solution of the problem $(PL_2(H))$. Let us suppose that G_2 it is not an optimal solution of the problem $(P_2(H))$. Two cases can occur:
1) there exists G_2^* a feasible solution of the problem $(P_2(H))$, such that $g(G_2^*) < g(G_2)$; or
2) there exists G_2^* a feasible solution of the problem $(P_2(H))$, such that $g(G_2) = g(G_2^*)$ and $F(G_1, G_2^*) < F(G_1, G_2)$.

As $\lambda > 0$, in the first case we obtain that

$$\lambda \cdot g(G_2^*) \; < \; \lambda \cdot g(G_2). \tag{3.26}$$

If $F(G_1, G_2^*) \leq F(G_1, G_2)$, then $\lambda \cdot g(G_2^*) + F(G_1, G_2^*) < \lambda \cdot g(G_2) + F(G_1, G_2)$. This contradicts the hypothesis that (G_1, G_2) it is an optimal solution of the problem $(PL_2(H))$.

Let us now suppose that $F(G_1, G_2^*) > F(G_1, G_2)$. As $g(G_2) \in \mathbb{N}$ and $g(G_2^*) \in \mathbb{N}$, based on $g(G_2^*) < g(G_2)$ we have that $g(G_2) - g(G_2^*) \geq 1$. Therefore,

$$\frac{F(G_1, G_2) - F(G_1, G_2^*)}{g(G_2) - g(G_2^*)} \leq \frac{F(G_1, G_2) - F(G_1, G_2^*)}{1} \leq F(G_1, G_2) \; < \; \lambda. \tag{3.27}$$

It follows that $\lambda \cdot g(G_2^*) + F(G_1, G_2^*) < \lambda \cdot g(G_2) + F(G_1, G_2)$, which contradicts the hypothesis that (G_1, G_2) it is an optimal solution of the problem $(PL_2(H))$.

If we consider case 2), then we immediately get that

$$\lambda \cdot g(G_2^*) + F(G_1, G_2^*) < \lambda \cdot g(G_2) + F(G_1, G_2),$$

which contradicts the optimality of (G_1, G_2) for the problem $(PL_2(H))$.

Sufficiency: Let G_2 be an optimal solution of the problem $(P_2(H))$ and let G_2^* be a feasible solution of the problem $(PL(H))$. Since G_2^* it is a feasible solution of the problem $(P_2(H))$, we get that

$$\begin{pmatrix} g(G_2) \\ F(G_1, G_2) \end{pmatrix} <_{\text{lex}} \begin{pmatrix} g(G_2^*) \\ F(G_1, G_2^*) \end{pmatrix}.$$

Three cases can occur:
1) $g(G_2) < g(G_2^*)$;

2) $g(G_2) = g(G_2^*)$ and $F(G_1, G_2^*) < F(G_1, G_2)$;
3) $g(G_2) = g(G_2^*)$ and $F(G_1, G_2^*) = F(G_1, G_2)$.

Since $g(G_2) \in \mathbb{N}$ and $g(G_2^*) \in \mathbb{N}$, in the first case we have $g(G_2) - g(G_2^*) \leq -1$. Therefore,

$$\begin{aligned}
&\lambda \cdot g(G_2) + F(G_1, G_2) - (\lambda \cdot g(G_2^*) + F(G_1, G_2^*)) = \\
&\lambda \cdot (g(G_2) - g(G_2^*)) + F(G_1, G_2) - F(G_1, G_2^*) \leq \\
&-\lambda + F(G_1, G_2) - F(G_1, G_2^*) \leq \\
&-1 - F(G_1, G_2^*) < 0.
\end{aligned}$$

In the cases 2) and 3) it results that $\lambda \cdot g(G_2) + F(G_1, G_2) \leq \lambda \cdot g(G_2^*) + F(G_1, G_2^*)$. Since G_2^* it is a feasible solution chosen arbitrary, it results that G_2 it is an optimal solution of the problem $(PL_2(H))$. ■

Remark 3.2.7 *Based on Theorem 3.2.6 we can reduce the solving of the problem* (PBG) *to solving a finite sequence of couples of problems* $(P_1(H), P_2(H))$*, where the parameter* H *belongs to the set* 2^I*, and, on the other hand, the solving of the problem* $(P_2(H))$ *can be replaced by solving the problem* $(PL_2(H))$*.*

We note that the results obtained within this chapter complete the results obtained by Ruuska S., Miettinen K. and Wiecek M.M. [100].

Chapter 4

Applications of Multilevel Optimization with Respect to Professional Training Programs

Education means more than human capital investment (Becker G., 1993) because it offers to the human-being the possibility to learn and achieve new skills and job qualifications and it leads to the formation of a new behavior, of some new beliefs and of a new role in society (Haveman R.H. and Wolfe B.L., 1984). The participation to the education process on a longer period offers the possibility to acquire new information, to learn how to take different decisions, which will have a positive impact in the human's future activities. Better qualification of human capital increases their chances of being employed (Iyigun M.F. and Owen A.L., 1999), reduces the unemployment rate (Kettunen J., 1997) and has a positive influence upon their earning possibilities. Education is not any more a process limited to the ages between 3 and 24, but becomes a life-long learning process. Thus imposes the need of professional training courses of the persons since well educated people with better qualification adapt more rapidly to technological changes and ensure growth of productivity on the long run. The education and life long learning has a positive impact on the individual because it seems to increase his chances of unemployment, to reduce the term of unemployment, to reduce the costs of finding a job, to offer to the individual the possibility to have a bigger wage, to contribute to the productivity growth and to the economic development of the society. With the elaboration of the Lisbon Strategy, European Union (EU) sets the orientation toward an economy and a society based on knowledge; this is why the need of life-long learning is a priority for the EU. For Romania, as a member-state of the EU, the employment at a high level of labor force is one of the most important priorities. On one hand, we should notice the employees attitudes concerning continuous professional training and their participation to such programs [45]. On the other hand, we should relieve the present level of employer's investment in the employees' education and professional training [46].

Taking into account the importance of professional training courses, Guţ C.M. and Bode O.R. studied in [47] the present level of employer's investment in the employees' education and professional training. This paper highlights the demand for continuous professional training among employers and employees in the manufacturing companies and the efficiency of these training programs. The authors pointed out some of the main constraints that met the employees and employers concerning investment and participation in professional training.

The employers' demand from low qualification towards medium and high ones is present in many countries. Guţ C.M., Vorzsak M., Chifu C.I. and Bode O.R. [48] noticed that every year the employers demand will be orientated towards persons with a medium level of qualification and high level of qualification.

Although there exists an increasing tendency of labor demand for workers with a medium and high level of qualification, there exists the risk of exclusion of the unemployed whose qualifications do not fit the employers' demand, and the possibility that unskilled workers or with a low level of qualification to become unemployed. That is why it is necessary that persons with a low level of qualification to participate to lifelong learning programs. In [49], Guţ C.M., Vorzsak M. and Bode (Tuns) O.R. investigates the way in which small and medium enterprises have contributed to job creation/destruction in the Cluj County during the period 2008-2010.

Keeping in mind the importance of well-qualified persons, we should notice that one of the major problems faced by the national institutions from our country and from abroad is the lack of the financial resources allocated for the professional training programs (PTPs) of the unemployed persons. The problem that arises very often is that the budget allocated for these institutions is not enough to offer for free PTPs from different areas to all unemployed persons. Therefore, different economic problems concerning the assignment of the persons to attend a PTP or the restriction of the budget allocated for these type of courses can be found in real life situations.

In the present chapter we study from the optimization point of view some of these economic problems that arise and which are solved in practice intuitively. The present chapter is organized as follows: in Section 4.1 we give a brief background concerning assignment problems. In Section 4.2 we formulate the two studied economic problems involving assigning unemployed persons to PTPs. In Section 4.3, respectively 4.4, we mathematically model and solve the first, respectively second, problem based on some given restrictions.

The scientific results within this chapter belong to the author and can be found in the paper authored by Tuns (Bode) O.R. and Neamţiu L. [116].

4.1 Brief Background Concerning Assignment Problems

In real life situations, we can find some concrete economic problems involving assigning unemployed persons to PTPs. Since we use as a mathematical tool the assignment problems we give a brief background concerning these type of problems.

The name *assignment problem* seems to have first appeared in 1952 in the paper authored by Votaw D.F. and Orden A. [122]. But the beginning of the development of practical solution methods for and variations on the classic assignment problem was the publication in 1955 of Kuhn's article on the Hungarian method for its solution [64]. Over the years, many variations on the classic assignment problem have been proposed, such as the bottleneck assignment problem, the lexicographic bottleneck assignment problem, the classic assignment problem recognizing agent qualification, the generalized assignment problem, the quadratic assignment problem and a variety of others. During the time, the assignment problems knew different generalizations. A very useful overview regarding the variety of models of the assignment problems can be found in Pentico D.W. [91]. This paper provides a comprehensive survey of the different variations on the assignment problem that have appeared in the literature.

Assignment problems involve matching the elements of two or more sets in such a way that some objective function is optimized. They consist of two components: the assignment as underlying combinatorial structure and an objective function modeling the best way. Classic assignment problems deal with the question how to assign n tasks to n machines (or workers) in the best possible way, i.e. to find a one-to-one matching between n tasks and n workers, while the objective being to minimize the total cost of the assignments. Classic examples involve such situations as assigning jobs to machines, jobs to workers, or workers to machines. The mathematical model for the classic assignment problem may be given by (4.1), where x_{ij} is the binary variable having the significance that $x_{ij} = 1$ if agent i is assigned to task j and $x_{ij} = 0$ otherwise; c_{ij} represents the cost of assigning agent i to task j. The constraints ensures that every task is assigned to only one agent and every agent is assigned to a task. A problem in which the objective function is to be maximized can be easily converted into a minimization problem by either multiplying all of the c_{ij}'s by -1, or replacing each c_{ij} by $c_{max} - c_{ij}$, where c_{max} is the maximum of the c_{ij}'s, thus converting the problem to one of minimizing regret. A problem that is not balanced (i.e., one for which the numbers of tasks and agents differ) can be easily converted into a balanced problem by adding a sufficient number of dummy tasks or agents (whichever is in shorter supply) with costs of 0.

$$\begin{cases} \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \to \min, \\ \sum_{i=1}^{n} x_{ij} = 1, \quad j \in \{1,\dots,n\}, \\ \sum_{j=1}^{n} x_{ij} = 1, \quad i \in \{1,\dots,n\}, \\ x_{ij} \in \{0,1\}, \end{cases} \tag{4.1}$$

A variation of the classic assignment problem is the classic assignment problem recognizing agent qualification that can be find in [17]. In their work, Caron G., Hansen P. and Jaumard B. [17] use a mathematical model for a variation of the classic assignment problem in which there are m agents and n tasks, not every agent is qualified to do every task, and the objective is utility maximization.

In 1953, Fulkerson D.R., Glicksberg I. and Gross O. [37] mentioned for the first time in literature the bottleneck assignment problem. While the objective of a classic assignment problem is to minimize or maximize the sum of the costs of the assignments of tasks to workers, the objective of the bottleneck assignment problem is to minimize the maximum of the costs of the assignments (or to maximize the minimum of the efficiency of the assignments). In Ford L.R. and Fulkerson D.R. [35] the authors give an example that involves assigning workers to jobs in such a way that the minimum efficiency of such assignments will be maximized. The bottleneck assignment problem has been also studied by Gross O. [43], Garfinkel R.S. [40], Ravindran A. and Ramaswami V. [97]. In Pentico D.W. [91] it can be find some of the variety of models of the bottleneck assignment problems, such as the lexicographic bottleneck problem, the assignment problem with side constraints and r-lexicographic multi-objective problem. The lexicographic bottleneck problem have been studied for example in Burkard R.E. and Rendl F. [15] or in Sokkalingam P.T. and Aneja Y.P. [104]. Another useful work for researchers and practitioners is the one authored by Burkard R., Dell'Amico M. and Martello S. [14]. It provides a comprehensive treatment of assignment problems from their conceptual beginnings through present day.

In [62] Khanmohammadi S., Hajiha A. and Jassbi J. introduce a qualification matrix used to classify and select the qualified individuals for different jobs to optimize the man power of the organization.

Based on the author's papers with a preponderant economic character (see [47], [48] and [49]), one of the critical problems that can be find in real life situations and must be solved is the problem of retraining the unemployed persons. Therefore, the mathematical models introduced by us in the present book in order to select suitable persons for different professional training programs are used to solve different types of optimization problems, which can be view as general assignment problems.

4.2 The Concrete Economic Problems

The problems to be discussed in the present chapter models some concrete economic problems and involve assigning unemployed persons to PTPs:

Problem I. Assume that in the same period of time there are organized different PTPs for the unemployed persons. For each PTP there is known its efficiency (defined from the point of view of finding a place to work by the unemployed persons after graduating it), the maximum number of the persons that can attend it and the score that each unemployed person has if attends it (this score was calculated based on historical data about each unemployed person taking into account his/her education or professional experience). The problem that arises is how to assign the registered unemployed persons to the PTPs, based on each person's score, such that the following restrictions to be fulfilled:

i) all unemployed persons to attend the PTPs (i.e. the case when the maximum number of the persons that can attend the PTPs is bigger than the total number of the registered unemployed persons which need to attend the courses);

ii) each unemployed person to attend exactly one PTP;

iii) the assignment of the unemployed persons to a PTP to be done such that to maximize the minimum score of the assignments;

iv) the efficiency of the PTP for which the minimum score is reached to be as small as possible and to be reached as few times as possible.

Let us denote by (AEP_1) this first concrete economic problem.

Problem II. Now, we consider the above problem but under the hypothesis that restriction i) is not fulfilled, i.e. the case when the maximum number of the persons that can attend the PTPs is smaller than the total number of the registered unemployed persons which need to attend the courses, while the above restrictions ii), iii) and iv) occur.

Let us denote by (AEP_2) this second concrete economic problem.

Furthermore, we give the mathematical model of each concrete economic problem, we study some properties of the optimal solutions of each problem and we propose an algorithm or a method for solving it, highlighted by different examples.

In each of the following sections, let us denote by:

-m the number of the total PTPs identified by the variable i, $i \in \{1, \dots, m\}$. Let $I = \{1, \dots, m\}$;

-e_i, $i \in I$, the efficiency of the PTP i;

-n the total number of the unemployed persons that need to attend the PTPs. Let $J = \{1, \dots, n\}$;

-a_i, $i \in I$, the maximum number of the persons that can participate to the PTP i, $i \in I$;

-r_{ij}, $i \in I$, $j \in J$, the score corresponding to each unemployed person j if attends the PTP i. Let $R \in \mathcal{M}_{m\times n}(\mathbb{R}_+^*)$ be the matrix which elements represent the scores r_{ij};

-y_{ij}, $i \in I$, $j \in J$, the binary variable having the significance $y_{ij} = 1$ if the unemployed person j will participate to the course i and $y_{ij} = 0$ otherwise;

4.3 The Study of the Problem (AEP_1)

The problem to be discussed in the present section represents a new kind of a generalized bottleneck assignment problem. It images the modeling of a concrete economic problem which involves assigning unemployed persons to PTPs under the circumstances that, on one hand, there exists no restriction regarding the budget allocated for it and, on the other hand, the only restrictions are the ones concerning the number of the unemployed persons which must attend the PTPs, the score of the assignments and the efficiency of each PTP.

4.3.1 Mathematical Modeling of the Problem (AEP_1)

Within the restrictions of our practical problem the values of the efficiencies of the PTPs does not interfere. It interferes just the arrangement of the efficiency of one PTP in relation to the other PTPs. Therefore, we assume that the arrangement of the PTPs was done in a descending order of their efficiency, i.e. $e_i \geq e_{i+1}$, $\forall\, i \in I$.

Let $\mathcal{Y}$ be the set of matrices $Y = [y_{ij}] \in M_{m\times n}(\mathbb{R})$ which fulfill the following conditions:

$$y_{ij} \in \{0, 1\}, \forall\, i \in I, \forall\, j \in J; \tag{4.2}$$

$$\sum_{i \in I} y_{ij} = 1, \,\forall\, j \in J; \tag{4.3}$$

$$\sum_{j \in J} y_{ij} \leq a_i, \,\forall\, i \in I. \tag{4.4}$$

Let $f = (f_1, f_2, f_3) : \mathcal{Y} \to \mathbb{R}^3$ be the function given by: $\forall\, Y \in \mathcal{Y}$,

$$f_1(Y) = \min\Big\{r_{ij} \,|\, i \in I,\, j \in J,\, y_{ij} = 1\Big\} = \min\Big\{r_{ij}y_{ij} \,|\, i \in I,\, j \in J\Big\}, \tag{4.5}$$

$$f_2(Y) = \min\Big\{i \in I \,|\, \exists\, j \in J \quad \text{such that} \quad r_{ij}y_{ij} = f_1(Y)\Big\}, \tag{4.6}$$

$$f_3(Y) = \sum_{(i,j)\in I\times J;\, r_{ij}y_{ij}=f_1(Y);\, i\geq f_2(Y)} y_{ij} = card\Big\{y_{ij} \,|\, i \in I,\, i \geq f_2(Y),\, j \in J,\, r_{ij}y_{ij} = f_1(Y)\Big\}. \tag{4.7}$$

Based on (4.3) and (4.4) we get that

$$n = \sum_{j\in J}\Big(\sum_{i\in I} y_{ij}\Big) = \sum_{i\in I}\sum_{j\in J} y_{ij} \leq \sum_{i\in I} a_i.$$

Hence, we work under the hypothesis that

$$\sum_{i\in I} a_i \geq n, \tag{4.8}$$

i.e. the total number of the persons that can attend the PTPs is greater than the total number of the registered unemployed persons which need to attend it. Condition (4.8) assures that $\mathcal{Y} \neq \emptyset$.

Indeed, if we set $a_0 = 0$ and we take

$$y^*_{ij} = \begin{cases} 1, \text{ if } j \in J,\ 1 + \sum\limits_{k=0}^{i-1} a_k \leq j \leq \sum\limits_{k=0}^{i} a_k \\ 0, \text{ if } j \in J,\ j \leq \sum\limits_{k=0}^{i-1} a_k \\ 0, \text{ if } j \in J,\ j \geq 1 + \sum\limits_{k=0}^{i} a_k \end{cases} ,\forall\, i \in I, \tag{4.9}$$

then $Y^* = [y^*_{ij}] \in \mathcal{Y}$. If condition (4.8) is not fulfilled, then $\mathcal{Y} = \emptyset$.

In order to give the mathematical model for the (AEP_1) problem we introduce in $\mathbb{R}^3$ the following order relation of lexicographic type, denoted by $<_{\max-\max-\min}$.

Definition 4.3.1 (TUNS (BODE) O.R. *and* NEAMŢIU L. [*116*]). *Let* $x = (x_1, x_2, x_3)$ *and* $y = (y_1, y_2, y_3)$ *be two elements from* $\mathbb{R}^3$.

We say that $x <_{\max-\max-\min} y$ *if and only if one of the following conditions occur:*

i) $x_1 < y_1$;

ii) $x_1 = y_1$ *and* $x_2 < y_2$;

iii) $x_1 = y_1,\ x_2 = y_2$ *and* $x_3 > y_3$.

Therefore, our problem can be graphically given by the following problem:

$$(PS) \quad \begin{cases} f(Y) \to \ lex - \max - \max - \min, \\ Y \in \mathcal{Y}. \end{cases} \tag{4.10}$$

Definition 4.3.2 (TUNS (BODE) O.R. *and* NEAMŢIU L. [*116*]). *A point $Y^0 \in \mathcal{Y}$ is said to be an optimal solution of the problem (PS) if there is no other point $Y \in \mathcal{Y}$ such that to have $f(Y^0) <_{\max - \max - \min} f(Y)$, i.e. neither one of the following restrictions to occur:*

i) $f_1(Y) > f_1(Y^0)$;

ii) $f_1(Y) = f_1(Y^0)$ *and* $f_2(Y) > f_2(Y^0)$;

iii) $f_1(Y) = f_1(Y^0)$, $f_2(Y) = f_2(Y^0)$ *and* $f_3(Y) < f_3(Y^0)$.

We remark that the mathematical model attached to the economic problem is a problem of lexicographic optimization type. Based on the restrictions (4.3) and (4.4), the problem (PS) can be view as a particular type of an unbalanced transportation problem of bottleneck type, having the property that all its variables have a boolean value. On the other hand, based on restriction (4.2), the problem (PS) can be view as a generalization of the bottleneck assignment problem. Also, the problem can be seen as a resources assignment problem [89]. Whatever we consider this problem, as far as we know a such type of problem have not been studied yet.

4.3.2 Necessary and Sufficient Optimality Conditions of (PS)

Let

$$\lambda = \min\{r_{ij} \mid i \in I,\ j \in J\}, \tag{4.11}$$

$$h = \min\{i \in I \mid \exists\, k \in J \,\text{such that}\, r_{ik} = \lambda\}, \tag{4.12}$$

$$J_{h,\lambda} = \{j \in J \mid r_{hj} = \lambda\}, \tag{4.13}$$

and

$$q = card(J_{h,\lambda}). \tag{4.14}$$

Proposition 4.3.3 (TUNS (BODE) O.R. *and* NEAMŢIU L. [*116*]). *If*

$$\sum_{i \in I} a_i < n + q, \tag{4.15}$$

then for each $Y \in \mathcal{Y}$ the following conditions occur:

$$f_1(Y) = \lambda \tag{4.16}$$

and

$$f_2(Y) = h. \tag{4.17}$$

Proof. From (4.15) we get that

$$\sum_{i\in I\setminus\{h\}} a_i + a_h - q < n. \tag{4.18}$$

Let $Y^0 = [y^0_{ij}] \in \mathcal{Y}$. For each $i \in I\setminus\{h\}$ the restriction (4.4) is fulfilled and we get

$$\sum_{i\in I\setminus\{h\}} \sum_{j\in J} y^0_{ij} \leq \sum_{i\in I\setminus\{h\}} a_i. \tag{4.19}$$

Based on restriction (4.3) we obtain that

$$\sum_{i\in I} \sum_{j\in J} y^0_{ij} = \sum_{j\in J} \sum_{i\in I} y^0_{ij} = \sum_{j\in J} 1 = n. \tag{4.20}$$

From (4.19) and (4.20) we have that

$$n - \sum_{j\in J} y^0_{hj} = \sum_{i\in I\setminus\{h\}} \sum_{j\in J} y^0_{ij} \leq \sum_{i\in I\setminus\{h\}} a_i. \tag{4.21}$$

Then,

$$n \leq \sum_{i\in I\setminus\{h\}} a_i + \sum_{j\in J} y^0_{hj}. \tag{4.22}$$

From (4.18) and (4.22) we obtain that

$$\sum_{i\in I\setminus\{h\}} a_i + a_h - q < \sum_{i\in I\setminus\{h\}} a_i + \sum_{j\in J} y^0_{hj},$$

or

$$a_h - q < \sum_{j\in J} y^0_{hj}.$$

Therefore, there exists $j \in J_{h,\lambda}$ such that

$$y^0_{hj} = 1. \tag{4.23}$$

Based on (4.5), (4.11) and (4.23) it results that $f_1(Y^0) = \lambda$. Then, from (4.12) we deduce that $f_2(Y^0) = h$.

As $Y^0 \in \mathcal{Y}$ was chosen arbitrary, it results that (4.16) and (4.17) are true, for each $Y \in \mathcal{Y}$. ■

Remark 4.3.4 *If (4.15) holds, then for each $Y^0 \in \mathcal{Y}$ an optimal solution of the problem (PS), we have that $f_1(Y^0) = \lambda$ and $f_2(Y^0) = h$.*

Proposition 4.3.5 (TUNS (BODE) O.R. *and* NEAMŢIU L. [*116*]). *Let $Y^0 \in \mathcal{Y}$ be an optimal solution of the problem (PS). If*

$$\sum_{i\in I} a_i \geq n + q, \tag{4.24}$$

then we have

$$y^0_{hj} = 0, \forall j \in J_{h,\lambda}. \tag{4.25}$$

Proof. Let us suppose that there exists $k \in J_{h,\lambda}$ such that $y^0_{hk} = 1$. Then, based on (4.11) we get that $r_{hk} = \lambda$. As $y^0_{hk} = 1$, from (4.5) and (4.11), respectively from (4.6) and (4.12) we get that

$$f_1(Y^0) = \lambda \tag{4.26}$$

and

$$f_2(Y^0) = h. \tag{4.27}$$

Now, we build a matrix $Y^* = [y^*_{ij}] \in \mathcal{Y}$ such that

$$f_1(Y^*) \geq \lambda \quad \text{or} \quad f_1(Y^*) = \lambda, \text{ but } f_2(Y^*) > h.$$

For this, without restricting the generality, we can suppose that $h = 1$ and $J_{h,\lambda} = \{1, \ldots, q\}$. Indeed, if $h > 1$, then we can interchange in the scores matrix R the line h with the first line, and then interchange the columns between them, such that the first q elements from the first line to be equal to λ. Based on the above, (4.14) implies that

$$r_{hj} > \lambda, \forall j \in J \backslash \{1, \ldots, q\}. \tag{4.28}$$

Hence, we set the matrix $Y^* = [y^*_{ij}]$ such that

$$y^*_{ij} = \begin{cases} 0, & \text{if} \quad i = 1 \quad \text{and} \quad j \in J \cap \{1, \ldots, \sum_{s=2}^m a_s\}, \\ 1, & \text{if} \quad i = 1 \quad \text{and} \quad j \in J \backslash \{1, \ldots, \sum_{s=2}^m a_s\}, \\ 1, & \text{if} \quad i = 2 \quad \text{and} \quad j \in J \cap \{1, \ldots, a_1\}, \\ 0, & \text{if} \quad i = 2 \quad \text{and} \quad j \in J \backslash \{1, \ldots, a_1\}, \\ 1, & \text{if} \quad 3 \leq i \leq m \quad \text{and} \quad j \in J \cap \{\sum_{s=2}^{i-1} a_s + 1, \ldots, \sum_{s=2}^{i} a_s\}, \\ 0, & \text{if} \quad 3 \leq i \leq m \quad \text{and} \quad j \in J \backslash \{\sum_{s=2}^{i-1} a_s + 1, \ldots, \sum_{s=2}^{i} a_s\}. \end{cases} \tag{4.29}$$

It is not difficult to see that $Y^* \in \mathcal{Y}$. From the way we have build the matrix Y^* and based on (4.11) and (4.28), we obtain that $f_1(Y^*) \geq \lambda$.

If $f_1(Y^*) > \lambda$, then (4.26) contradicts the optimality of Y^0.

If $f_1(Y^*) = \lambda$, then there exists $(i, j) \in I \times J$ such that $r_{ij} y^*_{ij} = \lambda$.

Let $u = \min\{i \in I \,|\, \exists\, j \in J \text{ such that } r_{ij} y^*_{ij} = \lambda\}$. As $y^*_{ij} = 0$, $\forall\, j \in \{1, \ldots, q\} \cap J$, from (4.29) it results that $u \geq 2$. Therefore, $f_2(Y^*) = u > 1 = f_2(Y^0)$. Again, the optimality of Y^0 is contradicted.

Hence, (4.25) must be true. ■

From Proposition 4.3.5, it results that if $q = n$, then to determine an optimal solution of the problem (PS) is equivalent to determine an optimal solution of a problem of the same type as the problem (PS), but in which in the scores matrix the line h does not appear. In what follows, we suppose that $q < n$.

Let us now consider the following lexicographic optimization problem:

$$(PM) \quad \begin{cases} \varphi(Y) = \begin{pmatrix} f_1(Y) \\ f_2(Y) \end{pmatrix} \to lex - \max - \max, \\ Y \in \mathcal{Y}. \end{cases} \tag{4.30}$$

Let $\tilde{Y}$ be an optimal solution of the problem (PM), $\lambda = f_1(\tilde{Y})$ and $h = f_2(\tilde{Y})$. We set the matrix $C_{\lambda,h} = [c_{ij}]$ such that

$$c_{ij} = \begin{cases} 0, & \text{if} \quad r_{ij} > \lambda, \\ 1, & \text{if} \quad r_{ij} = \lambda \quad \text{and} \quad i \geq h, \\ n+1, & \text{if} \quad (r_{ij} < \lambda) \quad \text{or} \quad (r_{ij} = \lambda \quad \text{and} \quad i < h). \end{cases} \tag{4.31}$$

Let $g_{\lambda,h} : \mathcal{Y} \to \mathbb{R}$ be the function given by

$$g_{\lambda,h}(Y) = \sum_{i \in I} \sum_{j \in J} c_{ij} y_{ij}, \; \forall\, Y \in \mathcal{Y}. \tag{4.32}$$

Now, let us consider the following optimization problem:

$$(AC_{\lambda,h}) \quad \begin{cases} g_{\lambda,h}(Y) \to \min, \\ Y \in \mathcal{Y}. \end{cases} \tag{4.33}$$

The problem $(AC_{\lambda,h})$ represents a transportation problem which can be solved using the potential plan algorithm.

Let

$$E^1_{\lambda,h} = \Big\{(i,j) \in I \times J \,|\, r_{ij} = \lambda, \, i \geq h\Big\} \tag{4.34}$$

and

$$E^2_{\lambda,h} = \Big\{(i,j) \in I \times J \,|\, (r_{ij} < \lambda) \quad \text{or} \quad (r_{ij} = \lambda \quad \text{and} \quad i < h)\Big\}. \tag{4.35}$$

We remark that

$$g_{\lambda,h}(Y) = \sum_{(i,j) \in E^1_{\lambda,h}} y_{ij} + (n+1) \sum_{(i,j) \in E^2_{\lambda,h}} y_{ij}. \tag{4.36}$$

Proposition 4.3.6 (Tuns (Bode) O.R. *and* Neamţiu L. [*116*]). *If* $\tilde{Y} \in \mathcal{Y}$ *is an optimal solution of the problem* (PS), $\lambda = f_1(\tilde{Y})$ *and* $h = f_2(\tilde{Y})$, *then* $\tilde{Y}$ *is an optimal solution of the problem* (PM) *and an optimal solution of the problem* $(AC_{\lambda,h})$.

Proof. Let $\tilde{Y}$ be an optimal solution of the problem (PS). Then, from (4.5) and (4.6) we get that

$$f_1(\tilde{Y}) = \min\Big\{r_{ij}\,|\, i \in I,\, j \in J,\, \tilde{y}_{ij} = 1\Big\} \tag{4.37}$$

and

$$f_2(\tilde{Y}) = \min\Big\{i \in I\,|\,\exists\, j \in J \quad \text{such that} \quad r_{ij}\tilde{y}_{ij} = f_1(\tilde{Y})\Big\}. \tag{4.38}$$

Based on (4.34) and (4.35) we have that

$$E^1_{f_1(\tilde{Y}),f_2(\tilde{Y})} = \Big\{(i,j) \in I \times J\,|\, r_{ij} = f_1(\tilde{Y}),\, i \geq f_2(\tilde{Y})\Big\} \tag{4.39}$$

and

$$E^2_{f_1(\tilde{Y}),f_2(\tilde{Y})} = \Big\{(i,j) \in I \times J\,|\,(r_{ij} < f_1(\tilde{Y})) \quad \text{or} \quad (r_{ij} = f_1(\tilde{Y}) \quad \text{and} \quad i < f_2(\tilde{Y}))\Big\}. \tag{4.40}$$

From (4.31) and (4.32) we get that

$$g_{f_1(\tilde{Y}),f_2(\tilde{Y})}(\tilde{Y}) = \sum_{(i,j)\in E^1_{f_1(\tilde{Y}),f_2(\tilde{Y})}} \tilde{y}_{ij} \leq \sum_{i\in I}\sum_{j \in J} \tilde{y}_{ij} = n. \tag{4.41}$$

On the other hand, let us remark that

$$g_{f_1(\tilde{Y}),f_2(\tilde{Y})}(\tilde{Y}) = \sum_{(i,j)\in E^1_{f_1(\tilde{Y}),f_2(\tilde{Y})}} \tilde{y}_{ij} \tag{4.42}$$

$$= card\Big(\Big\{(i,j) \in I \times J\,|\, r_{ij}\tilde{y}_{ij} = f_1(\tilde{Y}),\, i \geq f_2(\tilde{Y})\Big\}\Big) = f_3(\tilde{Y}).$$

Now, let us suppose that $\tilde{Y}$ is not an optimal solution of the problem (PM). Then, there exists $Y \in \mathcal{Y}$ such that

$$\Big(f_1(Y) > f_1(\tilde{Y})\Big) \quad \text{or} \quad \Big(f_1(Y) = f_1(\tilde{Y}) \text{ and } f_2(Y) > f_2(\tilde{Y})\Big),$$

which contradicts the optimality of $\tilde{Y}$ for the problem (PS). Therefore, $\tilde{Y}$ is an optimal solution of the problem (PM).

We prove now that $\tilde{Y}$ is an optimal solution of the problem $(AC_{\lambda,h})$, where $\lambda = f_1(\tilde{Y})$ and $h = f_2(\tilde{Y})$.

Let $Y \in \mathcal{Y}$. Since $\tilde{Y}$ is an optimal solution of the problem (PM), we have three possible cases:

i) $f_1(Y) < f_1(\tilde{Y})$;
ii) $f_1(Y) = f_1(\tilde{Y})$ and $f_2(Y) < f_2(\tilde{Y})$;
iii) $f_1(Y) = f_1(\tilde{Y}),\ f_2(Y) = f_2(\tilde{Y}) = n+1$.

In the first two cases it results that $\exists\,(i_0, j_0) \in E^2_{f_1(\tilde{Y}),f_2(\tilde{Y})}$ such that $y_{i_0 j_0} = 1$. Then, $c_{i_0 j_0} = n+1$ and, based on (4.32), we deduce that

$$g_{f_1(\tilde{Y}),f_2(\tilde{Y})}(Y) \geq c_{i_0 j_0} y_{i_0 j_0} = (n+1) \times 1 = n+1. \tag{4.43}$$

From (4.41) and (4.43) we obtain that

$$g_{f_1(\tilde{Y}),f_2(\tilde{Y})}(\tilde{Y}) \leq n < n+1 \leq g_{f_1(\tilde{Y}),f_2(\tilde{Y})}(Y).$$

Therefore, Y can not be an optimal solution of the problem $(AC_{\lambda,h})$.

In the third case, i.e.

$$f_1(Y) = f_1(\tilde{Y}),\ f_2(Y) = f_2(\tilde{Y}) = n+1, \tag{4.44}$$

we have that Y is an optimal solution of the problem (PM). Based on (4.31) we have

$$g_{f_1(\tilde{Y}),f_2(\tilde{Y})}(Y) = \sum_{i\in I}\sum_{j\in J} c_{ij}y_{ij} = \sum_{(i,j)\in I\times J; r_{ij} > f_1(\tilde{Y})} 0 \times y_{ij} \tag{4.45}$$

$$+ \sum_{(i,j)\in I\times J; r_{ij}=f_1(\tilde{Y}); i \geq f_2(\tilde{Y})} 1 \times y_{ij} + \sum_{(i,j)\in I\times J; (r_{ij}<f_1(\tilde{Y}))\text{ or }(r_{ij}=f_1(\tilde{Y})\text{ and } i<f_2(\tilde{Y}))} (n+1)y_{ij}$$
$$= card\Big(\Big\{(i,j) \in I \times J \,|\, r_{ij} = f_1(\tilde{Y}),\ i \geq f_2(\tilde{Y}) \text{ and } y_{ij} = 1\Big\}\Big)$$
$$= card\Big(\Big\{(i,j) \in I \times J \,|\, r_{ij}y_{ij} = f_1(\tilde{Y}),\ i \geq f_2(\tilde{Y})\Big\}\Big) = f_3(Y).$$

Based on (4.44), on the fact that $\tilde{Y}$ is an optimal solution of the problem (PS) and on Definition 4.3.2, we get that $f_3(\tilde{Y}) \leq f_3(Y)$. Then, from (4.42) and (4.45) we obtain that

$$g_{f_1(\tilde{Y}),f_2(\tilde{Y})}(\tilde{Y}) \leq g_{f_1(\tilde{Y}),f_2(\tilde{Y})}(Y).$$

Therefore, in all the above three possible cases we get that

$$g_{f_1(\tilde{Y}),f_2(\tilde{Y})}(\tilde{Y}) \leq g_{f_1(\tilde{Y}),f_2(\tilde{Y})}(Y),$$

i.e. $\tilde{Y}$ is an optimal solution of the problem $(AC_{\lambda,h})$. ■

Proposition 4.3.7 (Tuns (Bode) O.R. *and* Neamţiu L. [*116*]). *If* $\tilde{Y} \in \mathcal{Y}$ *is an optimal solution of the problem* (PM) *and an optimal solution of the problem* $(AC_{\lambda,h})$*, where* $\lambda = f_1(\tilde{Y})$ *and* $h = f_2(\tilde{Y})$*, then* $\tilde{Y}$ *is an optimal solution of the problem* (PS).

Proof. Since $\tilde{Y}$ is an optimal solution of the problem (PM), it results that $\forall\, Y \in \mathcal{Y}$ one of the following three cases can occur:
i) $f_1(Y) < f_1(\tilde{Y})$;
or
ii) $f_1(Y) = f_1(\tilde{Y})$ and $f_2(Y) < f_2(\tilde{Y})$;
or
iii) $f_1(Y) = f_1(\tilde{Y})$ and $f_2(Y) = f_2(\tilde{Y})$.

In the first two cases it is obvious that Y can not be an optimal solution of the problem (PS).

In the third case, it results that for all $(i,j) \in I \times J$, with $y_{ij} = 1$, we have that

$$r_{ij} \geq f_1(\tilde{Y}), \text{ and, if } r_{ij} = f_1(\tilde{Y}) \text{ then } i \geq f_2(\tilde{Y}). \tag{4.46}$$

But, in the same time there exists $(u,v) \in I \times J$ such that

$$y_{uv} = 1,\ r_{uv} = f_1(\tilde{Y}) \text{ and } u = f_2(\tilde{Y}). \tag{4.47}$$

Based on (4.7), (4.31) and (4.32) we get that

$$g_{f_1(\tilde{Y}),f_2(\tilde{Y})}(\tilde{Y}) = f_3(\tilde{Y}). \tag{4.48}$$

From (4.46) we deduce that there is no $(i,j) \in I \times J$ with $y_{ij} = 1$ such that $(i,j) \in E^2_{f_1(\tilde{Y}),f_2(\tilde{Y})}$. Then,

$$g_{f_1(\tilde{Y}),f_2(\tilde{Y})}(Y) = \sum_{(i,j)\in E^1_{f_1(\tilde{Y}),f_2(\tilde{Y})}} c_{ij} y_{ij} = f_3(Y). \tag{4.49}$$

As $\tilde{Y}$ is an optimal solution of $(AC_{\lambda,h})$, we have $g_{f_1(\tilde{Y}),f_2(\tilde{Y})}(\tilde{Y}) \leq g_{f_1(\tilde{Y}),f_2(\tilde{Y})}(Y)$. Then, from (4.48) and (4.49), it results that $f_3(\tilde{Y}) \leq f_3(Y)$.

Therefore, $\tilde{Y}$ is an optimal solution of the problem (PS). ■

From Propositions 4.3.6 and 4.3.7 it results that the solving of the problem (PS) can be reduced by solving two problems: on one hand, we have to solve the problem (PM) and, on the other hand, we have to verify the optimality of the solution $\tilde{Y}$ of the problem (PM) for the problem $(AC_{\lambda,h})$, where $\lambda = f_1(\tilde{Y})$ and $h = f_2(\tilde{Y})$. In order to verify the optimality of $\tilde{Y}$ for the problem $(AC_{\lambda,h})$, we have to apply the potential plan theorem. Then, we obtain the following result.

Remark 4.3.8 (Tuns (Bode) O.R. *and* Neamţiu L. [*116*]). *$\tilde{Y}$ is an optimal solution of the problem $(AC_{\lambda,h})$ if and only if*

$$\sum_{s\in I} c_{sj}\tilde{y}_{sj} \leq c_{ij},\ \forall\, i \in I,\ \forall\, j \in J.$$

If $\tilde{Y}$ is not an optimal solution of the problem $(AC_{\lambda,h})$, then applying the potential plan algorithm to the problem $(AC_{\lambda,h})$, we get an optimal solution of the problem (PS).

For this, we recall the problem $(AC_{\lambda,h})$, where c_{ij} are given by (4.31). The problem $(AC_{\lambda,h})$ can be view as an unbalanced transport problem.

Now, let us denote by $\overline{J} = J \cup \{n+1\}$. Let

$$b_j = 1, \forall\, j \in J,\ b_{n+1} = \sum_{s\in I} a_s - n, \tag{4.50}$$

and

$$c^*_{ij} = \begin{cases} c_{ij}, & \text{if} \quad i \in I,\ j \in J, \\ 0, & \text{if} \quad i \in I,\ j = n+1. \end{cases} \tag{4.51}$$

Let $\mathcal{Z} = \{Z = [z_{ij}] \in M_{m\times\{n+1\}}(\mathbb{N})\}$, where all the elements of $\mathcal{Z}$ are fulfilling the following conditions:

i) $z_{ij} \in \{0,1\}, \forall\, i \in I, \forall\, j \in J$;

ii) $\sum\limits_{j\in J} z_{ij} = a_i, \forall\, i \in I$;

iii) $\sum\limits_{i\in I} z_{ij} = b_j, \forall\, j \in J$.

Let us consider the balanced transport problem:

$$(AC^*_{\lambda,h}) \quad \begin{cases} f^*_c(Z) = \sum\limits_{i\in I}\sum\limits_{j\in \overline{J}} c^*_{ij} z_{ij} \to \min, \\ Z \in \mathcal{Z}. \end{cases} \tag{4.52}$$

Proposition 4.3.9 (Tuns (Bode) O.R. [?]). *If $Y = [y_{ij}]$ is a feasible solution of the problem $(AC_{\lambda,h})$, then taking $\mu_i = \max\Big\{0,\ a_i - \sum\limits_{j\in J} y_{ij}\Big\}$, the matrix $Z^Y = [z^Y_{ij}]$ with*

$$z^Y_{ij} = \begin{cases} y_{ij}, & \textit{if} \quad i \in I,\ j \in J, \\ \mu_i, & \textit{if} \quad i \in I,\ j = n+1, \end{cases} \tag{4.53}$$

is a feasible solution of the problem $(AC^*_{\lambda,h})$.

Proof. Indeed, we have $z_{ij} \in \{0,1\}$ for all $i \in I$ and for all $j \in J$. Also,

$$\sum_{j\in\overline{J}} z^Y_{ij} = \sum_{j\in J} y_{ij} + \max\Big\{0,\ a_i - \sum_{j\in J} y_{ij}\Big\} = \sum_{j\in J} y_{ij} + a_i - \sum_{j\in J} y_{ij} = a_i, \forall\, i \in I.$$

If $j \in J$, then we get that $\sum\limits_{i\in I} z^Y_{ij} = \sum\limits_{i\in I} y_{ij} = 1 = b_j$.

If $j = n+1$, then we obtain that

$$\sum_{i\in I} z^Y_{i,n+1} = \sum_{i\in I} \max\Big\{0,\ a_i - \sum_{j\in J} y_{ij}\Big\} = \sum_{i\in I}(a_i - \sum_{j\in J} y_{ij})$$

$$= \sum_{i \in I} a_i - \sum_{i \in I} \sum_{j \in J} y_{ij} = \sum_{i \in I} a_i - n = b_{n+1}.$$

Therefore, Z^Y is a feasible solution of the problem $(AC^*_{\lambda,h})$. ■

Proposition 4.3.10 (Tuns (Bode) O.R. [?]). *If $Z = [z_{ij}] \in \mathcal{Z}$, then taking*

$$y^Z_{ij} = z_{ij},\ \forall\, i \in I,\ \forall\, j \in J, \tag{4.54}$$

we get that $Y^Z = [y^Z_{ij}] \in \mathcal{Y}$.

Proof. Indeed, if $Z \in \mathcal{Z}$ then $z_{ij} \in \{0,1\}$, $\forall\, i \in I$, $\forall\, j \in J$.

It results that $y_{ij} \in \{0,1\}$, $\forall\, i \in I$, $\forall\, j \in J$.

From $\sum_{i \in I} z_{ij} = b_j$, $\forall\, j \in \overline{J}$, it results that $\sum_{i \in I} y^Z_{ij} = 1$, $\forall\, j \in J$.

From $\sum_{j \in \overline{J}} z_{ij} = a_i$, $\forall\, i \in I$, it results that $\sum_{j \in J} y^Z_{ij} + z_{i,n+1} = a_i$.

We obtain that $\sum_{j \in J} y^Z_{ij} \leq a_i$, $\forall\, i \in I$. Therefore, $Y^Z \in \mathcal{Y}$. ■

Proposition 4.3.11 (Tuns (Bode) O.R. [?]). *If $Y \in \mathcal{Y}$ is an optimal solution of the problem $(AC_{\lambda,h})$, then Z^Y is an optimal solution of the problem $(AC^*_{\lambda,h})$.*

Proof. From Proposition 4.3.9 we have that $Z^Y \in \mathcal{Z}$. Let us suppose that Z^Y is not an optimal solution of the problem $(AC^*_{\lambda,h})$. Then, $\exists\, Z \in \mathcal{Z}$ such that

$$f^*_c(Z) < f^*_c(Z^Y). \tag{4.55}$$

Let Y^Z be given by (4.54). From Proposition 4.3.10 we have that $Y^Z \in \mathcal{Y}$. Based on (4.51) and (4.54) we obtain that

$$f^*_c(Z) = \sum_{i \in I} \sum_{j \in J} c^*_{ij} z_{ij} = \sum_{i \in I} \sum_{j \in J} c_{ij} y^Z_{ij} = g_{\lambda,h}(Y^Z). \tag{4.56}$$

On the other hand, we have that

$$f^*_c(Z^Y) = \sum_{i \in I} \sum_{j \in J} c^*_{ij} z^*_{ij} = \sum_{i \in I} \sum_{j \in J} c_{ij} y_{ij} = g_{\lambda,h}(Y). \tag{4.57}$$

Based on (4.55) - (4.57) it results that $g_{\lambda,h}(Y) > g_{\lambda,h}(Y^Z)$, which contradicts the optimality of Y. Therefore, Z^Y is an optimal solution of the problem $(AC^*_{\lambda,h})$. ■

Proposition 4.3.12 (Tuns (Bode) O.R. [?]). *If $Z \in \mathcal{Z}$ is an optimal solution of the problem $(AC^*_{\lambda,h})$, then Y^Z is an optimal solution of the problem $(AC_{\lambda,h})$.*

Proof. Let us suppose that Y^Z is not an optimal solution of the problem $(AC_{\lambda,h})$. Then, $\exists\, Y^* \in \mathcal{Y}$ such that

$$g_{\lambda,h}(Y^*) < g_{\lambda,h}(Y^Z). \tag{4.58}$$

Taking Z^{Y^*} given by (4.53), it results that $Z^{Y^*} \in \mathcal{Z}$. Then, we get that

$$f_c^*(Z^{Y^*}) = \sum_{i\in I}\sum_{j\in J} c_{ij}^* z_{ij}^{Y^*} = \sum_{i\in I}\sum_{j\in J} c_{ij} y_{ij}^* = g_{\lambda,h}(Y^*). \tag{4.59}$$

On the other hand, we have that

$$f_c^*(Z) = \sum_{i\in I}\sum_{j\in \overline{J}} c_{ij}^* z_{ij} = \sum_{i\in I}\sum_{j\in J} c_{ij} z_{ij} = \sum_{i\in I}\sum_{j\in J} c_{ij} y_{ij}^Z = g_{\lambda,h}(Y^Z). \tag{4.60}$$

Based on (4.58) - (4.60) it results that $f_c^*(Z^{Y^*}) < f_c^*(Z)$, which contradicts the optimality of Z. ■

By using the the potential plan theorem, we have:

Theorem 4.3.13 *$Z = [z_{ij}] \in \mathcal{Z}$ is an optimal solution of the problem $(AC^*_{\lambda,h})$ if and only if is a potential plan, i.e. there exist the numbers $(u_1, \ldots, u_m, v_1, \ldots, v_{n+1})$ such that*

$$v_j - u_i \le c_{ij}^*,\ \forall\, i \in I,\ \forall\, j \in \overline{J} \tag{4.61}$$

and

$$v_j - u_i = c_{ij}^*,\ \forall\, i \in I,\ \forall\, j \in \overline{J} \text{ with } z_{ij} \neq 0. \tag{4.62}$$

Furthermore, by using the above result and the particularities of the problem $(AC_{\lambda,h})$, we give a necessary and sufficient condition such that $Y \in \mathcal{Y}$ to be an optimal solution of this problem.

Theorem 4.3.14 (Tuns (Bode) O.R. [?]). *$Y \in \mathcal{Y}$ is an optimal solution of the problem $(AC_{\lambda,h})$ if and only if*

$$\sum_{s\in I} c_{sj} y_{sj} \ \le\ c_{ij},\ \forall\ i \in I,\ \forall\ j \in J. \tag{4.63}$$

Proof.

Necessity: Let $Y \in \mathcal{Y}$ be an optimal solution of the problem $(AC_{\lambda,h})$. Then, Z^Y is an optimal solution of the problem $(AC^*_{\lambda,h})$, and, based on Theorem 4.3.13, it results that $\exists(u_1^*, \ldots, u_m^*, v_1^*, \ldots, v_n^*, v_{n+1}^*)$ such that

$$v_j^* - u_i^* \le c_{ij}^*,\ \forall\, i \in I,\ \forall\, j \in \overline{J} \tag{4.64}$$

and

$$v_j^* - u_i^* = c_{ij}^*,\ \forall\, i \in I,\ \forall\, j \in \overline{J},\ \text{such that } z_{ij} = 1. \tag{4.65}$$

Therefore, based on the way we have set Z^Y, we deduce that we have

$$v_j^* - u_i^* \leq c_{ij},\ \forall\, i \in I,\ \forall\, j \in J \tag{4.66}$$

and

$$v_j^* - u_i^* = c_{ij},\ \forall\, i \in I,\ \forall\, j \in J,\ \text{such that } y_{ij} = 1. \tag{4.67}$$

Taking $u_i = 0$ and $v_j = v_j^* - u_i^*$, based on (4.66) and (4.67), we get that

$$v_j \leq c_{ij},\ \forall\, i \in I,\ \forall\, j \in J \tag{4.68}$$

and

$$v_j = c_{ij},\ \forall\, i \in I,\ \forall\, j \in J,\ \text{such that } y_{ij} = 1. \tag{4.69}$$

We know that $\forall\, j \in J$, $\exists_u\, i$, denoted by i_j, such that

$$y_{i_j j} = 1 \text{ and } y_{ij} = 0,\ \forall\, i \in I \backslash \{i_j\}. \tag{4.70}$$

Therefore, (4.69) is equivalent with

$$v_j = c_{i_j j},\ \forall\, j \in J. \tag{4.71}$$

From (4.71) and (4.68) we obtain that

$$c_{i_j j} \leq c_{ij},\ \forall\, i \in I,\ \forall\, j \in J. \tag{4.72}$$

Based on (4.70) we deduce that

$$\sum_{s \in I} c_{sj} y_{sj} = c_{i_j j}. \tag{4.73}$$

Now, from (4.72) and (4.73) we get that

$$\sum_{s \in I} c_{sj} y_{sj} \leq c_{ij},\ \forall\, i \in I,\ \forall\, j \in J. \tag{4.74}$$

Sufficiency: Let $Y \in \mathcal{Y}$ be such that is satisfying (4.63). We prove that Y is an optimal solution of the problem $(AC_{\lambda,h})$.

Based on Propositions 4.3.9 and 4.3.10 it results that it is sufficient to prove that Z^Y is an optimal solution of the problem $(AC^*_{\lambda,h})$, i.e. Z^Y is a potential plan.

Let us consider

$$u_i^* = 0,\ \forall\, i \in I, \tag{4.75}$$

$$v_j^* = \sum_{s \in I} c_{sj} y_{sj},\ \forall\, j \in J \tag{4.76}$$

and

$$v_{n+1}^* = 0. \tag{4.77}$$

Therefore, $\forall\, i \in I$, $\forall\, j \in J$, we have

$$-u_i^* + v_j^* = \sum_{s\in I} c_{sj} y_{sj} \leq c_{ij} = c_{ij}^*. \tag{4.78}$$

Also, we have $-u_i^* + v_{n+1}^* = 0$, $\forall\, i \in I$.

On the other hand, from (4.61), we have that $c_{i,n+1}^* = 0$, $\forall\, i \in I$. Therefore,

$$-u_i^* + v_{n+1}^* = c_{i,n+1}^*,\ \forall\, i \in I. \tag{4.79}$$

From (4.78) and (4.79) it results that

$$-u_i^* + v_j^* \leq c_{ij}^*,\ \forall\, i \in I,\ \forall\, j \in \overline{J}. \tag{4.80}$$

Now, let $(i,j) \in I \times J$ be such that $z_{ij}^Y \neq 0$.

If $j = n+1$, then (4.79) holds.

If $j \in J$, then $\exists_u\, i_j \in I$ such that $y_{i_j j} \neq 0$ and $y_{ij} = 0$, $\forall\, i \in I \setminus \{i_j\}$.

Hence, based on (4.53), we get that $z_{i_j j}^Y \neq 0$ and $z_{ij}^Y = 0$, $\forall\, i \in I \setminus \{i_j\}$.

Based on (4.78) we have $-u_{i_j}^* + v_j^* = \sum\limits_{s\in I} c_{sj} y_{sj} = c_{i_j j} y_{i_j j} = c_{i_j j} z_{i_j j}$. It results that $(u_1^*, \ldots, u_m^*, v_1^*, \ldots, v_n^*, v_{n+1}^*)$ represents a potential system. Therefore, Z^Y is an optimal solution of the problem $(AC_{\lambda,h}^*)$.

In view of Proposition 4.3.12, Y is an optimal solution of the problem $(AC_{\lambda,h})$. ■

From Proposition 4.3.6, Proposition 4.3.7 and Theorem 4.3.14 it results the following important result.

Theorem 4.3.15 (Tuns (Bode) O.R. [?]). *The matrix $\tilde{Y} \in \mathcal{Y}$ is an optimal solution of the problem (PS) if and only if $\tilde{Y}$ it is an optimal solution of the problem (PM) and*

$$\sum_{s\in I} c_{sj} y_{sj} \ \leq\ c_{ij},\ \forall\, i \in I,\ \forall\, j \in J, \tag{4.81}$$

where

$$c_{ij} = \begin{cases} 0, & \text{if} \quad r_{ij} > f_1(\tilde{Y}), \\ 1, & \text{if} \quad r_{ij} = f_1(\tilde{Y}) \quad \text{and} \quad i \geq f_2(\tilde{Y}), \\ n+1, & \text{if} \quad (r_{ij} < f_1(\tilde{Y})) \quad \text{or} \quad (r_{ij} = f_1(\tilde{Y}) \ \text{and}\ i < f_2(\tilde{Y})). \end{cases} \tag{4.82}$$

4.3.3 A Technique for Solving the Problem (PM)

Based on Propositions 4.3.3 and 4.3.5 we give a polynomial technique for solving the problem (PM) (see Tuns (Bode) O.R. and Neamţiu L. [116]).

Let R^k be the work matrix that we set at each iteration. Initially, $R^1 = R$. The optimal solution of (PM) is memorized by the matrix $Y = [y_{ij}] \in \mathcal{M}_{m\times n}\{0,1\}$, $i \in I$, $j \in J$, and the optimal value of the function φ by the vector $F = (F_1, F_2)$. Initially, $Y = O_{m\times n}$ (null matrix).

The idea of the proposed technique is the following:

- at each iteration k it is determined the value $\lambda_k := \min\{r^k_{ij} \,|\, i \in I,\, j \in J\}$. If $\lambda_k = +\infty$, then we set $F = \varphi(Y)$ and we stop;

- at each iteration we pass through the work matrix R^k from left to right and from up to down. When we find the first element r^k_{ij} equal to λ_k we verify if the column and the line that contains it have more than two finite elements each one:

  - if *Yes*, i.e. $\exists\, p, r \in I$, $\exists\, q, s \in J$ s.t. $r^k_{pj} \neq +\infty, r^k_{rj} \neq +\infty, r^k_{iq} \neq +\infty, r^k_{ir} \neq +\infty$, then we set $r^{k+1}_{ij} := +\infty$;

  - if *No*, then we have the following three possible cases:

  (i) if the number of the finite elements from the column that contains it, is exactly two and the number of the finite elements from the line that contains it, is greater than two, i.e. $\exists\, p \in I$, $\exists\, q, s \in J$ s.t. $r^k_{pj} \neq +\infty, r^k_{hj} = +\infty$, $\forall h \in I \setminus \{i, p\}$, $r^k_{iq} \neq +\infty$, $r^k_{is} \neq +\infty$, then we set $r^{k+1}_{hj} := +\infty$, $\forall h \in I$, $y_{pj} := 1$,$a_p := a_p - 1$;

  (ii) if the number of the finite elements from the column that contains it, is exactly two and the number of the finite elements from the line that contains it, is exactly two, also, i.e. $\exists\, p \in I$, $\exists\, q \in J$ s.t. $r^k_{pj} \neq +\infty, r^k_{hj} = +\infty, \forall h \in I \setminus \{i, p\}$ and $r^k_{iq} \neq +\infty$, $r^k_{it} = +\infty, \forall t \in J \setminus \{j, q\}$, then we set $r^{k+1}_{it} := +\infty, \forall t \in J$, $r^{k+1}_{hj} := +\infty, \forall h \in I$, $y_{iq} := 1$, $y_{pj} := 1$, $a_i := a_i - 1$;

  (iii) if the number of the finite elements from the column that contains it, is greater than two and the number of the finite elements from the line that contains it, is exactly two, i.e. $\exists\, p, s \in I$, $\exists\, q \in J$ s.t. $r^k_{pj} \neq +\infty, r^k_{sj} \neq +\infty, r^k_{iq} \neq +\infty$,
  $r^k_{ih} = +\infty, \forall h \in J \setminus \{j, q\}$, then we set $r^{k+1}_{ih} := +\infty, \forall h \in I$, $y_{iq} := 1$, $a_i := a_i - 1$.

The efficiency of this technique results from the fact that we pass through the scores matrix R from up to down. The technique described above is presented below:

Input

the natural numbers m, n;

the elements of natural vector $a = (a_1, \ldots, a_m)$;
the elements of natural matrix $R = [r_{ij}],\ i \in \{1, \ldots, m\},\ j \in \{1, \ldots, n\}$;

Output

ok — *true* if a solution exists,
$Y = [y_{ij}],\ i \in \{1, \ldots, m\},\ j \in \{1, \ldots, n\}$ and $F = (F_1, F_2)$ — the solution

Algorithm

```
ok := false; sw := 0;
for j = 1 to n do
   s_j := 0;
   for i = 1 to m do
      y_ij := −1;
   end for
end for
I := {1, ..., m}; J := {1, ..., n};
while J ≠ ∅ do
   r := min{r_ij | i ∈ I, j ∈ J};
   for i = 1 to m do
      if i ∈ I then
        for j = 1 to n do
          if j ∈ J then
            if r_ij = r then
              s_j := s_j + 1; r_ij := +∞;
              if s_j = m then
                y_ij := 1;
                if sw = 0 then
                  F_1 := r_ij; F_2 := i; sw := 1;
                end if
                a_i := a_i − y_ij;
                if a_i < 1 then I := I \ {i};
                end if
                J := J \ {j};
              else
                y_ij := 0;
                if s_j = m − 1 then
                  s_j := m;
```

for $k = 1$ to m do
if $k \in I$ then
if $y_{kj} = -1$ then
$y_{kj} := 1$; $r_{kj} := +\infty$; $a_k := a_k - y_{kj}$;
if $a_k = 0$ then $I := I \setminus \{k\}$;
end if
end if
end if
end for
end if
end if
end if
end if
end for
end if
end for
end while
if $I = \emptyset$ then $OK := false$;
end if
if $OK = false$ then
output: there is an error in the technique;
else
output: $Y = [y_{ij}]$ is the optimal solution of the problem (PM) and $F = (F_1, F_2)$ is the optimal value of the function φ;
end if

End Algorithm

Now, an easiest example to point out how the technique works is given.

Example 4.3.16 (Tuns (Bode) O.R. *and* Neamţiu L. [*116*]). *Let* $m = 3$, $n = 6$, $a_1 = 2$, $a_2 = 3$, $a_3 = 3$ *and the matrices*

$$R = R^1 = \begin{bmatrix} 2 & 7 & 3 & 4 & 1 & 5 \\ 3 & 1 & 2 & 3 & 2 & 5 \\ 7 & 1 & 8 & 3 & 7 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Iteration 1: *We get* $a_1 = 1$, $a_2 = 3$, $a_3 = 3$,

$$R^2 = \begin{bmatrix} 2 & +\infty & 3 & 4 & +\infty & 5 \\ 3 & +\infty & 2 & 3 & 2 & 5 \\ 7 & +\infty & 8 & 3 & 7 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Iteration 2: *We get* $a_1 = 1$, $a_2 = 3$, $a_3 = 2$,

$$R^3 = \begin{bmatrix} +\infty & +\infty & 3 & 4 & +\infty & 5 \\ 3 & +\infty & +\infty & 3 & +\infty & 5 \\ 7 & +\infty & 8 & 3 & +\infty & +\infty \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Iteration 3: *We get* $a_1 = 0$, $a_2 = 2$, $a_3 = 0$,

$$R^4 = \begin{bmatrix} +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \\ +\infty & +\infty & +\infty & +\infty & +\infty & +\infty \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Therefore, Y *is the optimal solution of the problem* (PM) *and* $F = (4, 1)$ *is the optimal value of the function* φ.

4.4 The Study of the Problem (AEP_2)

The problem to be discussed in the present section images the modeling of a concrete economic problem which involves assigning unemployed persons to PTPs under the circumstances that, on one hand, there exists no restriction regarding the budget allocated for it and, on the other hand, under the circumstances that the maximum number of the persons that can attend the PTPs is smaller than the total number of the registered unemployed persons which need to attend the courses.

Under the above circumstances, furthermore we work under the hypothesis that

$$\sum_{i \in I} a_i < n. \tag{4.83}$$

Using the notations introduced in Subsection 4.2, we consider the following lexicographic optimization problem:

$$(PMR) \quad \begin{cases} \varphi_1(Y) = \begin{pmatrix} f_0(Y) \\ f_1(Y) \end{pmatrix} \to lex - \max - \max, \\ \sum\limits_{j \in J} y_{ij} \leq a_i, \, \forall\, i \in I, \\ \sum\limits_{i \in I} y_{ij} \leq 1, \, \forall\, j \in J, \\ y_{ij} \in \{0, 1\}, \, \forall\, i \in I, \, \forall\, j \in J, \end{cases} \tag{4.84}$$

where $f_0(Y) = \sum\limits_{i \in I} \sum\limits_{j \in J} y_{ij}$ and $f_1(Y)$ is given by (4.5).

Let us denote by Ω the set of feasible solutions of the problem (PMR), i.e.

$$\Omega = \Big\{Y = [y_{ij}] \in \mathcal{M}_{m\times n}(\{0,1\}) \,|\, \sum_{i\in I} y_{ij} \leq 1,\, \forall\, j \in J;\, \sum_{j\in J} y_{ij} \leq a_i,\, \forall\, i \in I\Big\}. \tag{4.85}$$

Based on the restrictions of the problem (PMR), we deduce that

$$\sum_{j\in J}\Big(\sum_{i\in I} y_{ij}\Big) \leq \sum_{j\in J} 1 \leq card(J) = n \tag{4.86}$$

and

$$\sum_{i\in I}\Big(\sum_{j\in J} y_{ij}\Big) \leq \sum_{i\in I} a_i. \tag{4.87}$$

Since we work under the hypothesis (4.83), the maximum value of the sum $\sum_{i\in I}\sum_{j\in J} y_{ij}$ is $\sum_{i\in I} a_i$.

We have that $Y^* = [y^*_{ij}] \in \Omega$, where

$$y^*_{ij} = \begin{cases} 1, \text{ if } i = 1,\, j \in \{1, \ldots, a_1\}, \\ 0, \text{ if } i = 1,\, j \in \{a_1 + 1, \ldots, \sum\limits_{i\in I} a_i\}, \\ 1, \text{ if } i \in I \setminus \{1\},\, j \in \Big\{\sum\limits_{k=1}^{i-1}(a_k + 1), \ldots, \sum\limits_{k=1}^{i} a_k\Big\}, \\ 0, \text{ if } i \in I \setminus \{1\},\, j \in \Big\{1, \ldots, \sum\limits_{k=1}^{i-1} a_k\Big\} \cup \Big\{\sum\limits_{k=1}^{i}(a_k + 1), \ldots, n\Big\}. \end{cases} \tag{4.88}$$

Hence, we get that

$$f_0(Y^*) = \sum_{i\in I}\sum_{j\in J} y^0_{ij} = a_1 + \ldots + a_i + \ldots + a_m.$$

Based on the above, the solving of the problem (PMR) is reduced to solving the following problem:

$$(PMR_1) \quad \begin{cases} f_1(Y) \to \max, \\ \sum\limits_{j\in J} y_{ij} = a_i,\, \forall\, i \in I, \\ \sum\limits_{i\in I} y_{ij} \leq 1,\, \forall\, j \in J, \\ y_{ij} \in \{0,1\},\, \forall\, i \in I,\, \forall\, j \in J. \end{cases} \tag{4.89}$$

Now, let

$$r_{m+1,j} := 1 + \max\{r_{ij} \,|\, i \in I,\, j \in J\},$$

$$a_{m+1} := n - \sum_{i\in I} a_i \quad \text{and} \quad \bar{I} := I \cup \{m+1\}.$$

Let us consider the problem:

$$(PMR_2) \quad \begin{cases} \min\left\{r_{ij}y_{ij} \mid i \in \bar{I},\, j \in J\right\} \to \max, \\ \sum_{j\in J} y_{ij} = a_i,\, \forall\, i \in \bar{I}, \\ \sum_{i\in \bar{I}} y_{ij} = 1,\, \forall\, j \in J, \\ y_{ij} \in \{0,1\},\, \forall\, i \in \bar{I},\, \forall\, j \in J. \end{cases} \tag{4.90}$$

Proposition 4.4.1 (Tuns (Bode) O.R. [?]). *i) If $\bar{Y} = [\bar{y}_{ij}] \in \mathcal{M}_{(m+1)\times n}(\{0,1\})$ is an optimal solution of the problem (PMR_2), then $Y^* = [y^*_{ij}] \in \mathcal{M}_{m\times n}(\{0,1\})$ is an optimal solution of the problem (PMR_1).*

ii) If $Y^* = [y^*_{ij}] \in \mathcal{M}_{m\times n}(\{0,1\})$ is an optimal solution of the problem (PMR_1), then taking

$$\bar{y}_{m+1,j} = \begin{cases} 0, \text{ if } \sum_{i\in I} y^*_{ij} = 1 \\ 1, \text{ if } \sum_{i\in I} y^*_{ij} = 0 \end{cases}, \forall\, j \in J, \tag{4.91}$$

and $\bar{y}_{ij} = y^*_{ij}$, $\forall\, i \in I$, $\forall\, j \in J$, we have that the matrix $\bar{Y} = [\bar{y}_{ij}] \in \mathcal{M}_{(m+1)\times n}(\{0,1\})$ is an optimal solution of the problem (PMR_2).

The solving of the problem (PMR_2) can be done applying the technique described in the above paragraph. We just give an easiest example to point out how the proposed technique works in this case.

Example 4.4.2 *Let $m = 2$, $n = 4$, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$ and the matrices*

$$R = \begin{bmatrix} 2 & 3 & 5 & 7 \\ 1 & 4 & 2 & 3 \end{bmatrix}$$

and

$$\bar{R} = \begin{bmatrix} 2 & 3 & 5 & 7 \\ 1 & 4 & 2 & 3 \\ 8 & 8 & 8 & 8 \end{bmatrix}.$$

We have that

$$\bar{Y} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

and

$$Y^* = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Chapter 5

Practical Applications Related to Portfolio Optimization

The complexity of economic-financial phenomena have determined that specialists in economics, finance and mathematics to appeal to different mathematical tools in order to simplify as much as possible their option evaluation. My professional experience acquired so far in the economic field proved that the mathematical tools can simplify very much the decision process of a firm's management, such that the results of its activity to be advantageous.

Decision is a very complex and difficult task. Each person must have special abilities so he/she can take a good decision in very complex cases. In real life situations, when the goal is very important, many persons does not know or are not sure what decision should take. Hence, they appeal to different help methods: informal methods (they read the horoscope, flip a coin) or formal methods (they ask for the help of some experts).

One of the many difficult tasks which assume taking an important decision is investment, because it supposes to use an amount of money in order to increase that amount. But, if investment is not done taking into account some restrictions, the result can be an disadvantageous one for the investor. That is the reason why in the present chapter my goal is to identify economic-financial problems related to portfolio theory area wherein by using the optimization theory we are guided to optimal solutions viable from the practical point of view. So, within the present chapter we focus our attention on mathematical modeling and solving problems associated with portfolio optimization.

In what follows, in Section 5.1 we begin our exposure with some basic notions related with the portfolio theory, then we recall the portfolio selection models of Markowitz type, emphasizing in mathematical terms the portfolio selection problem. Furthermore, we define the relation between the portfolio selection problem and the bicriteria optimization by introducing a new mathematical model for the portfolio selection problem. This is based on the fact that we introduce a new objective function for the investor and we treat the portfolio selection problem as a bicriteria optimization problem. Then, we introduce a boolean portfolio selection problem and propose a technique for solving it.

In Section 5.2, based on a concrete economic problem, we formulate a kind of portfolio selection problem. By using the bilevel optimization for modeling this problem, we consider two different cases which implies the mathematical study of two different problems. In each case, we mathematically model the economic problem and provide a method to solve it. For the second case we also propose a technique to solve it.

We note that the results within this chapter belong to the author and can be found in the papers authored by Lupşa L. and Tuns (Bode) O.R. [74], respectively by Tuns (Bode) O.R. [109], [110] and [111].

In [109] we treat the portfolio selection problem as a bicriteria optimization problem. This allows us to obtain a new perspective concerning the investor's objective. In this way we obtain a new mathematical model for the portfolio selection problem, in which the new objective function is equal to risk/return ratio, while adding some specified restrictions. In [74], based on a concrete economic problem, we study the case in which both the objective function of the upper level problem and of the lower level problem are linear, while the problem restrictions coincide with the ones of a E-type problem. In [110] and [111] a kind of bilevel optimization problem in 0-1 variables, based on the mathematical model attached by us to a concrete portfolio optimization problem, is analyzed. The upper level function is to be maximized, while the lower level function (which is a bicriteria function) is to be maximized-minimized in the lexicographic sense. The core idea of these papers is to present a way for solving the proposed bilevel problem by reducing it to a finite number of couples of linear pseudo boolean optimization problems. One of these last types of problems is an assignment problem.

5.1 A Relation between Portfolio Selection Problems and Bicriteria Optimization

5.1.1 Basic Notions Related to Portfolio Theory

Modern portfolio theory represents the scientific approach to investment. It deals with the selection of portfolios for investors who wish to maximize the expected return for the level of risk each investor is willing to assume.

The portfolio theory is rooted to Professor Harry Markowitz's work [80] and

[81]. H. Markowitz introduced the mathematical problem of portfolio optimization and was rewarded with a Nobel Prize in Economics in 1990 which he shared with Sharpe W. and Miller M. [82]. He presented an ingenuity approach concerning investments wherein, unlike traditional approaches that rely on technical and fundamental analysis, focuses on the entire portfolio performance analysis realized based on the return/risk ratio of its components. He formulated the portfolio selection problem as a portfolio choice based only on mean and variance of its return, and also formulated two of the fundamental models concerning the portfolio selection: minimizing the risk/variance in terms of a constant average return or maximizing the average return in terms of a constant variance. These two models led to the definition of an efficient frontier, which helps the investor to choose his/her optimal portfolio.

Among major works which describes *more mathematically* the portfolio theory we recall the papers authored by Huang C.F. and Litzenberger R.H. [53] and by Ingersoll J.E. [55]. Some papers with respect to this area have been published by Maringer D. [78], Meucci A. [86] and by Wang S. and Xia Y. [127]. An interesting and concise overview of the most important models used in portfolio optimization is given in Rădulescu M., Rădulescu S. and Rădulescu C.Z. [96].

The portfolio theory is used in real life situations in order to choose the optimal combination of investments objects such as the investor for a certain level of risk that is willing to assume to secure the biggest possible return, or for a certain level of return to assume the smallest possible risk. The portfolio wherewith the investor can achieve the above represents an *optimal portfolio*. This type of portfolio has an important place in the portfolio theory area. The selection of the optimal portfolios for each investor is not just a problem of finding attractive investments and cannot be done by human intuition only. It requires the use of some *mathematical tools*. These tools are used by many financial companies and institutions to select and diversify their portfolios, and by business and management schools for teaching and research.

The modern portfolio theory is the economists creation trying to understand the market as a whole and not investigating each investment individually. The goal is that each investor to identify the level of risk that is willing to assume and, then, to find for this level of risk the portfolio with the biggest return.

An investment portfolio represents all the investment objects owned by an investor in order to achieve an investment activity adequate to the investment strategy established. In other words, a portfolio is an adequate combination of different investment objects owned by a company or by a person. A portfolio can contain bank deposits, bonds, stocks, real estates and any other financial instrument which is accepted to hold its value in time. In the paper *Le choix des investissments* (Dunod, 1959), Masse P. said that the investment activity represents an exchange between an immediate and certain satisfaction which is gave up and a realized hope which base is represented by the object that invests.

Portfolio optimization is not something that can be done easily by small calculations. There are many mathematical models used in practice to identify how to build an optimal portfolio. For portfolios, optimization models includes an objective function (to be maximized or minimized) and several restrictions that limit achieving the proposed goal. Many models, although there were perfect in terms of logic and construction, when they were applied in practice, failed.

Portfolio optimization is also called mean-variance optimization because the term mean refers to the mean or the expected return of the investment, and the term variance refers to the measure of the risk associated with the portfolio. The mathematical problem can be formulated in many ways. We recall just three types of such ways: maximize the expected return for a specified risk, minimize the risk for a specified expected return, and minimize the risk and maximize the expected return using a specified risk aversion factor. All these problems could have equality and inequality constraints, or linear or nonlinear constraints.

We recall that the two main characteristics of a portfolio are the expected return and the risk measured by the variance (or the standard deviation, which is the square-root of the variance). It is well-known that any investor would like to have the highest return possible from an investment. But this has to be counterbalanced by the amount of risk the investor is willing to assume. Unfortunately, portfolios with high returns usually correlate with high risk.

As already mentioned, MARKOWITZ developed a model of portfolio selection introducing the term of *efficient portfolio*. This model has as starting points the risk and return of a diversified portfolio of securities. In his paper [80], MARKOWITZ presented the theory of optimal portfolio selection by an investor, under the circumstances of the balance between the risk and return, emphasizing the idea of portfolio diversification used as a method of risk reduction. This can be translated into popular language by the expression "do not keep all the eggs in one basket". Later, MARKOWITZ explained his ideas more clearly in his well-known works [81] and *Mean-Variance Analysis in Portfolio Choice and Capital Markets*, Basil Blackwell Ltd., Cambridge, Massachusetts, 1987.

Despite the fact that the portfolio selection models of Markowitz's type are considered to be simplified models because they take into account only the average and variance of a portfolio profit, they are the most commonly used nowadays, remaining the cornerstone of modern portfolio theory.

5.1.2 Portfolio Selection Models of Markowitz Type

Let us suppose there is a portfolio that consists of n financial assets. An investor, disposing by an amount of money M (positive number), wants to invest in these financial assets available on the market. The financial assets can be bonds, securities of the financial investment companies or can be stocks of the companies listed on the stock exchange. For each asset j, $j \in \{1, 2, ..., n\}$, we denote by s_j the sum invested in it. An important aspect in the portfolio selection theory is the infinite divisibility of the financial assets. This hypothesis suppose that the investor can purchase any quantity of a financial asset which is represented by a positive real number. In real life situations this is impossible since the investor can purchase from the stock exchange only a natural number of financial assets. Therefore, the amount invested in order to purchase a financial asset must be multiple of the financial asset value.

Let us denote by r_j the rate of return for each asset j, $j \in \{1, 2, ..., n\}$, (random variable), by μ_j the mean of the random variable r_j and by

$$c_{ij} = E[(r_i - \mu_i)(r_j - \mu_j)] \tag{5.1}$$

the covariance of the random variables r_i and r_j.

We remark that the vector $s = (s_1, s_2, ..., s_n) \in \mathbb{R}^n$ describes a possible investment policy only if it fulfills the following restrictions:

$$\begin{cases} \displaystyle\sum_{j=1}^{n} s_j = M, \\ s_j \geq 0, \ j \in \{1, ..., n\}. \end{cases} \tag{5.2}$$

As it was introduced by Markowitz H. [80], very used in the literature from this area of research and also pointed out by Rădulescu M., Rădulescu S. and Rădulescu C.Z. [96], the portfolio return is defined by $V(s) = \sum_{j=1}^{n} r_j s_j$. Hence, the mean value of the portfolio return is given by $E(V(s)) = \sum_{j=1}^{n} \mu_j s_j$. The risk of the investment is defined by the dispersion of the portfolio return $D^2(V(s)) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} s_i s_j$. The investor's goal is to get, concomitantly, the maximum return with a minimum level of the risk assumed.

Let us now consider Ω the set of all possible investment policies, i.e.

$$\Omega = \{s = (s_1, \ldots, s_n) \in \mathbb{R}_+^n | s_1 + \ldots + s_n = M\}.$$

Let $f_1 : \mathbb{R}_+^n \to \mathbb{R}_+$ be the risk function, given by

$$f_1(s) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} s_i s_j, \ \forall \ s = (s_1, ..., s_n) \in \Omega. \tag{5.3}$$

Let $f_2 : \mathbb{R}_+^n \to \mathbb{R}_+$ be the return function, given by

$$f_2(s) = \sum_{j=1}^{n} r_j s_j,\ \forall\ s = (s_1, ..., s_n) \in \Omega. \tag{5.4}$$

Based on the above, we formulate in mathematical terms the following portfolio selection problem, denoted by us with (PP): determine a point $s^0 \in \Omega$ such that

$$f_1(s^0) \ = \ \min_{s\in\Omega} f_1(s) \ \text{ and } \ f_2(s^0) \ = \ \max_{s\in\Omega} f_2(s).$$

In the case of existence, the point s^0 is called *the ideal point*. In real life situations, such a point does not often exists. That is why, in practice, even though maximizing the return of the portfolio might be tempting for the investor, he/she must take into account at all times the risk of the portfolio. Most investors want to have a balance between the maximum return and minimum risk. That is why there were formulated several models for an optimal portfolio. The three classic portfolio selection models of Markowitz type given in M. Rădulescu, S. Rădulescu and C.Z. Rădulescu [96] are:

I. *The problem to maximize the expected return given a specific risk level* $\overline{m}$*:*

$$(\tilde{\mathrm{P}}_{\max}) \qquad \begin{cases} f_2(s) \to \max, \\ f_1(s) \le \overline{m}, \\ s \in \Omega. \end{cases} \tag{5.5}$$

The $(\tilde{P}_{\max})$ problem is a quadratic optimization problem with linear objective function and with a quadratic restriction.

II. *The problem to minimize the risk given a specific expected return level* $\underline{\mathrm{m}}$*:*

$$(\tilde{\mathrm{P}}_{\min}) \qquad \begin{cases} f_1(s) \to \min, \\ f_2(s) \ge \underline{m}, \\ s \in \Omega. \end{cases} \tag{5.6}$$

The $(\tilde{P}_{\min})$ problem is a quadratic optimization problem with the objective function defined by a quadratic form semi-positive defined and with linear restrictions. It is very difficult to solve numerically this problem when n is greater than 500.

We remark that the portfolios which maximize the expected return given a specific risk level or minimize the risk given a specific expected return level are called *efficient portfolios*. The set of all efficient portfolios is called *efficient frontier*.

III. *The problem of balancing the return and the risk:*

$$(\tilde{\mathrm{P}}_{\alpha}) \begin{cases} (1-\alpha) f_1(s) + \alpha(-f_2)(s) \ \to \ \min, \\ s \in \Omega, \end{cases} \tag{5.7}$$

where $\alpha \in [0,1]$ is a parameter which reflects the importance that each term has.

We remark that if we set $\alpha = 0$ then the problem $(\tilde{P}_\alpha)$ becomes the problem $(\tilde{P}_{\min})$, and if we set $\alpha = 1$ then it becomes the problem $(\tilde{P}_{\max})$. In other words, the parameter α depends on the investors's attitude regarding the risk. For example, if $\alpha = 0$ then the investor wants to gain a certain profit even it has a lower value. If $\alpha = 1$ then the investor is willing to assume the risk, his only objective being to maximize the expected return.

Furthermore, we treat the portfolio selection problem as a bicriteria optimization problem. This allows us to obtain a new perspective concerning the investor's objective. In this way, we obtain a new mathematical model related to portfolio selection problem in which the new objective function is equal to risk/return ratio, while adding some specified restrictions.

5.1.3 Portfolio Selection Problem versus Bicriteria Optimization

Let $f = (f_1, f_2) : \Omega \to \mathbb{R}^2$ be a vector function, where f_1 is the risk function given by (5.3) and f_2 is the return function given by (5.4). We can view the general portfolio selection problem as a bicriteria minimization problem, denoted by (PV):

$$\text{(PV)} \qquad \begin{cases} (f(s)) = \begin{pmatrix} f_1(s) \\ -f_2(s) \end{pmatrix} \to \text{ v} - \min, \\ s \in \Omega. \end{cases} \tag{5.8}$$

We remark that a point $s^0 \in \Omega$ is a global optimal solution of the problem (PV) if s^0 is both a minimum point of f_1 with respect to Ω and a maximum point of f_2 with respect to Ω (or a minimum point of $-f_2$ with respect to Ω). Generally, there are rare cases when global optimal points exist. Most of the times the minimum point of f_1 with respect to Ω is not the maximum point of f_2 with respect to Ω and vice versa. Therefore, for the problem (PV) the global optimal solutions usually do not exist (only in very special cases).

Let us remark that if we use the ε-restrictive method (see [5] and [26]) for solving the (PV) problem, then for $\varepsilon = \underline{m}$ we find ourselves solving the $(\tilde{P}_{\min})$ problem and for $\varepsilon = \overline{m}$ we find ourselves solving the $(\tilde{P}_{\max})$ problem. If we use the weighted method with $w_1 = \alpha$ and $w_2 = 1 - \alpha$, where $\alpha \in [0,1]$, we come to solve the $(\tilde{P}_\alpha)$ problem.

Furthermore, in order to solve the portfolio selection problem we set a preference relation, in which we use a synthesis function and also the idea of the sv-balanced point given by Neamţiu L. [89]. We start from the remark that, in real life situations, usually the risk can not exceed a specified value $\overline{m}$ and the expected return must be greater than a given value $\underline{m}$. These values are determined based on historical data. Taking into account that all the time the goal is to obtain a positive return, we consider $\underline{m} > 0$.

Let

$$\tilde{\Omega} = \{s \in \Omega |\, f_1(s) \leq \overline{m},\ f_2(s) \geq \underline{m}\}.$$

Based on the idea given by STANCU-MINASIAN I.M. [107] (§ 1.1), we introduce the function $F : \Omega \to \overline{\mathbb{R}}$ given by

$$F(s) = \begin{cases} \dfrac{f_1(s)}{f_2(s)}, \text{ if } f_2(s) \neq 0, \\ +\infty, \text{ if } f_2(s) = 0. \end{cases}$$

Using the function F we introduce the following preference relation.

Definition 5.1.1 (TUNS (BODE) O.R. [*109*]). *We say that a point $s^0 \in \Omega$ is strictly preferred to another point $s \in \Omega$ with respect to relation $\succ$, and we denote by $s^0 \succ s$, if*
a) $s^0 \in \tilde{\Omega}$ and $s \notin \tilde{\Omega}$;
or
b) $s^0 \in \tilde{\Omega}$, $s \in \tilde{\Omega}$ and $F(s) > F(s^0)$.
A point $s^0 \in \Omega$ is called non-dominated if $s^0 \in \tilde{\Omega}$ and there is no point $s \in \Omega$ strictly preferred to s^0.

Proposition 5.1.2 (TUNS (BODE) O.R. [*109*]). *If the (PV) problem admits an ideal point $s^0 \in \Omega$ and the values $\overline{m}$ and $\underline{m}$ fulfill the natural conditions*

$$f_1(s^0) \leq \overline{m} \text{ and } f_2(s^0) \geq \underline{m}, \tag{5.9}$$

then s^0 is a non-dominated point with respect to the preference relation $\succ$.

Proof. From hypothesis, it results that $s^0 \in \tilde{\Omega}$. Let $s \in \Omega$. Two cases can occur:

(i) $s \notin \tilde{\Omega}$. Then, it results that $s^0 \succ s$.

(ii) $s \in \tilde{\Omega}$. Then, we have $f_2(s) \geq \underline{m} > 0$. Based on the fact that s^0 is an ideal point, it results that $f_1(s^0) \leq f_1(s)$ and $f_2(s^0) \geq f_2(s)$. Three cases can appear:

a) $f_1(s^0) = f_1(s)$ and $f_2(s^0) = f_2(s)$. Then, $F(s^0) = F(s)$. Therefore, $s \not\succ s^0$.

b) $f_1(s^0) = f_1(s)$ and $f_2(s^0) > f_2(s)$. Then, $F(s^0) = \frac{f_1(s^0)}{f_2(s^0)} = \frac{f_1(s)}{f_2(s^0)} < \frac{f_1(s)}{f_2(s)} = F(s)$. Therefore, $s^0 \succ s$.

c) $f_1(s^0) < f_1(s)$ and $f_2(s^0) \geq f_2(s)$. Then, $F(s^0) = \frac{f_1(s^0)}{f_2(s^0)} < \frac{f_1(s)}{f_2(s^0)} \leq \frac{f_1(s)}{f_2(s)} = F(s)$. Therefore, $s^0 \succ s$.

In all the above cases we get that s^0 is a non-dominated point with respect to the preference relation $\succ$. ■

Remark 5.1.3 *If $f_1(s^0) > \overline{m}$ then it results that $s^0 \notin \tilde{\Omega}$ and the choice of $\overline{m}$ it restricts the get of the ideal point. Analogously, if $f_2(s^0) < \underline{m}$ then it results that $s^0 \notin \tilde{\Omega}$ and the choice of $\underline{m}$ it restricts the get of the ideal point.*

From Definition 5.1.1 it results that a point $x^0 \in \Omega$ is a non-dominated point with respect to the preference relation $\succ$ if and only if is the optimal solution of the following optimization problem:

$$(\text{PO}) \qquad \begin{cases} F(s) \ \rightarrow \ \min, \\ \sum\limits_{j=1}^{n} s_j = M, \\ f_1(s) \leq \overline{m}, \\ f_2(s) \geq \underline{m}, \\ s \in \mathbb{R}^n_+. \end{cases}$$

Proposition 5.1.4 (Tuns (Bode) O.R. [*109*]). *If $s^0 \in \Omega$ is an optimal solution of the problem (PO) and $\underline{m} > 0$, then s^0 is a* min-*efficient point of the function $f = (f_1, -f_2)$ with respect to the set $\tilde{\Omega}$.*

Proof. Let s^0 be an optimal solution of the problem (PO). Let us suppose that it is not a min-efficient point of the function $f = (f_1, -f_2)$ with respect to $\tilde{\Omega}$. Then, there exists a point $s^* \in \tilde{\Omega}$ such that:
(i) $f_1(s^*) < f_1(s^0)$ and $f_2(s^*) \geq f_2(s^0)$
or
(ii) $f_1(s^*) = f_1(s^0)$ and $f_2(s^*) > f_2(s^0)$.
Based on $f_1(s^0) \geq 0$ and $f_2(s^0) \geq \underline{m} > 0$, in both cases we obtain that $F(s^*) < F(s^0)$. This contradicts the hypothesis. ■

Remark 5.1.5 *Based on the above and on the fact that, generally, the (PV) problem does not have a global optimal solution, we call optimal portfolio any point $s^0 \in \Omega$ which is non-dominated with respect to the preference relation $\succ$.*

5.1.4 A Kind of Boolean Portfolio Selection Problem

In the present section we emphasize the effect of the new type of objective function introduced by us by considering the particular case of the portfolio selection problem wherein s_j, $j \in \{1, \ldots, n\}$, has a boolean value, the risk function f_1 is given by (5.3) and the return function f_2 is given by (5.4). The new investor's objective function is equal to risk/return ratio.

Hence, we obtain the following optimization problem which returns the optimal investment policy:

$$
\text{(PB)} \quad \begin{cases} F(s) = \dfrac{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} c_{ij} s_i s_j}{\sum\limits_{i=1}^{n} r_i s_i} \to \min, \\ \sum\limits_{i=1}^{n} s_i = M, \\ \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} c_{ij} s_i s_j \le \overline{m}, \\ \sum\limits_{i=1}^{n} r_i s_i \ge \underline{m}, \\ s_i \in \{0,1\},\ \forall\, i \in \{1,\ldots,n\}. \end{cases} \tag{5.10}
$$

(PB) is a fractional pseudo boolean optimization problem, in which the numerator has a square shape. Making the change of variables:

$$
z_{\frac{2n(i-1)+2j-i^2-i}{2}} := s_i s_j,\ \forall\, i \in \{1,\ldots,n-1\},\ j \in \{i+1,\ldots,n\},
$$

we obtain the following fractional optimization problem, in which both, counts and denominator, are linear functions:

$$
\text{(PBL)} \quad \begin{cases} \dfrac{\sum\limits_{i=1}^{n} c_{ii} s_i + \sum\limits_{i=1}^{n-1}\sum\limits_{j=i+1}^{n} (c_{ij} + c_{ji}) z_{\frac{2n(i-1)+2j-i^2-i}{2}}}{\sum\limits_{i=1}^{n} r_i s_i} \to \min, \\ \sum\limits_{i=1}^{n} c_{ii} s_i + \sum\limits_{i=1}^{n-1}\sum\limits_{j=i+1}^{n} (c_{ij} + c_{ji}) z_{\frac{2n(i-1)+2j-i^2-i}{2}} \le \overline{m}, \\ \sum\limits_{i=1}^{n} r_i s_i \ge \underline{m}, \\ \sum\limits_{i=1}^{n} s_i = M, \\ s_i + s_j - 1 \le z_{\frac{2n(i-1)+2j-i^2-i}{2}} \le \dfrac{s_i + s_j}{2},\ \forall\, i \in \{1,\ldots,n-1\}, \\ \qquad j \in \{i+1,\ldots,n\}, \\ s_i \in \{0,1\}, i \in \{1,\ldots,n\},\ \ z_k,\ k \in \{1,\ldots,\frac{n(n-1)}{2}\}. \end{cases} \tag{5.11}
$$

In real life applications, when we use a small number of variables, we can solve the fractional pseudo boolean optimization problem (PBL) using the method given by Hammer P.L. and Rudeanu S. [50]. But this method is quite laborious. Therefore, we choose to solve the (PB) problem using an enumerative algorithm, in which we work in base two using strings and imposing a restriction of filter type. Furthermore, we give this algorithm (see Tuns (Bode) O.R. [109]).

Input

the natural number n;
the real positive numbers M, $\underline{m}$, $\overline{m}$;
the elements of positive vector $r = (r_1, \ldots, r_n)$;
the elements of positive matrix $C = [c_{ij}],\ i \in \{1, \ldots, n\},\ j \in \{1, \ldots, n\}$;

Output

ok — *true* if a solution exists,
$S0$ and $F0$ — the solution

Algorithm

$ok := false$;
$N := 2^n - 1$;
$F0 := 0$;
for i=0 to N do
  $s := (i)_2$; {the vector $s = (s_1, \ldots, s_n)$ is equal to the binary expression of i}
  $\Sigma := s_1 + s_2 + \ldots + s_n$;
  if $\Sigma = M$ then
    $f1 := c_{11}s_1 + c_{12}s_1s_2 + \ldots + c_{n,n-1}s_ns_{n-1} + c_{nn}s_n$;
    if $f1 \leq \overline{m}$ then
      $f2 := r_1s_1 + \ldots + r_ns_n$;
      if $f2 \geq \underline{m}$ then
        $F := f1/f2$;
        if $F < F0$ then
          $S0 := s$;
          $F0 := F$;
        end if;
      end if;
    end if;
  end if;
end for;
if $F0 > 0$ then
$ok := true$;
end if;
End Algorithm

The proposed algorithm is fairly successful when we used it in practical problems.

It shows a better performance for those portfolios which consists of a number of assets lower than 50. The performance of this algorithm results from the fact that we work with strings. The generated solution of this algorithm was accepted to be indeed a good practical solution.

5.2 A Kind of Portfolio Selection Problem

The problem of selecting a portfolio has been largely faced by many investors. The largest part of the portfolio selection models that can be found in the literature are based on the assumption of a perfect fractionability of the investments such that the portfolio fraction for each asset is represented by a real variable.

In the present section we study a kind of portfolio selection problem which represents the mathematical model attached to a concrete economic problem. By using the bilevel optimization for mathematically modeling this problem, we consider two different cases which implies the study of two different problems:

(i) a boolean portfolio selection problem based on a single period of investment and on the case when a stock portfolio contains stocks which have the same quotation on the capital market;

(ii) a boolean portfolio selection problem based on a period of investment divided in several subperiods of time and on the case when there are different stock portfolios on the capital market and there exists more restrictions concerning the investment.

In each case we provide a method to solve the problem. For the second problem we also propose an algorithm to solve it.

5.2.1 Formulation of the Economic Problem

We consider the following concrete economic problem:

Let S be a firm which owns n subsidiaries denoted by S_j, $j \in J = \{1, \ldots, n\}$. The firm needs to invest in some stock portfolios available on the capital market.

Let P_i, $i \in I = \{1, \ldots, m\}$, $m > n$, be the stock portfolios in which the firm S will invest. For each stock portfolio P_i, $i \in I$, the firm S has historical data based on which it can predict the expected return for a certain level of risk undertaken for a period of time T.

The firm S can make transactions with the stock portfolios in two different ways:

i) directly, through its n branches;

ii) indirectly, through p companies denoted by C_k, $k \in K = \{1, \ldots, p\}$, within a group of companies C specialized in financial investment services.

Therefore, the firm's return is equal to the sum of direct return (achieved from the investment made through its branches) and indirect return (achieved from the investment made through the specialized investment companies). In the second case, the company engaged by the firm to invest on its behalf will get its own return from the transactions made and, based on an agreed share, will yield a part of the return to the firm.

The problem that firm S needs to solve is how to choose n stock portfolios in which to invest directly (one portfolio through each branch) such that it maximizes its return.

The investment company C must invest through its subsidiaries C_k, $k \in K$, in all stock portfolios that were not chosen by the firm S to invest directly.

Therefore, the main objective of both firms is to maximize their returns, under the given circumstances.

We remark that both firms play a Stackelberg game: the leader (firm S) acts first and chooses those stock portfolios in which it will invest directly through its branches, after which the follower (company C) responds by its own transactions with the remained stock portfolios. We point out the game restrictions for both players:

- for the leader: each branch will transact with exactly one stock portfolio in such a way it maximizes its return;

-for the follower: the company (engaged by the firm to invest on its behalf) will get its own return from the transactions made and, based on an agreed share, will yield a part of the return to the firm. The investment company must transact with all stock portfolios that were not chosen by the leader to invest directly.

This economic problem leads us, thinking from the mathematical point of view, to a typical case of bilevel optimization problem.

5.2.2 Modeling and Solving the Portfolio Selection Problem

We first begin by considering the particular portfolio selection problem in the case there exists a single period of time of investment for each firm and a stock portfolio contains stocks which have the same quotation on the capital market.

Let us denote by:

- a_{ij}, the expected return of the firm S if it trades the stock portfolio P_i, $i \in I$, through its branch S_j, $j \in J$. Let $A = [a_{ij}] \in \mathbb{R}^{m \times n}$, $i \in I$, $j \in J$, be the matrix whose elements represent these expected returns;

- c_{ik}, the expected return of the company C_k, $k \in K$, if it trades the stock portfolio P_i, $i \in I$. Let $C = [c_{ik}] \in \mathbb{R}^{m \times p}$, $i \in I$, $k \in K$, be the matrix whose elements represent these expected returns;

- b_{ik}, the return yield by the company C_k, $k \in K$, to the firm S, in the case it will transact the stock portfolio P_i, $i \in I$. Let $B = [b_{ik}] \in \mathrm{I\!R}^{m\times p}$, $i \in I$, $k \in K$, be the matrix whose elements represent these yield returns;

- x_{ij}, $i \in I$, $j \in J$, the binary variable having the significance $x_{ij} = 1$ if the firm S trades through its branch S_j, $j \in J$, the stock portfolio P_i, $i \in I$, and $x_{ij} = 0$ otherwise. Let $X = [x_{ij}] \in \{0,1\}^{m\times n}$ be the matrix that represents any direct investment of the firm S;

- y_{ik}, $i \in I$, $k \in K$, the binary variable having the significance $y_{ik} = 1$ if the company C_k, $k \in K$, trades the stock portfolio P_i, $i \in I$, and $y_{ik} = 0$ otherwise. Let $Y = [y_{ik}] \in \{0,1\}^{m\times p}$ be the matrix that represents any indirect investment of the firm S.

We note that the main objective of each firm (leader and follower firm) is to maximize its return. For the leader firm, this return is get by summing up its direct return (gained through investment made through its own branches) and indirect return (gained through investment made through a specialized company). For the follower firm, this return is get by summing up the net returns obtained from the transactions made by each company C_k, $k \in K$.

Thinking in mathematical terms, we note that the mathematical model attached to this portfolio selection problem is a bilevel type problem. Furthermore, we give the mathematical model for the concrete economic problem.

Let $f_1 : \mathrm{I\!R}^{m\times n} \to \mathrm{I\!R}$ be the function given by

$$f_1(X) = \sum_{i\in I}\sum_{j\in J} a_{ij}x_{ij},\ \forall\, X = [x_{ij}] \in \mathrm{I\!R}^{m\times n}.$$

It represents the total return of the firm S gained by its direct investment.

Let $g : \mathrm{I\!R}^{m\times p} \to \mathrm{I\!R}$ be the function given by

$$g(Y) = \sum_{i\in I}\sum_{k\in K} c_{ik}y_{ik},\ \forall\, Y = [y_{ik}] \in \mathrm{I\!R}^{m\times p}.$$

It represents the total return of the companies C_k, $k \in K$.

Let $f_2 : \mathrm{I\!R}^{m\times p} \to \mathrm{I\!R}$ be the function given by

$$f_2(Y) = \sum_{i\in I}\sum_{k\in K} b_{ik}y_{ik},\ \forall\, Y = [y_{ik}] \in \mathrm{I\!R}^{m\times p}.$$

It represents the total return of the firm S gained by its indirect investment.

Let

$$\Lambda = \Big\{X = [x_{ij}] \in \{0,1\}^{m\times n} \mid \sum_{i\in I} x_{ij} = 1,\ \forall\, j \in J,\ \sum_{j\in J} x_{ij} \le 1, \forall\, i \in I\Big\}.$$

The set Λ represents the set of all possible solutions for investment of the firm S. Under the condition that if firm S chooses to invest in the portfolio P_i, $i \in I$, through its own branch then the value of x_{ij} is 1, otherwise is 0, X is an element of Λ with the restrictions that each branch of the firm S will invest in just one portfolio on the market and firm S chooses to invest in one portfolio from the total ones through its branch or not.

Now, for each $X \in \Lambda$, we set

$$U^X = \Big\{ Y^X = [y_{ik}^X] \in \{0,1\}^{m \times p} | \sum_{i \in I} y_{ik}^X = 1,\ \forall\, k \in K,\ \sum_{k \in K} y_{ik}^X = 1 - \sum_{j \in J} x_{ij},\ \forall\, i \in I \Big\}.$$

The set U^X represents the set of all possible solutions for investment of the firm C, knowing the portfolios in which firm S already invested by itself. Under the condition that if a branch C_k, $k \in K$, of the firm C chooses to invest in the portfolio P_i, $i \in I$, from the remained ones then the value of y_{ik} is 1, otherwise is 0, $Y^X \in U^X$ is a possible solution of the $(P2_X)$ problem under the restriction that each branch of the firm C must invest in just one portfolio from the remained ones on the market.

Furthermore, for each $X \in \Lambda$, let us denote by U^{*X} the set of optimal solutions of the problem

$$(P2_X) \quad \begin{cases} g(Y) \ = \displaystyle\sum_{i \in I} \sum_{k \in K} c_{ik} y_{ik} \ \rightarrow \ \max, \\ Y \in U^X. \end{cases}$$

Using the above notations, the mathematical model for the portfolio selection problem is given by the following problem:

$$(EB) \quad \begin{cases} f_1(X) + f_2(Y) \rightarrow \ \max, \\ X \in \Lambda, \\ Y \in U^{*X}. \end{cases}$$

We call the problem (EB) - *assignment bilevel cost type problem or E bilevel cost type problem.*

The particularity of the restrictions allows us to give a finite algorithm for solving the (EB) problem. The base is that if $X \in \Lambda$, then there are exactly n lines $i_1, \ldots, i_n$ such that

$$\sum_{j \in J} x_{i_h, j} = 1,\ \forall\, h \in J, \text{ and } x_{ij} = 0,\ \forall\, i \in I \setminus \{i_h | h \in J\}.$$

In economic terms, there exist exactly n portfolios such that for each portfolio there is one branch of the firm S to invest in it and this branch will not invest in the remained $p = m - n$ portfolios.

Therefore, the set U^{*X} have to be the set of optimal solutions of the problem

$$(P2_X) \quad \begin{cases} \sum\limits_{i \in (I \setminus \{i_h | h \in J\})} \sum\limits_{k \in K} c_{ik} \cdot y_{ik}^X \ \rightarrow \ \max, \\ \sum\limits_{k \in K} y_{ik}^X \ = \ 1, \quad \forall\, i \in I \setminus \{i_h | h \in J\}, \\ \sum\limits_{i \in I} y_{ik}^X \ = \ 1, \quad \forall\, k \in K, \\ y_{ik}^X = 0, \quad \forall\, i \in \{i_h | h \in J\},\, k \in K, \\ y_{ik}^X \ \in \{0,1\}, \quad \forall\, i \in I,\ k \in K. \end{cases} \tag{5.12}$$

If, in the problem (P2$_X$), we consider X as a parameter, there exists the possibility to split the set Λ in a finite number q of subsets, $q = C_m^n$. For this purpose, we introduce the set

$$V = \{v = (v_1, ..., v_m) \in \{0,1\}^m \,|\, v_1 + \ldots + v_m = n\}.$$

For each $v = (v_1, ..., v_m) \in V$, we set

$$\Lambda^v \ = \ \Big\{X = [x_{ij}] \in \Lambda \,|\, \sum_{j \in J} x_{ij} = v_i,\ \forall\, i \in I\Big\}$$

and

$$U^v \ = \ \Big\{Y = [y_{ik}] \in \{0,1\}^{m \times p} \,|\, \sum_{i \in I} y_{ik} = 1, \forall\, k \in K,\ \sum_{k \in K} y_{ik} = 1 - v_i, \forall\, i \in I\Big\}.$$

Hence,

$$\Lambda^{v'} \bigcap \Lambda^{v''} = \emptyset,\ \forall\, v', v'' \in V,\ v' \neq v'' \ \text{ and } \ \bigcup_{v \in V} \Lambda^v = \Lambda.$$

In what follows, for each $v \in V$, we consider the problems

$$(P_1^v) \quad \begin{cases} f_1(X) \ = \ \sum\limits_{i \in I} \sum\limits_{j \in J} a_{ij} x_{ij} \ \rightarrow \ \max, \\ X \in \Lambda^v \end{cases}$$

and

$$(P_3^v) \quad \begin{cases} g^v(Y) \ = \ \sum\limits_{i \in (I \setminus M_v)} \sum\limits_{k \in K} c_{ik} y_{ik} \ \rightarrow \ \max, \\ Y \in U^v, \end{cases}$$

where $M_v \ = \ \{i \in I \,|\, v_i = 1\}$.

Let F_1^v be the maximum value of f_1 on Λ^v, $\mathcal{X}^v$ the set of optimal solutions of the problem (P_1^v), G^v the maximum value of g^v on U^v, and $\mathcal{Y}^v$ the set of optimal solutions of the problem (P_3^v).

Theorem 5.2.1 (Tuns (Bode) O.R. [*110*]). *If (X^0, Y^0) is an optimal solution of the problem (EB), then there exists $v^0 = (v_1^0, \ldots, v_m^0) \in V$ such that X^0 is an optimal solution of the problem $(P_1^{v^0})$ and Y^0 is an optimal solution of the problem $(P_3^{v^0})$.*

Proof. If we consider

$$v_i^0 = \sum_{j \in J} x_{ij}^0, \ \forall\, i \in I, \tag{5.13}$$

then we have that $X^0 \in \Lambda^{v^0}$ and $U^{X^0} = U^{v^0}$. As (X^0, Y^0) is an optimal solution of the (EB) problem, it follows that $Y^0 \in U^{*v^0}$. This implies that Y^0 is an optimal solution of the $(\mathrm{P}_3^{v^0})$ problem.

Let us suppose that X^0 is not an optimal solution of the $(P_1^{v^0})$ problem. It implies that there exists $X^* \in \Lambda^{v^0}$ such that

$$f_1(X^*) > f_1(X^0). \tag{5.14}$$

As $X^* \in \Lambda^{v^0}$ and (X^0, Y^0) is an optimal solution of the problem (EB), (5.13) implies that $Y^0 \in U^{X^*}$. Based on (5.14) we get that $f_1(X^*) + f_2(Y^0) > f_1(X^0) + f_2(Y^0)$. This contradicts the hypothesis that (X^0, Y^0) is an optimal solution of the (EB) problem. Therefore, X^0 is an optimal solution of the $(P_1^{v^0})$ problem. ■

Now, let $F : V \to \mathbb{R}$, $F(v) = F_1^v + \max\{f_2(Y) | Y \in U^v\}$ and let us consider the problem:

$$(EBV) \quad \begin{cases} F(v) \to \ \max, \\ v \in V. \end{cases}$$

Theorem 5.2.2 (Tuns (Bode) O.R. [*110*]). *If function g is injective and v^0 is an optimal solution of the problem (EBV), then (X^0, Y^0) is an optimal solution of the problem (EB), for each $X^0 \in \mathcal{X}^{v^0}$ and $Y^0 \in \mathcal{Y}^{v^0}$.*

Proof. Let v^0 be an optimal solution of the problem (EBV). As g is an injective function, the set $\mathcal{Y}^{v^0}$ has just one element. Let us denote by Y^0 this element. Let $X^0 \in \mathcal{X}^{v^0}$. We prove that (X^0, Y^0) is an optimal solution of the problem (EB). As $U^{X^0} = U^{v^0}$ and $U^{*v^0} = \{Y^0\}$, it follows that (X^0, Y^0) is a feasible solution of the problem (EB). Let us suppose that is not an optimal solution. Then, there exist $X^* \in \Lambda$ and $Y^* \in U^{*X^*}$ such that

$$f_1(X^0) + f_2(Y^0) \ < \ f_1(X^*) + f_2(Y^*). \tag{5.15}$$

Taking $v_i^* = \sum\limits_{j \in J} x_{ij}^*$, for all $i \in I$, we have $v^* \in V$. As $U^{X^*} = U^{v^*}$ and $Y^* \in U^{*X^*}$, we deduce that $Y^* \in U^{*v^*}$, i.e. $f_2(Y^*) = \max\{f_2(Y) | Y \in U^{v^*}\}$. Based on (5.15) we get that

$$F(v^*) \geq f_1(X^*) + f_2(Y^*) > f_1(X^0) + f_2(Y^0) = f_1(X^0) + \max\{f_2(Y) | Y \in U^v\} = F(v^0),$$

which contradicts the optimality of v^0. Hence, (X^0, Y^0) is an optimal solution of the problem (EB). ∎

Theorems 5.2.1 and 5.2.2 allows us to reduce the solving of the problem (EB) by solving C_m^n couples of classical E-type problems. In what follows, by using an easy example, we present the way of solving the problem (EBV).

Example 5.2.3 (LUPŞA L. *and* TUNS (BODE) O.R. [74]). *Let* $m = 4$, $n = 2$, $p = 2$,

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 1 \\ 3 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \ \textit{and} \ C = \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 2 \\ 3 & 0 \end{bmatrix}.$$

Let V *be the set given by*

$$V = \Big\{ v^1 = (1,1,0,0),\ v^2 = (1,0,1,0),\ v^3 = (1,0,0,1),\ v^4 = (0,1,1,0),$$

$$v^5 = (0,1,0,1),\ v^6 = (0,0,1,1) \Big\}.$$

By solving the problems (P_1^v) and (P_3^v), $v \in V$, we obtain that:

$$\mathcal{X}^{v^1} = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \ \mathcal{Y}^{v^1} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}, \ F_1^{v^1} = 4, \ \ F(v^1) = 6.$$

$$\mathcal{X}^{v^2} = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}, \ \mathcal{Y}^{v^2} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}, \ F_1^{v^2} = 3, \ F(v^2) = 7.$$

$$\mathcal{X}^{v^3} = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}, \ \mathcal{Y}^{v^3} = \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}, \ F_1^{v^3} = 6, \ F(v^3) = 8.$$

$$\mathcal{X}^{v^4} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}, \ \mathcal{Y}^{v^4} = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}, \ F_1^{v^4} = 2, \ F(v^4) = 5.$$

$$\mathcal{X}^{v^5} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \right\},\ \mathcal{Y}^{v^5} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \right\},\ F_1^{v^5} = 5,\ F(v^5) = 7.$$

$$\mathcal{X}^{v^6} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},\ \mathcal{Y}^{v^6} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right\},\ F_1^{v^6} = 4,\ F(v^6) = 8.$$

As $\max\{F(v)|v \in V\} = 8$ and the function g is injective, in view of the Theorem 5.2.2 the optimal solution of the problem (EB) from Example 5.2.3 is

$$\left(X^0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix},\ Y^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right).$$

Remark 5.2.4 (Lupşa L. *and* Tuns (Bode) O.R. [74]). *The condition that g is an injective function is essential in the hypothesis of the Theorem 5.2.2.*

Indeed, if in Example 5.2.3, we take

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 2 \\ 3 & 0 \end{bmatrix},$$

then it is not difficult to prove that $\max\{F(v)|v \in V\} = 9$, corresponding to $v^4 = (0, 1, 1, 0)$. We have

$$\mathcal{X}^{v^4} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \right\},\ \mathcal{Y}^{v^4} = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},\ F_1^{v^4} = 2 \text{ and}$$

$$F(v^4) = \max\left\{ f_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, f_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \max\{3,\ 7\} = 7.$$

Therefore, the pair

$$\left(X^0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, Y^0 \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \right)$$

is not an optimal solution of the problem (EB). If g is not an injective function, then we have only the following result:

Corollary 5.2.5 (Lupşa L. *and* Tuns (Bode) O.R. [74]). *If v^0 is an optimal solution of the problem (EBV), then there exists $Y^0 \in \mathcal{Y}^{v^0}$ such that (X^0, Y^0) is an optimal solution of the problem (EB), for each $X^0 \in \mathcal{X}^{v^0}$.*

5.2.3 Modeling and Solving the Extended Portfolio Selection Problem

In the present section we consider the portfolio selection problem attached to the concrete economic problem in the case that: the period of time of investment is divided in several subperiods of time, there are different stock portfolios on the capital market and there exists more restrictions concerning the investment (such as, the risk that each firm is willing to assume when does the investment).

Therefore, we extend the concrete economic problem by imposing some new restrictions:

(i) we consider that firm S will invest in different stock portfolios available on the capital market. We mention that the stocks enclosed in a portfolio have the same quotation on the market. We define as *different stock portfolios* each portfolio available on the capital market that has a different market quotation in time. For some stock portfolios the quotation has an ascending trend. For other stock portfolios the price increases or decreases frequently and significantly, even if the long-term trend is known as being ascending, descending or constant;

(ii) the biggest risk that firm S is willing to assume by a direct investment shall not exceed a defined value e;

(iii) the investment company C splits the medium period of time T in shorter subperiods of time T_h, $h \in H = \{1, \ldots, s\}$, subdivisions of T. The goal of each investment company C_k, $k \in K$, is to maximize the return if it will trade the stock portfolio P_i, $i \in I$, in the subperiod of time T_h, $h \in H$, while the biggest risk that is willing to assume shall not exceed a defined value e_h, shall be as small as possible and impacts the smallest number of stock portfolios. We remark that $m - n \leq p \leq (m - n)s$.

We point out the new game restrictions for both players:

- for the leader firm: each branch will transact with exactly one stock portfolio in such a way it maximizes its return, while the biggest risk that is willing to assume itself shall not exceed the maximum value e;

- for the follower firm: the company (engaged by the leader firm to invest on its behalf) will get its own return from the transactions made and, based on an agreed share, will yield a part of the return to the leader firm. The follower firm must transact with all stock portfolios that were not chosen by the leader firm to invest directly. Also, considers the medium period of time divided in s subperiods of time T_h, $h \in H$, and one branch of the company must transact at least once with one stock portfolio in the whole period of time. The same stock portfolio can be traded in several subperiods of time, but only once in the same subperiod. As well as in the case of the leader firm, each branch of the follower firm has the restriction that the biggest risk that is willing to assume itself when transacts with one stock portfolio in a subperiod T_h, $h \in H$, shall not exceed the maximum value e_h, $h \in H$, shall be as small as possible and impacts the smallest number of stock portfolios.

Now, let us denote by:

- a_{ij}, the expected return of the firm S if it trades the stock portfolio P_i, $i \in I$, through its branch S_j, $j \in J$. Let $A = [a_{ij}] \in \mathbb{R}^{m \times n}$, $i \in I$, $j \in J$, be the matrix whose elements represent these expected returns;

- r_{ij}, the risk that firm S is willing to assume, through its branch S_j, $j \in J$, in the above case. Let $R = [r_{ij}] \in \mathbb{R}^{m \times n}$, $i \in I$, $j \in J$, be the matrix whose elements represent the values of these risks;

- c_{ikh}, the expected return of the company C_k, $k \in K$, if it trades the stock portfolio P_i, $i \in I$, in the subperiod of time T_h, $h \in H$. Let $C_h = [c_{ikh}] \in \mathbb{R}^{m \times p \times s}$, $i \in I$, $k \in K$, $h \in H$, be the matrix whose elements represent these expected returns;

- d_{ikh}, the risk that company C_k is willing to assume in the above case. Let $D_h = [d_{ikh}] \in \mathbb{R}^{m \times p \times s}$, $i \in I$, $k \in K$, $h \in H$, be the matrix whose elements represent the values of these risks;

- b_{ik}, the return yield by the company C_k, $k \in K$, to the firm S, in the case it will transact once the stock portfolio P_i, $i \in I$, in a subperiod of time T_h, $h \in H$. If this portfolio will be transact in w subperiods of time, $w \leq s$, by the company C_k, $k \in K$, the return yield to the firm S is equal to $w \cdot b_{ik}$. Let $B = [b_{ik}] \in \mathbb{R}^{m \times p}$, $i \in I$, $k \in K$, be the matrix whose elements represent these yield returns;

- x_{ij}, $i \in I$, $j \in J$, the binary variable having the significance $x_{ij} = 1$ if the firm S trades through its branch S_j, $j \in J$, the stock portfolio P_i, $i \in I$, and $x_{ij} = 0$ otherwise;

- y_{ikh}, $i \in I$, $k \in K$, $h \in H$, the binary variable having the significance $y_{ikh} = 1$ if the company C_k, $k \in K$, trades the stock portfolio P_i, $i \in I$, in the subperiod of time T_h, $h \in H$, and $y_{ikh} = 0$ otherwise.

The main objective of the follower firm is to obtain a biggest net return. This return is get by summing up the net returns obtained from the transactions made by each company C_k, $k \in K$. The return of one company C_k, $k \in K$, is get by summing up the returns obtained in each subperiod of time T_h, $h \in H$, and decreasing the share agreed to be yield to the leader firm.

Thinking in mathematical terms, we note that the mathematical model attached to this portfolio selection problem is a bilevel type problem. Furthermore, we give the mathematical model for the concrete economic problem.

Any direct investment of the firm S can be given by a matrix $X = [x_{ij}] \in \{0,1\}^{m\times n}$, and any indirect investment of the firm S by a matrix $Y = [y_{ikh}] \in \{0,1\}^{m\times p\times s}$. Therefore, an investment of the firm S is given by the pair (X, Y), where X represents the direct investment of the firm and Y represents the indirect investment of the firm.

The return of the firm S is given by the function

$$f : \{0,1\}^{m\times n} \times \{0,1\}^{m\times p\times s} \to \mathrm{IR},$$

$$f(X,Y) \;=\; \sum_{i\in I}\sum_{j\in J} a_{ij}x_{ij} \;+\; \sum_{i\in I}\sum_{k\in K} b_{ik}\Big(\sum_{h\in H} y_{ikh}\Big), \tag{5.16}$$

for all $\Big(X = [x_{ij}], Y = [y_{ikh}]\Big) \in \{0,1\}^{m\times n} \times \{0,1\}^{m\times p\times s}$.

Let $f_1 : \{0,1\}^{m\times n} \to \mathrm{IR}$ be the function given by

$$f_1(X) \;=\; \sum_{i\in I}\sum_{j\in J} a_{ij}x_{ij},\ \forall\, X \in \{0,1\}^{m\times n} \tag{5.17}$$

and $f_2 : \{0,1\}^{m\times p\times s} \to \mathrm{IR}$ be the function given by

$$f_2(Y) \;=\; \sum_{i\in I}\sum_{k\in K} b_{ik}\Big(\sum_{h\in H} y_{ikh}\Big),\ \forall\, Y \in \{0,1\}^{m\times p\times s}. \tag{5.18}$$

Obviously, $f(X,Y) = f_1(X) + f_2(Y)$.

The economic restrictions of the problem enforces the following restraints:

$$(i)\max\{r_{ij}x_{ij}\,|\, i \in I, j \in J\} \;\le\; e; \tag{5.19}$$

$$(ii)\sum_{i\in I} x_{ij} = 1,\ \text{for each } j \in J; \tag{5.20}$$

$$(iii)\sum_{j\in J} x_{ij} \;+\; \mathrm{sgn}\Big(\sum_{k\in K}\sum_{h\in H} y_{ikh}\Big) \;=\; 1,\ \text{for each } i \in I; \tag{5.21}$$

$$(iv)\max\{d_{ikh}y_{ikh}\,|\, i \in I, k \in K\} \;\le\; e_h,\ \text{for each } h \in H; \tag{5.22}$$

$$(v) \sum_{i \in I} \sum_{h \in H} y_{ikh} \geq 1, \text{ for each } k \in K; \tag{5.23}$$

$$(vi) \sum_{k \in K} y_{ikh} \leq 1, \text{ for each } h \in H, \ i \in I. \tag{5.24}$$

For the follower firm the return is given by the function

$$\varphi_1 : \{0,1\}^{m \times p \times s} \to \mathrm{I\!R}, \ \varphi_1(Y) = \sum_{i \in I} \sum_{k \in K} \sum_{h \in H} c_{ikh} y_{ikh}, \tag{5.25}$$

for all $Y = [y_{ikh}] \in \{0,1\}^{m \times p \times s}$.

The function

$$\varphi_2 : \{0,1\}^{m \times p \times s} \to \mathrm{I\!R}, \ \varphi_2(Y) = \max\{d_{ikh} y_{ikh} \,|\, i \in I, k \in K, h \in H\}, \tag{5.26}$$

for all $Y = [y_{ikh}] \in \{0,1\}^{m \times p \times s}$, gives us the maximum risk that the company C is willing to assume.

Let us denote by

$$\Omega_1 = \left\{ X = [x_{ij}] \in \{0,1\}^{m \times n} | \sum_{i \in I} x_{ij} = 1, \forall j \in J, \sum_{j \in J} x_{ij} \leq 1, \forall i \in I \right\} \tag{5.27}$$

and by

$$\mathcal{X} = \left\{ X \in \Omega_1 | \max\{r_{ij} x_{ij} \,|\, i \in I, \ j \in J\} \leq e \right\}.$$

The set Ω_1 represents the set of all possible direct investments of the firm S.

For each $X \in \Omega_1$ we set

$$\begin{aligned} W(X) = & \left\{ Y \in \{0,1\}^{m \times p \times s} \mid \operatorname{sgn}\left(\sum_{i \in I} \sum_{h \in H} y_{ikh} \right) = 1, \forall k \in K, \right. \\ & \operatorname{sgn}\left(\sum_{k \in K} \sum_{h \in H} y_{ikh} \right) = 1 - \sum_{j \in J} x_{ij}, \forall i \in I, \\ & \left. \sum_{k \in K} y_{ikh} \leq 1, \ \forall h \in H, \ \forall i \in I \right\}. \end{aligned} \tag{5.28}$$

The set $W(X)$ is the set of all possible investment solutions of the company C, not taking into account the portfolios already chosen by the firm S to invest directly.

For each $X \in \mathcal{X}$ let

$$\mathcal{Y}(X) = \left\{ Y \in W(X) \mid \max\{d_{ikh} y_{ikh} \,|\, i \in I, \ k \in K\} \leq e_h, \ \forall h \in H \right\}. \tag{5.29}$$

Using the above notations, the mathematical model attached to this economic problem is given by:

$$(EBCT)\begin{cases} f(X,Y) \;=\; \sum\limits_{i\in I}\sum\limits_{j\in J} a_{ij}x_{ij} \;+\; \sum\limits_{i\in I}\sum\limits_{k\in K} b_{ik}(\sum\limits_{h\in H} y_{ikh}) \to \max, \\ X=[x_{ij}]\in\Omega_1, \\ \max\{r_{ij}x_{ij}\,|\,i\in I,\, j\in J\}\;\leq\; e, \\ Y=[y_{ikh}]\in\mathcal{Y}^*(X)\,\text{with}\,Y\in\{0,1\}^{m\times p\times s}. \end{cases}$$

The notation $\mathcal{Y}^*(X)$ is explained later in this chapter.

Our aim is to give a method for solving the (EBCT) problem, based on its particularities (5.19) - (5.24).

The (EBCT) problem is a bilevel optimization problem for which the lower level function is bicriteria of cost-bottleneck type. As far as we know, this kind of bilevel optimization problem have not been discussed in the literature. The bilevel optimization problem presented above it is new by its particular structure of the objective function of the lower level, on one side, being of discrete variables and, on the other side, allowing the elaboration of an easiest method to solve it, under the hypothesis that all the elements of the matrices A, B, C_h, D_h, respectively R, and e, e_h, $h\in H$, are natural non-zero numbers.

Furthermore, we adapt the notion of $\max -p\min -\max$ point in the context of our problem. Then, by using the splitting technique, a method for solving the (EBCT) problem is proposed. Several necessary and sufficient optimal conditions (useful for solving the problems gained after applying the splitting technique) are proved. In the end, an algorithm for solving the initial problem is proposed and an example is given.

Let $\varphi=(\varphi_1,\varphi_2,f_2):\{0,\,1\}^{m\times p\times s}\to \mathrm{I\!R}^2$ be the vector function given by

$$\varphi_1(Y) \;=\; \sum_{i\in I}\sum_{k\in K}\sum_{h\in H} c_{ikh}y_{ikh}, \tag{5.30}$$

$$\varphi_2(Y) \;=\; \max\{d_{ikh}y_{ikh}\,|\,i\in I, k\in K, h\in H\}, \tag{5.31}$$

$$f_2(Y) \;=\; \sum_{i\in I}\sum_{k\in K} b_{ik}\Big(\sum_{h\in H} y_{ikh}\Big), \tag{5.32}$$

for all $Y=[y_{ikh}]\in\{0,\,1\}^{m\times p\times s}$.

The set $\varphi_2(\{0,1\}^{m\times p\times s})$ is a finite set. Let q be the cardinal of this set and let

$$\varphi_2(\{0,1\}^{m\times p\times s}) \;=\; \{z_1,\ldots,z_q\}.$$

We suppose that

$$z_t \;>z_{t+1}, \quad \text{for} \quad \text{each} \quad t\in\{1,\ldots,q-1\}. \tag{5.33}$$

For each $t \in \{1, \ldots, q\}$, we set

$$L_t := \{(i,k,h) \in I \times K \times H \,|\, d_{ikh} = z_t\}. \tag{5.34}$$

On the set $\mathcal{Y}(X)$ we define the following strictly ordering relation, obtained by using the function φ:
if for a given two points $Y^0, Y \in \mathcal{Y}(X)$ we have:

(i) $\varphi_1(Y^0) > \varphi_1(Y)$,
or

(ii) $\varphi_1(Y^0) = \varphi_1(Y)$ and $\varphi_2(Y^0) < \varphi_2(Y)$,
or

(iii) $\varphi_1(Y^0) = \varphi_1(Y)$, $\varphi_2(Y^0) = \varphi_2(Y) = z_\tau$ and there is a natural number $r \in \{\tau, ..., q\}$ such that $\sum\limits_{(i,k,h)\in L_r} y^0_{ikh} < \sum\limits_{(i,k,h)\in L_r} y_{ikh}$ and $\sum\limits_{(i,k,h)\in L_t} y^0_{ikh} = \sum\limits_{(i,k,h)\in L_t} y_{ikh}$ for each $t \in \{\tau, ..., r-1\}$,
or

(iv) $\varphi_1(Y^0) = \varphi_1(Y)$, $\varphi_2(Y^0) = \varphi_2(Y) = z_\tau$, $\sum\limits_{(i,k,h)\in L_t} y^0_{ikh} = \sum\limits_{(i,k,h)\in L_t} y_{ikh}$, for each $t \in \{\tau, ..., q\}$, and $f_2(Y^0) > f_2(Y)$,

then we say that Y^0 is strictly better than Y and we denote this by

$$\varphi(Y^0) >_{\max - p \, \min - \max} \varphi(Y).$$

Definition 5.2.6 (TUNS (BODE) O.R. [*111*]). *A point $Y^0 \in \mathcal{Y}(X)$ is said to be a* max −p min − max *point of $\mathcal{Y}(X)$ with respect to the function φ if we have $\varphi(Y^0) >_{\max - \mathrm{p} \, \min - \max} \varphi(Y)$ or $\varphi(Y^0) = \varphi(Y)$, for all $Y \in \mathcal{Y}(X)$. We denote the set of all points of $\mathcal{Y}(X)$ which are max-*p *min-max points by*

$$arg\Big(\max - \mathrm{p} \, \min - \max\{\varphi(Y) | Y \in \mathcal{Y}(X)\}\Big) \quad \text{or,} \quad \text{simple,} \quad \text{by} \quad \mathcal{Y}^*(X).$$

The particularity of the constraints allows us to give a finite algorithm for solving the (EBCT) problem. The main important fact is that if $X \in \Omega_1$, then there are exactly n lines $i_1, \ldots, i_n$ such that

$$\sum_{j\in J} x_{i_\nu, j} = 1, \quad \forall \nu \in J \quad \text{and} \quad \sum_{j\in J} x_{ij} = 0, \quad \forall i \in I \setminus \{i_\nu | \nu \in J\}.$$

If we consider X as a parameter, then there exists the possibility of splitting the set Ω_1 in a finite number of subsets, less or equal to C_m^n. Hence, we introduce the set

$$V = \Big\{ v = (v_1, ..., v_m) \in \{0,1\}^m \,|\, v_1 + \ldots + v_m = n \Big\}.$$

For each $v = (v_1, ..., v_m) \in V$ we set

$$U^v = \{i \in I | v_i = 1\}, \quad \bar{U}^v = \{i \in I | v_i = 0\} = I \setminus U^v,$$

$$\Lambda^v = \Big\{ X = [x_{ij}] \in \{0,1\}^{m \times n} \,|\, \sum_{i \in I} x_{ij} = 1, \, \forall j \in J, \quad \sum_{j \in J} x_{ij} = v_i, \, \forall \, i \in I \Big\}$$

and

$$\mathcal{X}^v = \Big\{ X = [x_{ij}] \in \Lambda^v \,|\, \max\{r_{ij} x_{ij} \,|\, i \in U^v, \, j \in J\} \le e \Big\}.$$

It is not difficult to see that

$$\Lambda^{v'} \bigcap \Lambda^{v''} = \emptyset, \quad \forall \, v', v'' \in V, \, v' \neq v'' \quad \text{and} \quad \bigcup_{v \in V} \Lambda^v = \Omega_1.$$

Hence, if we set $\mathcal{V}_1 = \{v \in V \,|\, \mathcal{X}^v \neq \emptyset\}$, then we have $\bigcup_{v \in \mathcal{V}_1} \Lambda^v = \mathcal{X}$. Also, for each $v \in V$, we set

$$\begin{aligned} W^v = \Big\{ & Y = [y_{ikh}] \in \{0,1\}^{m \times p \times s} \,| \\ & \operatorname{sgn}\Big(\sum_{i \in \bar{U}^v} \sum_{h \in H} y_{ikh} \Big) = 1, \, \forall \, k \in K, \\ & \operatorname{sgn}\Big(\sum_{k \in K} \sum_{h \in H} y_{ikh} \Big) = 1, \, \forall \, i \in \bar{U}^v, \\ & \sum_{k \in K} y_{ikh} \le 1, \, \forall \, i \in \bar{U}^v, \, \forall \, h \in H, \\ & y_{ikh} = 0, \forall \, i \in U^v, \, \forall \, k \in K, \, \forall \, h \in H \Big\} \end{aligned}$$

and

$$\mathcal{Y}^v = \Big\{ Y \in W^v \,|\, \max\{d_{ikh} y_{ikh} \,|\, i \in \bar{U}^v, k \in K\} \le e_h, \, \forall h \in H \Big\}.$$

Let $\mathcal{V}_2 = \{v \in V \,|\, \mathcal{Y}^v \neq \emptyset\}$ and $\mathcal{V} = \mathcal{V}_1 \cap \mathcal{V}_2$.

Remark 5.2.7 (Tuns (Bode) O.R. [*111*]). *For a given $v \in V$ and $X \in \Lambda^v$, it is obvious that:*

$$x_{ij} = 0, \quad \forall \, i \in \bar{U}^v, \quad \forall \, j \in J; \tag{5.35}$$

$$\sum_{j \in J} x_{ij} = 1, \forall \, i \in U^v, \quad \sum_{i \in U^v} x_{ij} = 1, \quad \forall \, j \in J; \tag{5.36}$$

$$\max\{r_{ij} x_{ij} \,|\, i \in I, \, j \in J\} = \max\{r_{ij} x_{ij} \,|\, i \in U^v, \, j \in J\}. \tag{5.37}$$

Remark 5.2.8 (Tuns (Bode) O.R. [*111*]). *Based on the above equalities, we deduce that:*

(i) If $X \in \mathcal{X}$, $Y \in \mathcal{Y}(X)$ and $v = (\sum_{j\in J} x_{1j}, \ldots, \sum_{j\in J} x_{mj})$, then $v \in \mathcal{V}$, $X \in \mathcal{X}^v$ and $Y \in \mathcal{Y}^v$.

(ii) If $v \in \mathcal{V}$, $X \in \mathcal{X}^v$ and $Y \in \mathcal{Y}^v$, then $X \in \mathcal{X}$ and $Y \in \mathcal{Y}(X)$.

(iii) If $v \in \mathcal{V}$ and X', $X'' \in \mathcal{X}^v$, then $\mathcal{Y}^*(X') = \mathcal{Y}^*(X'')$.

Statements (i) and (ii) are trivial. Let us prove statement (iii). As X', $X'' \in \mathcal{X}^v$ we have

$$1 - \sum_{j\in J} x'_{ij} = 1 - \sum_{j\in J} x''_{ij} = v_i, \quad \forall\, i \in I.$$

Then $W(X') = W(X'')$. It follows that $\mathcal{Y}(X') = \mathcal{Y}(X'')$. Hence, $\mathcal{Y}^*(X') = \mathcal{Y}^*(X'')$.

In what follows, for each $v \in \mathcal{V}$ we consider the problem

$$(\mathrm{P}_1^v) \quad \begin{cases} f_1(X) \to \max, \\ X \in \mathcal{X}^v, \end{cases}$$

where $f_1(X)$ is given by (5.17). Let us denote by $\mathcal{X}_0^v$ the set of optimal solutions of the problem (P_1^v).

We denote by (P_2^v) the problem of determining the $\max - p\ \min - \max$ points of the vector function φ with respect to the set $\mathcal{Y}^v$ and by $\mathcal{Y}_0^v$ the set of such kind of points.

Theorem 5.2.9 (Tuns (Bode) O.R. [*111*]). *If (X^0, Y^0) is an optimal solution of the (EBCT) problem and*

$$v^0 = \Big(\sum_{j\in J} x^0_{1j}, \ldots, \sum_{j\in J} x^0_{ij}, \ldots, \sum_{j\in J} x^0_{mj} \Big), \tag{5.38}$$

then $v^0 \in \mathcal{V}$, $X^0 \in \mathcal{X}_0^{v^0}$ and $Y^0 \in \mathcal{Y}_0^{v^0}$.

Proof. If (X^0, Y^0) is an optimal solution of the (EBCT) problem, then in view of Remark 5.2.8, i), we get that $v^0 \in \mathcal{V}$, $X^0 \in \mathcal{X}^{v^0}$ and $Y^0 \in \mathcal{Y}^{v^0}$.

Let us suppose that X^0 is not an optimal solution of $(\mathrm{P}_1^{v^0})$ problem and let X be a feasible solution of this problem such that

$$f_1(X) > f_1(X^0). \tag{5.39}$$

As X, $X^0 \in \mathcal{X}^{v^0}$ and $Y^0 \in \mathcal{Y}^*(X^0)$, in view of Remark 5.2.8, iii), we get that $Y^0 \in \mathcal{Y}^*(X)$, too. Therefore, (X, Y^0) is a feasible solution of the (EBCT) problem.

As $f(X, Y^0) = f_1(X) + f_2(Y^0) > f_1(X^0) + f_2(Y^0) = f(X^0, Y^0)$, the optimality of (X^0, Y^0) is contradicted. Therefore, X^0 is an optimal solution of the $(\mathrm{P}_1^{v^0})$ problem.

Let us suppose that Y^0 is not a max $-p$ min $-$ max point of the vector function φ with respect to the set $\mathcal{Y}^{v^0}$ and let $Y \in \mathcal{Y}^{v^0}$ be such that

$$\varphi(Y) >_{\max - p \min - \max} \varphi(Y^0). \tag{5.40}$$

Two cases can occur:

Case 1. If $\Big(\varphi_1(Y) > \varphi_1(Y^0)\Big)$ or $\Big(\varphi_1(Y) = \varphi_1(Y^0)$ and $\varphi_2(Y) < \varphi_2(Y^0)\Big)$ or $\Big(\varphi_1(Y) = \varphi_1(Y^0), \varphi_2(Y) = \varphi_2(Y^0) = z_\tau$ and there exists a natural number $r \in \{\tau, ..., q\}$ such that $\sum\limits_{(i,k,h)\in L_t} y^0_{ikh} = \sum\limits_{(i,k,h)\in L_t} y_{ikh}$, for all $t \in \{\tau, ..., r-1\}$, and $\sum\limits_{(i,k,h)\in L_r} y^0_{ikh} < \sum\limits_{(i,k,h)\in L_t} y_{ikh}\Big)$, then (5.40) contradicts the fact that $Y^0 \in \mathcal{Y}^*(X^0)$.

Case 2. Now, let us suppose that $\varphi_1(Y) = \varphi_1(Y^0)$, $\varphi_2(Y) = \varphi_2(Y^0) = z_\tau$, $\sum\limits_{(i,k,h)\in L_t} y^0_{ikh} = \sum\limits_{(i,k,h)\in L_t} y_{ikh}$, for each $t \in \{\tau, ..., q\}$, and $f_2(Y) > f_2(Y^0)$. As $v^0 \in \mathcal{V}$, we have that $X^0 \in \mathcal{X}^{v^0}$ and Y is a feasible solution of the $(\mathrm{P}_2^{v^0})$ problem, i.e. $Y \in \mathcal{Y}^{v^0}$.

In view of Remark 5.2.8, ii), we get that (X^0, Y) is a feasible solution of the (EBCT) problem. But, $f(X^0, Y) = f_1(X^0) + f_2(Y) > f_1(X^0) + f_2(Y^0) = f(X^0, Y^0)$. This contradicts the optimality of (X^0, Y^0).

As, in both cases, we get a contradiction, Y^0 is a max $-p$ min $-$ max point of the vector function φ with respect to the set $\mathcal{Y}^{v^0}$. ∎

We remark that if $\mathcal{V} \neq \emptyset$, then the values of the function f_1, for all $X \in \mathcal{X}_0^v$, are the same. The vector function φ has the same value for all $Y \in \mathcal{Y}_0^v$, too.

Therefore, we can consider the function $F : \mathcal{V} \to \mathbb{R}$ given by

$$F(v) = \sum_{i\in U^v} \sum_{j\in J} a_{ij} x_{ij} + \sum_{i\in \bar{U}^v} \sum_{k\in K} b_{ik} \left(\sum_{h\in H} y_{ikh} \right), \quad \forall\, v \in \mathcal{V}, \tag{5.41}$$

where $X \in \mathcal{X}_0^v$ and $Y \in \mathcal{Y}_0^v$ are chosen arbitrary.

Now, let us consider the problem

$$(EBVT) \quad \begin{cases} \text{Find} \quad v^0 \in \mathcal{V} \quad \text{such} \quad \text{that} \\ F(v^0) = \max\{F(v) \,|\, v \in \mathcal{V}\}. \end{cases}$$

Theorem 5.2.10 (Tuns (Bode) O.R. [*111*]). *If v^0 is an optimal solution of the (EBVT) problem, then the couple (X^0, Y^0) is an optimal solution of the (EBCT) problem, for each $X^0 \in \mathcal{X}_0^{v^0}$ and $Y^0 \in \mathcal{Y}_0^{v^0}$.*

Proof. From (5.41) we get that

$$F(v^0) \;=\; f_1(X^0) \;+\; f_2(Y^0). \tag{5.42}$$

Let us suppose that (X^0, Y^0) is not an optimal solution of the $(EBCT)$ problem, and let (X, Y) be an optimal solution of this problem. Then,

$$f(X^0, Y^0) < f(X, Y). \tag{5.43}$$

Now, let us consider the vector $v = (v_1, \ldots, v_m)$, where $v_i = \sum_{j \in J} x_{ij}$, for each $i \in I$.

As (X, Y) is an optimal solution of the (EBCT) problem, in view of Theorem 5.2.9, we get that X is an optimal solution of the (P_1^v) problem and Y is a max $-p$ min $-$ max point of the vector function φ with respect to the set $\mathcal{Y}^v$. Therefore, based on (5.41), we get that

$$F(v) = f_1(X) + f_2(Y). \tag{5.44}$$

From (5.42)-(5.44) we obtain that $F(v^0) < F(v)$, which contradicts the optimality of v^0. ■

Theorems 5.2.9 and 5.2.10 allow us to reduce the solving of the (EBCT) problem by solving at most C_m^n couples of classical pseudo boolean optimization problems. In real cases, the number of such couples of problems is decreased based on the following restrictions:

$$\max\{r_{ij}x_{ij} \,|\, i \in U^v,\; j \in J\} \;\le\; e,$$

$$\max\{d_{ikh}y_{ikh} \,|\, i \in \bar{U}^v,\; k \in K\} \le e_h, \text{ for each}\, h \in H.$$

Furthermore, we analyze the way in which the (P_1^v) and (P_2^v) problems can be transformed in new problems such that, after applying the transformations, it become easier to solve. Therefore, we apply some transformations to the matrices A and C_h, $h \in H$.

Let us consider the number

$$\mu \;=\; 1 \;+\; \sum_{i \in I} \sum_{j \in J} a_{ij}. \tag{5.45}$$

We set the matrices $\tilde{A} = [\tilde{a}_{ij}]$ and $\tilde{C}_h = [\tilde{c}_{ikh}]$ such that

$$\tilde{a}_{ij} \;=\; \begin{cases} \mu \,+\, a_{ij}, & \text{if} \quad r_{ij} \le e \\ 0, \quad \text{if} \quad r_{ij} > e \end{cases} \quad , \quad \text{for} \quad \text{each} \quad i \in I,\, j \in J \tag{5.46}$$

and

$$\tilde{c}_{ikh} \;=\; \begin{cases} c_{ikh}, & \text{if} \quad \mathrm{d_{ikh} \le e_h} \\ 0, \quad \text{if} \quad \mathrm{d_{ikh} > e_h} \end{cases} \quad , \quad \text{for} \quad \text{each} \quad \mathrm{i \in I,\, k \in K,\, h \in H}. \tag{5.47}$$

Remark 5.2.11 (Tuns (Bode) O.R. [*111*]). *Let $v \in V$. The following sentences are true:*

(i) If there exists $i \in U^v$ such that $\sum_{j\in J} \tilde{a}_{ij} = 0$, then the problem (P_1^v) is inconsistent. Indeed, in this case we have $r_{ij} > e$, for all $j \in J$.

(ii) If there exists $j \in J$ such that $\sum_{i\in U^v} \tilde{a}_{ij} = 0$, then the problem (P_1^v) is inconsistent. Indeed, in this case we have $r_{ij} > e$, for all $i \in U^v$.

(iii) If there exists $i \in \bar{U}^v$ such that $\sum_{k\in K}\sum_{h\in H} \tilde{c}_{ikh} = 0$, then the problem (P_2^v) is inconsistent.

(iv) If there exists $k \in K$ such that $\sum_{i\in \bar{U}^v}\sum_{h\in H} \tilde{c}_{ikh} = 0$, then the problem (P_2^v) is inconsistent.

Now, let $\tilde{f}_1 : \{0,\,1\}^{m\times n} \to \mathbb{R}$ be the function given by

$$\tilde{f}_1(X) = \sum_{i\in U^v}\sum_{j\in J} \tilde{a}_{ij}x_{ij}, \quad \forall\, X \in \{0,\,1\}^{m\times n}. \tag{5.48}$$

Let $X \in \Lambda^v$. If we set

$$\Delta(X) = \{(i,j) \in U^v \times J \,|\, x_{ij} = 1, \quad r_{ij} \le e\},$$

$$\bar{\Delta}(X) = \{(i,j) \in U^v \times J \,|\, x_{ij} = 1, \quad r_{ij} > e\},$$

then $0 \le \mathrm{card}(\Delta(\mathrm{X})) \le \mathrm{n}$ and

$$\tilde{f}_1(X) = \sum_{(i,j)\in\Delta(X)} \tilde{a}_{ij}x_{ij} + \sum_{(i,j)\in\bar{\Delta}(X)} \tilde{a}_{ij}x_{ij} = \mu\cdot \mathrm{card}(\Delta(\mathrm{X})) + \sum_{(\mathrm{i,j})\in\Delta(\mathrm{X})} \mathrm{a_{ij}}. \tag{5.49}$$

For each $v \in V$ we consider the problem

$$(\mathrm{PM_1^v}) \quad \begin{cases} \tilde{f}_1(X) \to \max, \\ x \in \Lambda^v. \end{cases} \tag{5.50}$$

Theorem 5.2.12 (Tuns (Bode) O.R. [*111*]). *Let $v \in V$.*

(i) The problem (P_1^v) has feasible solutions if and only if there exists $X^v \in \Lambda^v$ such that

$$\tilde{f}_1(X^v) > n\mu. \tag{5.51}$$

(ii) If $v \in \mathcal{V}_1$, then the problems (P_1^v) and (PM_1^v) have the same optimal solutions.

Proof. First we remark that, from $a_{ij} > 0$ for all $i \in I$, $j \in J$, we have that $0 < f_1(X) < \mu$, for all $X \in \Lambda^v$.

(i) *Necessity:* Let X^v be a feasible solution of the (P_1^v) problem. Then, card(Δ(X$^\mathrm{v}$)) = n. In view of (5.49) we obtain that $\tilde{f}_1(X^v) = n\mu + f_1(X^v) > n\mu$.

Sufficiency: Now, let $X^v \in \Lambda^v$ be such that (5.51) is fulfilled. If we suppose that $X^v \notin \mathcal{X}^v$, then there exists $(i,j) \in U^v \times J$ such that $x_{ij} = 1$ and $r_{ij} > e$. It follows that $\tilde{a}_{ij} = 0$ and card$(\Delta(X^v))\leq n-1$. Therefore, $\tilde{f}_1(X^v) \leq (n-1)\mu + f_1(X^v) < n\mu$. This contradicts (5.51).

(ii) Let $v \in \mathcal{V}_1$. Then, $\mathcal{X}^v \neq \emptyset$. Let X^v be an optimal solution of the (P_1^v) problem. Hence,

$$f_1(X^v) \geq f_1(X), \quad \forall\, X \in \mathcal{X}^v. \tag{5.52}$$

In view of (5.49) we have that $\tilde{f}_1(X) = n\mu + f_1(X)$, for all $X \in \mathcal{X}^v$. On the other hand, for each $X \in \Lambda^v \setminus \mathcal{X}^v$ we have that $\tilde{f}_1(X) < n\mu$. It follows that

$$\max\{\tilde{f}_1(X) \,|\, X \in \Lambda^v\} \;=\; \tilde{f}_1(X^v).$$

Therefore, X^v is an optimal solution of the problem (PM_1^v).

Let $v \in \mathcal{V}$ and let X^v be an optimal solution of the problem (PM_1^v).

Then, $\tilde{f}_1(X^v) \geq \tilde{f}_1(X)$, $\forall\, X \in \mathcal{X}^v \subseteq \Lambda^v$.

Based on (5.49) we get that $n\mu + f_1(X^v) \geq n\mu + f_1(X)$, $\forall\, X \in \mathcal{X}^v$. Therefore, X^v is an optimal solution of the problem (P_1^v). ■

In view of Theorem 5.2.12 the solving of the (P_1^v) problem is reduced to solving a classical assignment problem. This problem can be solved by using the exact algorithms considered by Bertsekas D.P. [9] and by Kuhn H.W. [65], or genetic algorithms considered by Sahu A. and Tapadar R. [101].

In what follows, we consider the function $\tilde{\varphi} = (\tilde{\varphi}_1, \varphi_2, f_2)$ where

$$\tilde{\varphi}_1(Y) \;=\; \sum_{i\in \bar{U}^v}\sum_{k\in K}\sum_{h\in H} \tilde{c}_{ikh}y_{ikh}, \quad \forall\, Y \in \{0,1\}^{m\times p\times s}. \tag{5.53}$$

Let $v \in V$. By (PM_2^v) we denote the problem of determining the max $-p$ min $-$ max points of the vector function $\tilde{\varphi}$ with respect to the set $\tilde{W}^v$, where

$$\tilde{W}^v \;=\; \{Y \in W^v \,|\, \varphi(Y) - \tilde{\varphi}(Y) = 0\}.$$

Let $\tilde{\mathcal{Y}}_0^v$ be the set of all max $-p$ min $-$ max points of the vector function $\tilde{\varphi}$ with respect to the set $\tilde{W}^v$.

Let $v \in V$ and $Y \in \tilde{W}^v$. We set

$$\Gamma(Y) \;=\; \Big\{(i,k,h) \in \bar{U}^v \times K \times H \,|\, y_{ikh} = 1 \quad \text{and} \quad \mathrm{d_{ikh}} \leq \mathrm{e_h}\Big\}$$

and

$$\bar{\Gamma}(Y) = \Big\{(i,k,h) \in \bar{U}^v \times K \times H \,|\, y_{ikh} = 1 \quad \text{and} \quad \mathrm{d_{ikh}} > \mathrm{e_h}\Big\}.$$

Then,

$$\tilde{\varphi}_1(Y) = \sum_{(i,k,h)\in\Gamma(Y)} \tilde{c}_{ikh} + \sum_{(i,k,h)\in\bar{\Gamma}(X)} \tilde{c}_{ikh} = \sum_{(i,k,h)\in\Gamma(Y)} c_{ikh}. \tag{5.54}$$

Theorem 5.2.13 (Tuns (Bode) O.R. [*111*]). *Let $v \in V$. Then we have:*

$$\mathcal{Y}^v = \tilde{W^v}; \tag{5.55}$$

$$\mathcal{Y}_0^v = \tilde{\mathcal{Y}}_0^v. \tag{5.56}$$

Proof. Let $Y^v \in \mathcal{Y}^v$. Then, $\bar{\Gamma}(Y^v) = \emptyset$ and $\Gamma(Y^v) = \{(i,k,h) \in \bar{U}^v \times K \times H \,|\, y^v_{ikh} = 1\}$. In view of (5.54) we get that

$$\tilde{\varphi}_1(Y^v) = \sum_{(i,k,h)\in\Gamma(Y^v)} \tilde{c}_{ikh} = \sum_{(i,k,h)\in\Gamma(Y^v)} c_{ikh} = \varphi_1(Y^v).$$

Therefore,

$$\mathcal{Y}^v \subseteq \tilde{W^v}. \tag{5.57}$$

Now, let $Y^v \in \tilde{W^v}$. If we suppose that $Y^v \notin \mathcal{Y}^v$, then it results that there exists $(i_0, k_0, h_0) \in \bar{U}^v \times K \times H$ such that $y_{i_0k_0h_0} = 1$ and $d_{i_0k_0h_0} > e_h$. Then,

$$\begin{aligned}\tilde{\varphi}_1(Y^v) &= \sum_{(i,k,h)\in\Gamma(Y^v)} c_{ikh} < \sum_{(i,k,h)\in\Gamma(Y^v)} c_{ikh} + c_{i_0k_0h_0} \\ &\leq \sum_{(i,k,h)\in\Gamma(Y^v)} c_{ikh} + \sum_{(i,k,h)\in\bar{\Gamma}(Y^v)} c_{ikh} = \varphi_1(Y^v).\end{aligned}$$

This contradicts the fact that $Y^v \in \tilde{W^v}$. Therefore,

$$\tilde{W^v} \subseteq \mathcal{Y}^v. \tag{5.58}$$

Hence, (5.57) and (5.58) implies (5.55).

It is easy to see that $\tilde{\varphi}(Y) = \varphi(Y)$, for all $Y \in \tilde{W^v}$. Then, (5.55) implies that (5.56) is valid, too. ■

The problem (PM_2^v) is a kind of pseudo boolean three objective lexicographic optimization problem. We can reduce the solving of this problem to solving a linear pseudo boolean optimization problem. To do this, we define a sequence of numbers $M, M_0, M_1, \ldots, M_q$, where q is the cardinal of the set $\varphi_2(\{0,1\}^{m\times p\times s})$. Let

$$M = 1 + \sum_{i\in I}\sum_{k\in K} b_{ik} \geq 1 + \max\{f_2(Y) | Y \in \{0,\,1\}^{m\times p\times s}\}. \tag{5.59}$$

By using the above notation, we set the numbers M_k, $k \in \{q, q-1, \ldots, 0\}$, such that

$$M_q = 1 \tag{5.60}$$

and

$$M_t = 1 + \sum_{\nu=t+1}^{q} M_\nu \cdot \mathrm{card}\,(\mathrm{L}_\nu), \quad \text{for} \quad \text{all} \quad \mathrm{t} \in \{\mathrm{q}-1, \ldots, 0\}. \tag{5.61}$$

Obviously,

$$M_0 = 1 + \sum_{\nu=1}^{q} M_\nu \cdot \mathrm{card}(\mathrm{L}_\nu). \tag{5.62}$$

We suppose that $c_{ikh} \in \mathbb{N}$, for all $(i,k,h) \in I \times K \times H$. It is not difficult to see that if Y' and Y'' are two elements of $\mathcal{Y}(X)$, then the following statements occur:

$$(i) \quad \text{if} \quad \varphi_1(\mathrm{Y}') > \varphi_1(\mathrm{Y}''), \quad \text{then} \quad -(\varphi_1(\mathrm{Y}') - \varphi_1(\mathrm{Y}'')) \leq -1; \tag{5.63}$$

$$(ii) \quad \text{if} \quad \varphi_2(\mathrm{Y}') = \mathrm{z_r} \in \{\mathrm{z_1}, ..., \mathrm{z_q}\}, \quad \text{then} \quad \sum_{(\mathrm{i,k,h}) \in \mathrm{L_r}} \mathrm{y_{ikh}} \geq 1; \tag{5.64}$$

and

$$(iii) \quad \sum_{(i,k,h) \in L_t} (y'_{ikh} - y''_{ikh}) \leq \sum_{(i,k,h) \in L_t} y'_{ikh} \leq \mathrm{card}(\mathrm{L_t}), \quad \text{for} \quad \text{all} \quad \mathrm{t} \in \{1, ..., \mathrm{q}\}. \tag{5.65}$$

Now, let $FL : \{0,\, 1\}^{m \times p \times s} \to \mathbb{R}$ be the function given by

$$FL(Y) = -M \cdot M_0 \cdot \tilde{\varphi}_1(Y) + M \sum_{t=1}^{q} \Big(M_t \cdot \sum_{(i,k,h) \in L_t} y_{ikh} \Big) - f_2(Y), \tag{5.66}$$

for all $Y \in \{0,\, 1\}^{m \times p \times s}$.

Let us consider the problem:

$$(\mathrm{PL}^v) \quad \begin{cases} FL(Y) \to \min, \\ Y \in \tilde{W}^v. \end{cases}$$

Let us denote by $\tilde{W}_0^v$ the set of all optimal solutions of the (PL^v) problem.

Theorem 5.2.14 (Tuns (Bode) O.R. [*111*]). *Let $v \in \mathcal{V}$. An element $Y^0 \in \tilde{W}^v$ is an optimal solution of the problem (PL^v) if and only if is a* max $-$p min $-$ max *point of the function $\tilde{\varphi}$ with respect to the set $\tilde{W}^v$.*

Proof. *Necessity:* Let $Y^0 = [y_{ikh}] \in \tilde{W}_0^v$. Supposing that Y^0 is not a max $-p$ min $-$ max point of $\tilde{\varphi}$ with respect to $\tilde{W}^v$, the following four cases can occur:

Case 1. There exists $Y \in \tilde{W}^v$ such that $\tilde{\varphi}_1(Y) > \tilde{\varphi}_1(Y^0)$;
Case 2. There exists $Y \in \tilde{W}^v$ such that $\tilde{\varphi}_1(Y) = \tilde{\varphi}_1(Y^0)$, but $\varphi_2(Y) < \varphi_2(Y^0)$;
Case 3. There exists $Y \in \tilde{W}^v$ such that $\tilde{\varphi}_1(Y) = \tilde{\varphi}_1(Y^0)$, $\varphi_2(Y) = \varphi_2(Y^0) = z_\tau$ and there exists $r \in \{\tau, \ldots, q\}$ such that $\sum_{(i,k,h)\in L_t} y_{ikh} = \sum_{(i,k,h)\in L_t} y^0_{ikh}$, for each $t \in \{\tau, \ldots, r-1\}$, but $\sum_{(i,k,h)\in L_r} y_{ikh} < \sum_{(i,k,h)\in L_r} y^0_{ikh}$;
Case 4. There exists $Y \in \tilde{W}^v$ such that $\tilde{\varphi}_1(Y) = \tilde{\varphi}_1(Y^0)$, $\varphi_2(Y) = \varphi_2(Y^0) = z_\tau$ and $\sum_{(i,k,h)\in L_t} y_{ikh} = \sum_{(i,k,h)\in L_t} y^0_{ikh}$, for each $t \in \{\tau, \ldots, q\}$, but $f_2(Y) < f_2(Y^0)$.
Let us analyse each of these possible cases.

Case 1. Since $\tilde{\varphi}_1(Y) > \tilde{\varphi}_1(Y^0)$ and $f_2(Y) \geq 0$, based on (5.59), (5.62), (5.63) and (5.65), it results that

$$\begin{aligned}
FL(Y) - FL(Y^0) &= \\
&= -M_0 \cdot M\Big(\tilde{\varphi}_1(Y) - \tilde{\varphi}_1(Y^0)\Big) + M \sum_{t=1}^{q} \Big(M_t \sum_{(i,k,h)\in L_t} (y_{ikh} - y^0_{ikh})\Big) \\
&\quad -\Big(f_2(Y) - f_2(Y^0)\Big) \\
&\leq -M_0 \cdot M + M \sum_{t=1}^{q} \Big(M_t \cdot \mathrm{card}(\mathrm{L_t})\Big) + \mathrm{f_2(Y^0)} \\
&\leq -M_0 \cdot M + M \sum_{t=1}^{q} \Big(M_t \cdot \mathrm{card}(\mathrm{L_t})\Big) + \mathrm{M} - 1 \\
&= -M - M \sum_{t=1}^{q} \Big(M_t \cdot \mathrm{card}(\mathrm{L_t})\Big) + \mathrm{M} \sum_{\mathrm{t}=1}^{\mathrm{q}} \Big(\mathrm{M_t} \cdot \mathrm{card}(\mathrm{L_t})\Big) + \mathrm{M} - 1 \\
&= -1 < 0.
\end{aligned}$$

Therefore, Y^0 can not be an optimal solution of the (PL^v) problem.

Case 2. Let $\varphi_2(Y) = z_\tau$ and $\varphi_2(Y^0) = z_r$. Since $\varphi_2(Y) < \varphi_2(Y^0)$ and the sequence $(z_t)_{t=1}^q$ is strictly increasing, we obtain that $r < \tau$. Therefore,

$$\begin{aligned}
&\sum_{t=1}^{q} \Big(M_t \sum_{(i,k,h)\in L_t} (y_{ikh} - y^0_{ikh})\Big) = \\
&\qquad = -M_r \sum_{(i,k,h)\in L_r} y^0_{ikh} + \sum_{t=r+1}^{q} \Big(M_t \cdot \sum_{(i,k,h)\in L_t} (y_{ikh} - y^0_{ikh})\Big).
\end{aligned}$$

Then, based on (5.64) and (5.65) it results that

$$\sum_{t=1}^{q} \Big(M_t \sum_{(i,k,h)\in L_t} (y_{ikh} - y^0_{ikh})\Big) \leq -M_r + \sum_{t=r+1}^{q} M_t \cdot \mathrm{card}(\mathrm{L_t}),$$

and, based on (5.61) it results that

$$\begin{aligned}
\sum_{t=1}^{q} \Big(M_t \sum_{(i,k,h)\in L_t} (y_{ikh} - y^0_{ikh})\Big) &\leq -1 - \sum_{t=r+1}^{q} M_t \cdot \mathrm{card}(\mathrm{L_t}) \\
&\quad + \sum_{t=r+1}^{q} M_t \cdot \mathrm{card}(\mathrm{L_t}) = -1.
\end{aligned}$$

Therefore,

$$\begin{aligned} FL(Y) - FL(Y^0) \ &= \ M \sum_{t=1}^{q} \Big(M_t \cdot \sum_{(i,k,h)\in L_t} (y_{ikh} \ - \ y^0_{ikh}) \Big) - \Big(f_2(Y) - f_2(Y^0) \Big) \\ &\le \ -M \ + \ f_2(Y^0) \ \le \ -M + M - 1 \ = \ -1 \ < \ 0, \end{aligned}$$

which contradicts that Y^0 is an optimal solution of the (PL^v) problem.

Case 3. Let us remark that from $\varphi_2(Y) = \varphi_2(Y^0) = z_\tau$ it results that if $\tau > 1$ then for each $t \in \{1, \ldots, \tau - 1\}$ we have that $y_{ikh} = y^0_{ikh} = 0$, for all $(i,k,h) \in L_t$. Since there exists $r \in \{\tau, \ldots, p\}$ such that

$$\sum_{(i,k,h)\in L_r} y^0_{ikh} \ > \sum_{(i,k,h)\in L_r} y_{ikh}, \tag{5.67}$$

and if $r > \tau$ then

$$\sum_{(i,k,h)\in L_t} y^0_{ikh} \ = \sum_{(i,k,h)\in L_t} y_{ikh}, \quad \text{for} \quad \text{all} \quad \mathrm{t} \in \{\tau, \ldots, \mathrm{r}-1\},$$

the condition (5.67) implies that

$$\sum_{(i,k,h)\in L_r} (y_{ikh} - y^0_{ikh}) \ \le \ -1.$$

Therefore, if $r = q$ then we get that

$$\begin{aligned} FL(Y^0) - FL(Y) \ &= \ M \cdot M_q \cdot \sum_{(i,k,h)\in L_q} (y_{ikh} - y^0_{ikh}) - (f_2(Y) - f_2(Y^0)) \\ &\le \ -M \cdot M_q + f_2(Y^0) \ \le \ -M + M - 1 \ = \ -1 \ < \ 0. \end{aligned}$$

If $r < q$ then we have that

$$\begin{aligned} FL(Y) - FL(Y^0) \ &= \ M \sum_{t=\tau}^{q} \Big(M_t \cdot \sum_{(i,k,h)\in L_t} (y_{ikh} - y^0_{ikh}) \Big) - \Big(f_2(Y) - f_2(Y^0) \Big) \\ &\le \ M \cdot M_r \cdot \sum_{(i,k,h)\in L_r} (y_{ikh} - y^0_{ikh}) \ + \ M \sum_{t=r+1}^{q} \Big(M_t \cdot \\ &\quad \sum_{(i,k,h)\in L_t} (y_{ikh} - y^0_{ikh}) \Big) + f_2(Y^0) \\ &\le \ -M \cdot M_r \ + \ M \sum_{t=r+1}^{q} \Big(M_t \cdot \mathrm{card}(\mathrm{L_t}) \Big) \ + \ \mathrm{f_2(Y^0)} \\ &\le \ -M \ + \ M - 1 \ = \ -1 \ < \ 0. \end{aligned}$$

In both cases we get the contradiction that Y^0 is an optimal solution of the (PL^v) problem.

Case 4. In this case $FL(Y) - FL(Y^0) \ = \ -f_2(Y) + f_2(Y^0) \ < \ 0$, which contradicts the optimality of Y^0.

In all the above cases we get a contradiction. It results that Y^0 is a $\max -p \min - \max$ point of $\tilde{\varphi}$ with respect to $\tilde{W}^v$.

Sufficiency: Let $Y^0 \in \tilde{W}_0^v$ and $Y \in \tilde{W}^v$. We prove that $FL(Y) \leq FL(Y^0)$. Since $\tilde{\varphi}(Y^0) \geq_{\max -p \min - \max} \tilde{\varphi}(Y)$, five cases can occur:

Case 1. $\tilde{\varphi}_1(Y) < \tilde{\varphi}_1(Y^0)$;

Case 2. $\tilde{\varphi}_1(Y) = \tilde{\varphi}_1(Y^0)$ and $\varphi_2(Y) > \varphi_2(Y^0)$;

Case 3. $\tilde{\varphi}_1(Y) = \tilde{\varphi}_1(Y^0)$, $\varphi_2(Y) = \varphi_2(Y^0) = z_\tau$ and there exists $r \in \{\tau, \ldots, q\}$ such that $\sum\limits_{(i,k,h)\in L_r} y_{ikh} = \sum\limits_{(i,k,h)\in L_r} y^0_{ikh}$, for each $t \in \{\tau, \ldots, r-1\}$, but $\sum\limits_{(i,k,h)\in L_t} y_{ikh} > \sum\limits_{(i,k,h)\in L_t} y^0_{ikh}$;

Case 4. $\tilde{\varphi}_1(Y) = \tilde{\varphi}_1(Y^0)$, $\varphi_2(Y) = \varphi_2(Y^0) = z_\tau$, $\sum\limits_{(i,k,h)\in L_t} y_{ikh} = \sum\limits_{(i,k,h)\in L_t} y^0_{ikh}$, for each $t \in \{1, \ldots, q\}$ and $f_2(Y) < f_2(Y^0)$;

Case 5. $\tilde{\varphi}_1(Y) = \tilde{\varphi}_1(Y^0)$, $\varphi_2(Y) = \varphi_2(Y^0) = z_\tau$, $\sum\limits_{(i,k,h)\in L_t} y_{ikh} = \sum\limits_{(i,k,h)\in L_t} y^0_{ikh}$, for each $t \in \{1, \ldots, q\}$ and $f_2(Y) = f_2(Y^0)$.

For the first four cases we get that $FL(Y) < FL(Y^0)$, by an analogously way as in the Necessity. In the fifth case, we have that $FL(Y) = FL(Y^0)$. Therefore, Y^0 is a maximum point of FL with respect to $\tilde{W}^v$. ∎

Theorem 5.2.14 can be used to determine a $\max -p \min - \max$ point of φ with respect to $\mathcal{W}^v$.

We remark that the problem (PL^v) is a linear pseudo boolean optimization problem. Therefore, it can be solved by using the techniques presented by Berthold T., Heinz S. and Pfetsch M.E. [8], by Hammer P.L. and Rudeanu S. [50], and by Manquinho V. and Marques-Silva J. [76].

Furthermore, based on Theorems 5.2.9, 5.2.10, 5.2.12, 5.2.13, 5.2.14 and on the above Remarks, an algorithm for solving the (EBCT) problem is given. We use the same notations as above. In addition, G and sw are two variables used in the algorithm. If the (EBCT) problem has optimal solutions and (X^0, Y^0) is an optimal solution of it, then X^0 is memorized by $Z1$, and Y^0 by $Z2$. The number $F(X^0, Y^0)$ is memorized by F, $\varphi_1(Y^0)$ by $\varphi 1$, and $\varphi_2(Y^0)$ by $\varphi 2$.

In what follows, we give the algorithm (see Tuns (Bode) O.R. [111]):

Input

the natural numbers m, n, p and s;

the positive numbers e, e_h, $\quad h \in \{1, \ldots, s\}$;

the elements of matrices:

$A = [a_{ij}], \quad i \in \{1, \ldots, m\}, \quad j \in \{1, \ldots, n\}$;

$R = [r_{ij}], \quad i \in \{1, \ldots, m\}, \quad j \in \{1, \ldots, n\}$;

$B = [b_{ik}], \quad i \in \{1, \ldots, m\}, \quad k \in \{1, \ldots, p\}$;

$C_h = [c_{ikh}], \quad i \in \{1, \dots, m\}, \quad k \in \{1, \dots, p\}, \quad h \in \{1, \dots, s\};$
$D_h = [d_{ikh}], \quad i \in \{1, \dots, m\}, \quad k \in \{1, \dots, p\}, \quad h \in \{1, \dots, s\};$

Output

ok — *true* if a solution exists;
$Z1$, $Z2$ — the optimal solution, if such a solution exists;
F — the value of the upper level (leader) function;
$\varphi 1, \varphi 2$ — the value of the lower level (follower) vector function;

Algorithm

$I := \{1, \dots, m\}$; $J := \{1, \dots, n\}$; $K := \{1, \dots, p\}$; $H := \{1, \dots, s\}$;
$\mu := \sum_{i \in I} \sum_{j \in J} a_{ij}$;
$$\forall i \in I, \quad \forall j \in J, \quad \tilde{a}_{ij} := \begin{cases} \mu + a_{ij}, & \text{if} \quad \mathrm{r_{ij}} \le \mathrm{e}; \\ 0, & \text{if} \quad \mathrm{r_{ij}} > \mathrm{e}; \end{cases}$$
$$\forall i \in I, \quad \forall k \in K, \quad \forall h \in H, \quad \tilde{c}_{ikh} := \begin{cases} c_{ikh}, & \text{if} \quad \mathrm{d_{ikh}} \le \mathrm{e_h}; \\ 0, & \text{if} \quad \mathrm{d_{ikh}} > \mathrm{e_h}; \end{cases}$$
$q = \mathrm{card}(\{\mathrm{d_{ikh}} | (\mathrm{i, k, h}) \in \mathrm{I \times K \times H}\})$;
$L_0 := \emptyset$;
$\forall t \in \{1, \dots, q\}, \qquad z_t := \max(\{d_{ikh} | (i, k, h) \in ((I \times K \times H) \setminus \cup_{l=0}^{t-1} L_l)\};$
$\qquad L_t := \{(i, k, h) \in I \times K \times H | d_{ikh} = z_t\};$
$F := -1$;
$V := \{v = (v_1, \dots, v_m) \in \{0, 1\}^m \,|\, v_1 + \ldots + v_m = n\}$;
while $V \neq \emptyset$ do
    choose $v \in V$;
    $U := \{i \in I | v_i = 1\}; \quad \bar{U} := I \setminus U$;
    sw:=1;
    $G := U$;
    if $G \neq \emptyset$ then
        choose $i \in G$;
        if $\sum_{j \in J} \tilde{a}_{ij} = 0$ then
            $G := \emptyset$;
            $sw := 0$;
            else $G := G \setminus \{i\}$;
        end if;
    end if;
    if $sw \neq 0$ then
        $G := J$;
        if $G \neq \emptyset$ then
            choose $j \in G$;

if $\sum_{i \in U} \tilde{a}_{ij} = 0$ then
  $G := \emptyset$;
  $sw := 0$;
  else $G := G \setminus \{j\}$;
end if;
end if;
if $sw \neq 0$ then
  $G := \bar{U}$;
    if $G \neq \emptyset$ then
      choose $i \in G$;
      if $\sum_{k \in K} \sum_{h \in H} c_{ikh} = 0$ then
        $G := \emptyset$;
        $sw := 0$;
        else $G := G \setminus \{i\}$;
      end if;
    end if;
    if $sw \neq 0$ then
      $G := K$;
      if $G \neq \emptyset$ then
        choose $k \in G$;
        if $\sum_{i \in \bar{U}} \sum_{h \in H} c_{ikh} = 0$ then
          $G := \emptyset$;
          $sw := 0$;
          else $G := G \setminus \{i\}$;
        end if;
      end if;
      if $sw \neq 0$ then
        solve the problem PM^v;
        denote by $X = [x_{ij}]$ an optimal solution;
        if $\sum_{i \in U} \sum_{j \in J} \tilde{a}_{ij} x_{ij} \geq n\mu$ then
          solve the problem PL^v;
            if PL^v has optimal solution then
              denote by Y an optimal solution;
              if $F < f_1(X) + f_2(Y)$ then
                $F := f_1(X) + f_2(Y)$;
                $Z1 := X$; $Z2 := Y$;
                $\varphi 1 := \varphi_1(Z2); \varphi 2 := \varphi_2(Z2)$;

end if;
end if;
end if;
end if;
end if;
end if;
end if;
$V := V \setminus \{v\}$;
end while
if $F = -1$ then
$ok := false$;
else $ok := true$;
end if;
End Algorithm

Now, we give an example to highlight how the above algorithm works.

Example 5.2.15 (Tuns (Bode) O.R. [*111*]).
Let $m = 5$, $n = 2$, $p = 3$, $s = 2$, $e = 2$, $e_1 = 3$, $e_2 = 2$ *and*

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \\ 5 & 1 \\ 1 & 3 \\ 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 4 \\ 4 & 2 & 3 \\ 3 & 4 & 1 \\ 2 & 1 & 3 \end{bmatrix}, \quad C_{h=1} = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 4 & 1 \\ 2 & 7 & 3 \\ 1 & 7 & 2 \\ 4 & 3 & 1 \end{bmatrix}, \quad C_{h=2} = \begin{bmatrix} 3 & 1 & 5 \\ 2 & 4 & 7 \\ 4 & 3 & 2 \\ 4 & 5 & 4 \\ 7 & 2 & 3 \end{bmatrix},$$

$$R = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad D_{h=1} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 5 & 3 \\ 3 & 2 & 2 \\ 4 & 5 & 4 \\ 2 & 1 & 4 \end{bmatrix}, \quad D_{h=2} = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 2 & 1 \\ 4 & 2 & 2 \\ 3 & 3 & 4 \\ 1 & 1 & 3 \end{bmatrix}.$$

As $\mu = 34$, *applying the transformations given by (5.46) and (5.47), we obtain that*

$$\tilde{A} = \begin{bmatrix} 36 & 38 \\ 37 & 36 \\ 0 & 0 \\ 35 & 37 \\ 35 & 39 \end{bmatrix}, \quad \tilde{C}_{h=1} = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 0 & 1 \\ 2 & 7 & 3 \\ 0 & 0 & 0 \\ 4 & 3 & 0 \end{bmatrix} \text{ and } \tilde{C}_{h=2} = \begin{bmatrix} 3 & 0 & 5 \\ 2 & 4 & 7 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \\ 7 & 2 & 0 \end{bmatrix}.$$

We have that $q = 5$, $z_1 = 5$, $z_2 = 4$, $z_3 = 3$, $z_4 = 2$, $z_5 = 1$, *card*(L_1)=*2*, *card*(L_2)=*6*, *card*(L_3)=*6*, *card*(L_4)=*9*, *card*(L_5)=*7*. *Therefore,* $M_5 = 1$, $M_4 = 3$, $M_3 = 21$, $M_2 = 147$,

$M_1 = 1470$, $M_0 = 11760$ and $M = 39$. We obtain that

$$V = \Big\{v^1 = (1,1,0,0,0),\ v^2 = (1,0,1,0,0),\ v^3 = (1,0,0,1,0),\ v^4 = (1,0,0,0,1),$$
$$v^5 = (0,1,1,0,0), v^6 = (0,1,0,1,0),\ v^7 = (0,1,0,0,1),\ v^8 = (0,0,1,1,0),$$
$$v^9 = (0,0,1,0,1),\ v^{10} = (0,0,0,1,1)\Big\}.$$

It is easy to see that for the vectors v^1, v^2, v^4, v^5, v^7, v^8 and v^9, at least one of the following restrictions

$$\max\{r_{ij}x_{ij} \,|\, i \in U^v,\ j \in J\} \ \leq\ e$$

or

$$\max\{d_{ikh}y_{ikh} \,|\, i \in \bar{U}^v,\ k \in K\} \leq e_h, \quad \text{for} \quad \text{each}\, \mathrm{h} \in \mathrm{H},$$

is not satisfied.

For v^3, the ($PM_1^{v^3}$) problem is

$$\begin{cases} 2x_{11} + 4x_{12} + 7x_{41} + 3x_{42} \quad \rightarrow \quad \max, \\ x_{11} + x_{12} = 1, \\ x_{41} + x_{42} = 1, \\ x_{11} + x_{41} = 1, \\ x_{12} + x_{42} = 1, \\ x_{ij} \in \{0,\, 1\}, \quad i \in \{1,4\}, \quad j \in \{1,2\}, \end{cases} \quad \text{its optimal solution being}\, \mathrm{X}^{\mathrm{v}^3} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The optimal solution of (PL^{v^3}) is

$$Y_{h=1}^{v^3} = Y_{h=2}^{v^3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Therefore, $F(v^3) = 27$.

Analogously, we get that $F(v^6) = 21$ and $F(v^{10}) = 26$. As $27 = \max\{27, 21, 26\}$, it results that (X^{v^3}, Y^{v^3}) is the optimal solution of the (EBVT) problem; therefore, it is an optimal solution of the (EBCT) problem.

Chapter 6

Applications of Multilevel Optimization in the Technology Transfer Area

Technology transfer, also called *Transfer of Technology* or *Technology Commercialization*, is the process of skill transferring, knowledge, technologies, methods of manufacturing, samples of manufacturing and facilities among governments or universities and other institutions to ensure that scientific and technological developments are accessible to a wide range of users who can then further develop and exploit the technology into new products, processes, applications, materials or services. It is closely related to knowledge transfer (can be considered a subset of it [**?**]).

Technology transfer is defined in a lot of different ways, depending on one hand on the discipline of the research and on the other hand on the purpose of the research. That is the reason why the definition of technology transfer differs significantly from one discipline to other. For example, economists define technology on the basis of the properties of generic knowledge, focusing particularly on variables that relate to production and design; sociologists tend to connect technology transfer to innovation and to view technology, including social technology, as "a design for instrumental action that reduces the uncertainty of cause effect relationships involved in achieving a desired outcome"; anthropologists view technology transfer in most cases within the context of cultural change and the ways in which technology affects change.

Everyone studying technology transfer figures out how complicated it can be. Vastly, this happens because the framework of the technology transfer process is difficult because there are so many concurrent processes.

In the present chapter we study some types of particular multilevel optimization problems generated by concrete economic problems related to technology transfer area. The scientific research results within this chapter belong to the author and can be found in papers authored by Ferreira F. and Bode O.R. [31] and [32], or by Tuns (Bode) O.R. [112]. In [31] we consider a differentiated Stackelberg model, when the leader firm engages itself in a research and development ($R\&D$) process that gives an endogenous cost-reducing innovation. The aim is to study the licensing of the cost-reduction by a two-part tariff. By using comparative static analysis, we conclude that the degree of the differentiation of goods plays an important role in the results. We also do a direct comparison between the Stackelberg duopoly model and Cournot duopoly model. In [32] by considering the same differentiated Stackelberg duopoly model, when the leader firm engages itself in a $R\&D$ process that gives an endogenous cost-reducing innovation, we study the licensing of the cost-reduction by a per-unit royalty and a fixed-fee. We analyse the implications of these types of licensing contracts over the $R\&D$ effort, the profits of the firms, the consumer surplus and the social welfare. By using comparative static analysis, we also conclude that the degree of the differentiation of goods plays an important role in the results. In [112] we study the case when two firms compete on the market in a differentiated Stackelberg model and there is no technology transfer between the innovator firm and the follower firm. A mathematical model is attached to this particular economic problem and an optimal solution is found. For that, we consider a multilevel parametric optimization problem in which both the upper and the lower level functions are to be maximized under some given conditions.

The present chapter is organized as follows: we begin our exposure in Section 6.1 with a brief background concerning technology transfer. In Section 6.2 we formulate the concrete economic problem studied. Then, in Sections 6.3, respectively 6.4, a mathematical model is attached to the concrete economic problem in the benchmark case, respectively in the per-unit royalty licensing case. We solve and determine the optimal solution of each problem from both mathematical and economic point of view. We remark that the mathematical model attached to each of the above two economic problems is an optimization problem that can be remodeled as a multilevel optimization problem. In Sections 6.5, respectively 6.6, we solve the economic problem in the case of licensing by means of a fixed-fee, respectively by means of a two-part tariff.

6.1 Brief Background Concerning Technology Transfer

Technology transfer between firms can be done with many significant methods. One of these methods is *licensing*. This is one of the many reasons that makes the licensing be an important phenomenon, because it is seen as a tool for managing the intellectual property of firms in high technology industries. Licensing can be defined as the granting of permission to use intellectual property rights (such as patents, trademarks or technology) under defined conditions. Among time, licensing activity has been the subject of much theoretical inquiry as we can see in the papers authored by ANAND B.N. and KHANNA T. [2] or CHOI J.P. [19].

As can be found in the literature, technology licensing represents a major economic activity and plays an important role for growth of firms and economy. Getting a new technology by technology licensing is a low risk access to increase the corporation profit. Although R&D is a good way to stimulate the growth of the firm profit, it needs not only to invest a lot of money, but also to spend a lot of time. Many firms have not enough capital or time to engage themselves in R&D activity, so they choose to adopt a new technology and get a technology licensing. A license allows an intellectual property rights holder (the licensor) to make money from an invention or creative work by charging a user (the licensee) for product use.

Licensing contracts cover a wide range of well-known situations. For example, a production firm might achieve the license for a proprietary production technology from another firm which owns it, in order to gain a competitive edge, rather than expending the time and money trying to develop its own technology. So, a licensing agreement is a legal contract between two parties: the licensor (the part who gives someone the license to produce and sell its products) and the licensee (the part to whom or to which the license is granted). Hence, in a typical licensing agreement the licensor grants the licensee the right to produce and sell goods, apply a brand name or trademark, or use patented technology owned by the licensor. In exchange, the licensee usually submits to a series of conditions regarding the use of the licensor's property and agrees to make payments under one of the followings forms: a royalty on per-unit of output produced with the patented technology, a fixed-fee that is independent of the quantity produced with the patented technology, and a two-part tariff, i.e. a fixed-fee plus royalty.

There exists vast literature focusing on the decision of the optimal licensing contract by the patentee. We recall the papers authored by CHANG M.C., LIN C.H. and HU J.L. [18], FERREIRA F.A. [30], FILIPPINI L. [33], FOSFURI A. and ROCA E. [36] or

Kamien M.I., Oren S.S. and Tauman Y. [58]. Nowadays, patent licensing is an important area of research which is becoming increasingly relevant because of the present trend of globalization and technology transfer between firms across countries. It takes place in many industries. It can be seen as a source of profit for the patentee (innovator) who earns rent from the licensee by transferring a new technology. Zuniga M.P. and Guellec D. [134] made an interesting and useful study concerning the intensity of licensing to affiliated and non-affiliated companies, its evolution, the characteristics, motivations and obstacles met by companies doing or willing to license, pointing out at the end the fact that patent licensing is widespread.

From economic point of view, each type of licensing contract (i.e. per-unit royalty, fixed-fee and two-part tariff) can be studied in different duopoly models: Cournot, Bertrand or Stackelberg models. Therefore, there can exists real life situations when two firms compete on the market in a differentiated Cournot, Bertrand or Stackelberg model. One of the two firms can engages itself in a R&D process and that reduces its unitary production cost, so it is called the innovator firm. Furthermore, the innovation process happens before the market competition. Hence, on one hand it can be study the pre-licensing case (also called benchmark case, i.e.the case when the technology transfer does not occurs) and, on the other hand, in case the technology transfer occurs, the following three cases of the licensing contract: per-unit royalty, fixed-fee and two-part tariff.

We briefly recall the basic notions concerning each type of market competition:

I. *Cournot competition* is an economic model used to describe an industry structure in which firms compete on the amount of output they will produce, which they decide independently of each other and at the same time. So, when competing in a Cournot model, firms: do not cooperate (i.e. there is no collusion); choose simultaneously the quantity of output it will produce in the market for a specific good; have market power (i.e. each firm's output decision affects the good's price); and are economically rational and act strategically, usually seeking to maximize profit given their competitors' decisions. An essential assumption of this model is the not conjecture that each firm aims to maximize profits, based on the expectation that its own output decision will not have an effect on the decisions of its rivals. Price is a commonly known decreasing function of total output. All firms know the total number of firms in the market and take the output of the others as given. The market price is set at a level such that demand equals the total quantity produced by all firms. Each firm takes the quantity set by its competitors as a given, evaluates its residual demand and then behaves as a monopoly.

II. *Bertrand competition* is an economic theory which indicates how two firms will compete in a duopoly market using price as a *weapon*. This theory was actually made to criticize the Cournot model which stated that firms will compete with each other by choosing their output. So, when competing in a Bertrand model: firms do not cooperate (i.e. there is no collusion) and compete with each other by setting prices simultaneously and not on the basis of a mutual agreement; all the firms will be supplying to all the market demands at the changed price; consumers buy everything from either of the firms, depending on who sells at the lowest price; if all the firms charge an equal price then consumers will buy products on random selection. As advantages of this model we remind that it can be applied when the output of a firm can be efficiently changed and when the cost involved in turning customers away is very high.

As critical evaluation of this model, we mention the followings:

- practically this model cannot hold true as the sales of a good does not just depend on its price but also on other factors, like brand name, product differentiation etc.;

- if one company is selling the same commodity at a higher price, then it is implied in the theory that he will sell nothing, but this is not true;

- if the market demand is greater than the market supply, then companies would like to increase the price of their goods to gain higher profits;

- it ignores capacity constraints, i.e. it may not be possible for a firm to be able to serve the entire market demand;

- instead of competing on the basis of price, a firm can choose to compete with other firms through product differentiation;

- for a market with homogenous goods, the Bertrand model becomes a paradox, since the equilibrium prices are equal to the unitary production costs.

This theory has been criticized for its unrealistic approach and has not been termed as a better model than the Cournot model. In fact, both models cater to different market structures. Even though this model cannot be applied to real life models, it still holds an important place in economics as they explain how firms can compete.

It is important to notice that even though each of Cournot and Bertrand model was approved or criticized along time, neither model is necessarily *better*. The accuracy of the predictions of each model will vary from industry to industry, depending on the closeness of each model to the industry situation. If capacity and output can be easily changed, then Bertrand is a better model of duopoly competition. If output and capacity are difficult to adjust, then Cournot is generally a better model. However, when number of firms goes to infinity, Cournot model gives the same result as Bertrand model: market price is pushed to marginal cost level.

III. The *Stackelberg competition* is a dynamic leadership model. It is a strategic game in economics in which the leader firm moves first and then the follower firms move sequentially, in a quantity competition. It contrast to the Cournot model in which the firms choose simultaneously their quantities. In the Stackelberg model the decisions are made sequentially.

As we already mentioned, literature focused on the optimal patent licensing in a wide variety of situations is vast. Also, there exists a lot of studies that analyze various aspects of patent licensing. The theoretical literature regarding patent licensing reveals two types of patentees, namely the outsider patentee and the insider patentee. The patentee is an outsider patentee when it is an independent R&D organization (for instance, a university or a research institute) and not a competitor of the licensee in the product market. On the other hand, when the patentee competes with the licensee it becomes an insider patentee. Based on this, it has been discussed in the literature about the nature of licensing that should take place between the patentee and licensee(s). The studies of insider and outsider patentee have been done in different models. In the standard models, in a complete information framework, if the patentee happens to be an outsider then it can be said that fixed-fee licensing is optimal to the patentee (see Kamien M.I. [57], Kamien M.I. and Tauman Y. [59] or Katz M. and Shapiro C. [61]). The reverse happens when the patentee is an insider that is a competitor, i.e. per-unit royalty licensing is optimal to the patentee (see Kamien M.I. and Tauman Y. [60], Marjit S. [79], Rockett K. [98], Wang X.H. [124] and [125] or Wang X.H. and Yang B.Z. [126]).

In 2004, Poddar S. and Sinha U.B. [92] opened up a new avenue of research related to patent licensing. By studying the optimal patent licensing strategy of an outsider patentee as well as of an insider patentee in a new environment, they contradict the existing results in the literature. They introduced the study of patent licensing in a spatial framework and not in a standard framework of price and quantity competition as it was done before. In this way, two important research areas, patent licensing and competition in a spatial model, were bringed in one platform.

So far, two main remarks must be mentioned: on one hand, no study has been done to reconcile the two results above and, on the other hand, in general a new technology is transferred from a firm who is at least as cost efficient as the recipient firm and in many cases it is the more efficient one.

In 2005, Poddar S. and Sinha U.B. [93] studied optimal licensing contract when the new technology is transferred from a firm which is relatively cost-inefficient in the pre-innovation stage compared to the recipient firm and provided a framework to bridge the literature on external and internal patentee. A similar study can be found in paper authored by Ferreira F.A. [29], but in this paper the inverse demand function is more general and some dependencies between fixed-fee and slope parameter, respectively between royalty and slope parameter, are proved.

Ferreira F. and Bode O.R. [31] studied the case of a patent licensing contract when the patentee is an insider and the innovation size is endogenous, in a differentiated-good Stackelberg model. In 2010, Li C. and Ji X. [67] develop a differentiated-good duopoly model where one of the firms engages itself in an endogenous cost-reducing innovation and licenses its innovation to its rival firm. But, on one hand, the authors only consider the license contract made by a two-part tariff and, on the other hand, only in the Cournot and Bertrand models.

Recalling the paper [31], we remark that we considered a differentiated Stackelberg model, when the leader firm engages itself in a R&D process that gives an endogenous cost-reducing innovation. The aim was to study the licensing of the cost-reduction innovation. The mathematical model attached to this economic problem is an optimization problem that can be remodeled as a multilevel optimization problem. In this chapter, we give a generalization of this problem, seen from the mathematical point of view.

In the present book we work under the hypothesis that the market competition holds in a Stackelberg model since, from the mathematical point of view, this leads us to multilevel optimization problems. This aspect results from the papers [31], [32], [110], [111] and [112] which represent author's scientific results obtained by herself or as a joint work. But we remark that the author's research is also regarding the Bertrand and Cournot competition, as we can see in paper [**?**].

6.2 Basic Framework of the Studied Economic Problem

We introduce the concrete economic problem studied in the present chapter:

Let us consider a duopoly model where two firms, denoted by F^1 and F^0, produce n differentiated goods. The inverse demand functions are given by

$$p^h = 1 - q^h - < d, q^{1-h} >,$$

where:

- $p^h = (p^h_1, \dots, p^h_n) \in \mathbb{R}^n$ represents the price of the firm F^h, $h = 0, 1$;
- $q^h = (q^h_1, \dots, q^h_n) \in \mathbb{R}^n$ and $q^{1-h} = (q^{1-h}_1, \dots, q^{1-h}_n) \in \mathbb{R}^n$ represent the outputs of firms F^h and F^{1-h}, respectively, where $h = 1$;
- d represents the degree of the differentiation of goods, $d = (d_1, \dots, d_n) \in \mathbb{R}^n$, with $d_j \in (0, 1)$, $\forall\, j \in \{1, \dots, n\}$.

The duopoly market is modeled as a Stackelberg competition: the leader firm F^1 choose its output level and then the follower firm F^0 is free to choose its optimal output taking into account the leader's output; so, the firms do not decide simultaneously the level of their outputs. Initially, both firms have identical unit production cost $c =$

$(c_1, \ldots, c_n) \in \mathbb{R}^n$, with $c_j \in (0, 1)$, $\forall\, j \in \{1, \ldots, n\}$. We consider that firm F^1, the leader firm, can engages itself in a R&D process in order to improve its technology. This allows a reduction of its production costs by an amount called *innovation size*. The cost-reducing innovation creates a new technology that reduces innovating firm's unit cost by the amount of k, while the amount invested in R&D is $k^2/2$.

In case there is a technology transfer between the two firms, we consider the following five stages game. In the first stage, the innovator firm decides the value of the innovation size (or, equivalently, the amount to invest in R&D). In the second stage, the innovator firm decides whether to license the technology or not, because licensing reduces the marginal cost of the licensee firm. If it decides to license the new technology, then it charges a payment from the licensee (either a per-unit royalty rate, a fixed-fee or a combination of both royalty and fixed-fee). In the third stage, the licensee firm decides whether to accept or reject the offer made by the licensor. Then, both firms represents the players of a Stackelberg game. Therefore, in the fourth stage the leader firm decides its output and in the last stage the follower firm, being aware of the leader's output, chooses the output to produce.

In the present chapter we consider both situations:

i) when there is no technology transfer between firms (benchmark case);

ii) when there is a technology transfer between firms based either on a per-unit royalty contract, a fixed-fee contract or a two-part tariff contract.

A mathematical model is attached to each particular economic problem and an optimal solution is found. For that, we consider different multilevel parametric optimization problems in which both the upper and the lower level functions are to be maximized under some given conditions.

6.3 Benchmark Case: No-licensing Case

6.3.1 Modeling and Solving the Economic Problem

We note that the results in this paragraph belong to the author and can be found in Tuns (Bode) O.R. [112].

Let $n \in \mathbb{N}^*$ be a natural number, $J = \{1, \ldots, n\}$, and let $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n) \in \mathbb{R}^n$. Let $T \subseteq \mathbb{R}^n$ be the set of variation of the parameter $d = (d_1, d_2, ..., d_n) \in \mathbb{R}^n$. Using d we can set the diagonal matrix $D \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}.$$

For each $d \in T$, let $f_d : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $F_d : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $g_d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the functions given, respectively, by

$$f_d(x, y, z) \; = < \gamma - x - Dy + z, x >, \; \forall \, (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n,$$

$$F_d(x, y, z) \; = < \gamma - x - Dy + z, x > -\frac{1}{2}\|z\|^2 \; = \; f_d(x, y, z) - \frac{1}{2}\|z\|^2, \; \forall \, (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$$

and

$$g_d(x, y) \; = < \gamma - y - Dx, y >, \; \forall \, (x, y) \in \mathbb{R}^n \times \mathbb{R}^n.$$

For each $x \in \mathbb{R}^n_+$ let $S^*_d(x) := argmax\{g_d(x, y) \,|\, y \in \mathbb{R}^n\}$. The elements of the set $S^*_d(x)$ will be generically denoted by y^x.

For each $z \in \mathbb{R}^n$ let $S^*_d(z) := argmax\{f_d(x, y^x, z) \,|\, x \in \mathbb{R}^n_+\}$. The elements of the set $S^*_d(z)$ will be generically denoted by x^z.

Let us consider the three-level parametric optimization problem

$$(P; T) \quad \begin{cases} F_d(x, y, z) \to \max \\ y \in S^*_d(x) \\ x \in S^*_d(z) \\ z \in \mathbb{R}^n \end{cases}, \; d \in T.$$

For each $d \in T$, by (P_d) we denote the three-level optimization problem obtained from $(P; T)$ if the parameter is fixed to d.

Remark 6.3.1 *If $T =]0, 1[^n$, then the problem $(P; T)$ is the mathematical model attached to the basic economic problem described above.*

Determining the Set $S^*_d(x)$

Let $d = (d_1, d_2, ..., d_n) \in T$ and $x \in \mathbb{R}^n_+$. We consider the problem:

$$(P^1_{d,x}) \quad \begin{cases} \varphi_{d,x}(y) \to \max, \\ y \in \mathbb{R}^n, \end{cases}$$

where $\varphi_{d,x}(y) = g_d(x, y) = < \gamma - y - Dx, y >, \; \forall \, y \in \mathbb{R}^n$.

We get that $\nabla\varphi_{d,x}(y) = -2y + \gamma - Dx$, $\forall\, y \in \mathbb{R}^n$. Since

$$\nabla^2\varphi_{d,x}(y) = \begin{pmatrix} -2 & 0 & \cdots & 0 \\ 0 & -2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -2 \end{pmatrix}, \ \forall\, y \in \mathbb{R}^n,$$

it results that the function $\varphi_{d,x}$ is strictly concave. So, we get that

$$y^x = \frac{1}{2}(\gamma - Dx)$$

is the maximum point of $\varphi_{d,x}$. Therefore, recalling the problem (P_d), $d \in T$, it results that for all $z \in \mathbb{R}^n$, if $S_d^*(z) \neq \emptyset$ and $x \in S_d^*(z)$, then the set $S_d^*(x)$ has just one element, i.e.

$$S_d^*(x) = \{y^x\} = \Big\{\frac{1}{2}(\gamma - Dx)\Big\}. \tag{6.1}$$

Determining the Set $S_d^*(z)$

Now, let $d = (d_1, d_2, ..., d_n) \in T$ and $z \in \mathbb{R}^n$. We consider the following optimization problem:

$$(P^2_{d,z}) \quad \begin{cases} \phi_{d,z}(x) \to \max, \\ x \in \mathbb{R}^n_+, \end{cases}$$

where $\phi_{d,z}(x) = f_d(x, y^x, z) =< \gamma - \frac{1}{2}D\gamma + z, x > + < (D^2 - I_n)x, x >$, $\forall\, x \in \mathbb{R}^n_+$, I_n being the identity matrix in n dimensions.

We remark that

$$\phi_{d,z}(x) = \sum_{j\in J}\Big(-(1 - \frac{d_j^2}{2})x_j^2 + (\gamma_j - \frac{d_j\gamma_j}{2} + z_j)x_j\Big), \ \forall\, x \in \mathbb{R}^n. \tag{6.2}$$

For $d \in T$ fixed, let us denote by

$$J_+^d = \Big\{j \in J \,|\, d_j^2 > 2\Big\}, \quad J_-^d = \Big\{j \in J \,|\, d_j^2 < 2\Big\}, \quad J_0^d = \Big\{j \in J \,|\, d_j^2 = 2\Big\}.$$

For $d \in T$ and $z \in \mathbb{R}^n$, both fixed, we set

$$J_{0+}^d(z) := \Big\{j \in J_0^d \,|\, \gamma_j - \frac{d_j\gamma_j}{2} + z_j > 0\Big\},$$

$$J_{00}^d(z) := \Big\{j \in J_0^d \,|\, \gamma_j - \frac{d_j\gamma_j}{2} + z_j = 0\Big\},$$

$$J_{0-}^d(z) := \Big\{j \in J_0^d \,|\, \gamma_j - \frac{d_j\gamma_j}{2} + z_j < 0\Big\}.$$

Proposition 6.3.2 (TUNS (BODE) O.R. [112]). *Let $d \in T$.*
(i) If $J_+^d \neq \emptyset$, then the function $\phi_{d,z}$ is upper unbounded on $\mathbb{R}_+^n$, for each $z \in \mathbb{R}^n$;
(ii) If $J_+^d = \emptyset$ and $z \in \mathbb{R}^n$ such that $J_{0+}^d(z) \neq \emptyset$, then the function $\phi_{d,z}$ is upper unbounded on $\mathbb{R}_+^n$.

Proof. a) Let us suppose that $J_+^d \neq \emptyset$ and let $h \in J_+^d$. Then $d_h^2 > 2$. If we consider the sequence $(x^k)_{k\geq 1}$, with $x^k = ke^h$ for each $k \in \mathbb{N}$, where e^h represents the h-vector of the canonical base, i.e. $e^h = (e_1^h, e_2^h, \ldots, e_n^h) \in \mathbb{R}^n$ with

$$e_j^h = \begin{cases} 0, \text{ if } j \in J \backslash h, \\ 1, \text{ if } j = h, \end{cases}$$

then we have $x^k \in \mathbb{R}_+^n$, $\forall\, k \in \mathbb{N}^*$, and

$$\lim_{k\to\infty} \phi_{d,z}(x^k) = \lim_{k\to\infty} \Big(-(1 - \frac{d_h^2}{2})k^2 + (\gamma_h - \frac{d_h\gamma_h}{2} + z_h)k \Big) = +\infty.$$

Therefore, the function $\phi_{d,z}$ is upper unbounded on $\mathbb{R}_+^n$.

b) Now, let us suppose that $J_+^d = \emptyset$ and let $z \in \mathbb{R}^n$ be such that $J_{0+}^d(z) \neq \emptyset$. Let $h \in J_{0+}^d(z)$. If we consider the sequence $(x^k)_{k\geq 1}$, with $x^k = ke^h$, for each $k \in \mathbb{N}$, where e^h represents the h-vector of the canonical base, then $x^k \in \mathbb{R}_+^n$, for all $k \in \mathbb{R}$, and we have

$$\lim_{k\to\infty} \phi_{d,z}(x^k) = \lim_{k\to\infty} (\gamma_h - \frac{d_h\gamma_h}{2} + z_h)k = +\infty.$$

Therefore, the function $\phi_{d,z}$ is upper unbounded on $\mathbb{R}_+^n$. ■

Let $d \in T$ and $z \in \mathbb{R}^n$ be such that $J_+^d = \emptyset$ and $J_{0+}^d(z) = \emptyset$. Let us denote by $p = card(J_{00}^d(z))$ and by $q = card(J_{0-}^d(z))$. Let $m = n - p - q = card(J_-^d)$.

Remark 6.3.3 (TUNS (BODE) O.R. [112]). *It is not difficult to see that, if $m = 0$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}_+^n$, then $x^z = (x_1^z, \ldots, x_n^z) \in \mathbb{R}_+^n$, with*

$$x_j^z = \begin{cases} 0, \ \textit{if } j \in J_{0-}^d(z), \\ \lambda_j, \ \textit{if } j \in J_{00}^d(z), \end{cases}$$

is a maximum point of $\phi_{d,z}$.

Indeed, in this case, (6.2) implies that

$$\phi_{d,z}(x) = \sum_{j\in(J_{00}^d(z)\cup J_{0-}^d(z))} \Big(\gamma_j - \frac{d_j\gamma_j}{2} + z_j\Big)x_j = \sum_{j\in J_{0-}^d(z)} \Big(\gamma_j - \frac{d_j\gamma_j}{2} + z_j\Big)x_j \leq 0, \forall\, x \in \mathbb{R}^n.$$

Then, we must have $x_j = 0$ for each $j \in J_{0-}^d(z)$. On the other hand, if $j \in J_{00}^d(z)$, then the value of $\phi_{d,z}(x)$ does not depends of x_j; therefore, x_j may take any positive value.

Under the hypothesis that $m > 0$, let

$$J_-^d = \{j_1, \ldots, j_m\}, \text{ where } 1 \leq j_1 < \ldots < j_m \leq n.$$

We consider the function $\tilde{\phi}_{d,z} : \mathbb{R}^m \to \mathbb{R}$,

$$\tilde{\phi}_{d,z}(x_{j_1}, \ldots, x_{j_m}) = -\sum_{h=1}^{m}(1-\frac{d_{j_h}^2}{2})x_{j_h}^2 + \sum_{h=1}^{m}(\gamma_{j_h} - \frac{d_{j_h}\gamma_{j_h}}{2} + z_{j_h})x_{j_h}, \ \forall\, (x_{j_1}, \ldots, x_{j_m}) \in \mathbb{R}^m.$$

Proposition 6.3.4 (Tuns (Bode) O.R. [*112*]). *The function $\tilde{\phi}_{d,z} : \mathbb{R}^m \to \mathbb{R}$ is strictly concave and its unique maximum point is $\tilde{x} = (\tilde{x}_{j_1}, ..., \tilde{x}_{j_m})$, where*

$$\tilde{x}_{j_h} = \frac{2\gamma_{j_h} - d_{j_h}\gamma_{j_h} + 2z_{j_h}}{2(2 - d_{j_h}^2)}, \textit{ for each } h \in \{1, \ldots, m\}.$$

Proof. It is easy to see that, for each $h \in \{1, \ldots, m\}$, we have

$$\frac{\partial\tilde{\phi}_{d,z}}{\partial x_{j_h}}(x_{j_1}, \ldots, x_{j_m}) = -2(1 - \frac{d_{j_h}^2}{2})x_{j_h} + \gamma_{j_h} - \frac{d_{j_h}\gamma_{j_h}}{2} + z_{j_h}, \ \forall\, (x_{j_1}, \ldots, x_{j_m}) \in \mathbb{R}^m \quad (6.3)$$

and

$$\nabla^2\tilde{\phi}_{d,z}(x) = \begin{pmatrix} -2 + d_{j_1}^2 & 0 & \cdots & 0 \\ 0 & -2 + d_{j_2}^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -2 + d_{j_m}^2 \end{pmatrix}, \forall\, (x_{j_1}, \ldots, x_{j_m}) \in \mathbb{R}^m.$$

As $j_h \in J_-^d$ we have $-2 + d_{j_h}^2 < 0$, for all $h \in \{1, \ldots, m\}$. Therefore, the function $\tilde{\phi}_{d,z}$ is strictly concave. Then, it has an unique maximum point given by $(\tilde{x}_{j_1}, \ldots, \tilde{x}_{j_m})$. From (6.3) we get that $\tilde{x}_{j_h} = \frac{2\gamma_{j_h} - d_{j_h}\gamma_{j_h} + 2z_{j_h}}{2(2-d_{j_h}^2)}$, for each $h \in \{1, \ldots, m\}$. ■

Let $d \in T$ and $z \in \mathbb{R}^n$.

We set $J_{--}^d(z) = \left\{j \in J_-^d \mid \gamma_j - \frac{d_j\gamma_j}{2} + z_j < 0\right\}$, $J_{-+}^d(z) = \left\{j \in J_-^d \mid \gamma_j - \frac{d_j\gamma_j}{2} + z_j > 0\right\}$ and $J_{-0}^d(z) = \left\{j \in J_-^d \mid \gamma_j - \frac{d_j\gamma_j}{2} + z_j = 0\right\}$.

We note that $J_-^d = J_{--}^d(z) \cup J_{-+}^d(z) \cup J_{-0}^d(z)$.

From Remark 6.3.3 and Proposition 6.3.4 we obtain the following result.

Corollary 6.3.5 (Tuns (Bode) O.R. [*112*]). *If $d \in T$ and $z \in \mathbb{R}^n$ such that $J_+^d = \emptyset$ and $J_{0+}^d(z) = \emptyset$, then, for all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}_+^n$, the point $x^z = (x_1^z, \ldots, x_n^z)$, where*

$$x_j^z = \begin{cases} \lambda_j, \ \textit{if } j \in J_{00}^d(z), \\ \\ 0, \ \textit{if } j \in (J_{0-}^d(z) \cup J_{--}^d(z) \cup J_{-0}^d(z)), \\ \\ \dfrac{2\gamma_j - d_j\gamma_j + 2z_j}{2(2 - d_j^2)}, \ \textit{if } j \in J_{-+}^d(z), \end{cases} \quad (6.4)$$

is a maximum point of the function $\phi_{d,z}$.

Based on the fact that for a multilevel optimization problem the objective functions of the lower level must have maximum (respectively, minimum) points, Proposition 6.3.2 implies that the set T^* of feasibility of parameter d has the following property:

$$T^* \subseteq \{d \in T \,|\, J_+^d = \emptyset \,\text{and}\, J_0^d = \emptyset\} = \{d \in T \,|\, d_j^2 < 2, \,\forall\, j \in J\}. \tag{6.5}$$

Indeed, if there exists $s \in J$ such that $d_s^2 > 2$, then taking for example $z^0 = 0_n$ and applying Proposition 6.3.2, we get that the function ϕ_{d,z^0} is unbounded on $\mathbb{R}_+^n$. Also, if there exists $s \in J$ such that $d_s^2 = 2$, then taking $z^0 = (z_1^0, \ldots, z_n^0) \in \mathbb{R}^n$ such that $z_j^0 = 0$ for $j \in J \setminus \{s\}$ and $z_s^0 = -\gamma_s + \frac{d_s\gamma_s}{2} + 1$, and applying Proposition 6.3.2, we get that the function ϕ_{d,z^0} is unbounded on $\mathbb{R}_+^n$.

In what follows, we consider that (6.5) holds. Under this hypothesis, for each $z \in \mathbb{R}$, the set $S_d^*(z)$ has exactly one element, i.e. we have $S_d^*(z) = \{x^z = (x_1^z, x_2^z, ..., x_n^z)\}$, where

$$x_j^z = \begin{cases} 0, \text{ if } j \in J_{--}^d(z) \cup J_{-0}^d(z), \\ \\ \frac{2\gamma_j - d_j\gamma_j + 2z_j}{2(2-d_j^2)}, \text{ if } j \in J_{-+}^d(z). \end{cases} \tag{6.6}$$

Let us remark that if we consider x_j^z as function of z_j, then this function is continuous on $\mathbb{R}$ since

$$\lim_{z_j \to \frac{2\gamma_j - d_j\gamma_j}{2}} \frac{2\gamma_j - d_j\gamma_j + 2z_j}{2(2-d_j^2)} = 0.$$

From (6.1) and (6.6) we obtain that

$$y_j^{x_j^z} = \begin{cases} \frac{\gamma_j}{2}, \text{ if } j \in J_{--}^d(z) \cup J_{-0}^d(z), \\ \\ \frac{\gamma_j(4-d_j^2) - 2d_j(\gamma_j + z_j)}{4(2-d_j^2)}, \text{ if } j \in J_{-+}^d(z). \end{cases} \tag{6.7}$$

Also, we remark that if we consider $y_j^{x_j^z}$ as function of z_j, then this function is continuous on $\mathbb{R}$ since

$$\lim_{z_j \to \frac{2\gamma_j - d_j\gamma_j}{2}} \frac{\gamma_j(4-d_j^2) - 2d_j(\gamma_j + z_j)}{4(2-d_j^2)} = \frac{\gamma_j}{2}.$$

The Optimal Solution of (P_d) Problem

Under the hypothesis that (6.5) holds, we have $J = J_-^d$. Solving the initial problem (P_d) is equivalent to determining the set $argmax\{F_d(x^z, y^{x^z}, z) \,|\, z \in \mathbb{R}^n\}$. Therefore, now we solve the problem

$$(P^3) \quad \begin{cases} \theta_d(z) \to \max, \\ z \in \mathbb{R}^n, \end{cases}$$

where $\theta_d(z) = F_d(x^z, y^{x^z}, z) \; = < \gamma - x^z - Dy^{x^z} + z, x^z > - \frac{1}{2}\|z\|^2$.

As $J = J^d_-$, we have

$$\theta_d(z) = \phi(x^z, y^{x^z}, z) - \frac{1}{2}\sum_{j \in J} z_j^2$$

or

$$\theta_d(z) = -\frac{1}{2}\sum_{j \in (J^d_{--}(z) \cup J^d_{-0}(z))} z_j^2 + \frac{1}{8}\sum_{j \in J^d_{-+}(z)} \frac{(2\gamma_j - d_j\gamma_j + 2z_j)^2 - 4z_j^2(2 - d_j^2)}{2 - d_j^2}. \tag{6.8}$$

Proposition 6.3.6 (TUNS (BODE) O.R. [112]). *If there exists $j \in J$ such that:*
(i) $2 > d_j^2 > 1$ or
(ii) $d_j^2 = 1$ and $\gamma_j \neq 0$,
then the function θ_d is upper unbounded on $\mathbb{R}^n$.

Proof. (i) Let $s \in J$ be such that $2 > d_s^2 > 1$. Let us consider the sequence $(z^k)_{k \in \mathbb{N}^*}$, where

$$z_j^k \; = \begin{cases} -\gamma_j + \dfrac{d_j\gamma_j}{2}, \text{ if } j \in J \setminus \{s\}, \\ -\gamma_j + \dfrac{d_j\gamma_j}{2} + k, \text{ if } j = s. \end{cases}$$

Then,

$$\theta_d(z^k) \; = \; -\frac{1}{2}\sum_{j \in J \setminus \{s\}} \Big(-\gamma_j + \frac{d_j\gamma_j}{2}\Big)^2 + \frac{4k^2 - 4(2 - d_s^2)(-\gamma_s + \dfrac{d_s\gamma_s}{2} + k)^2}{8(2 - d_s^2)} =$$

$$-\frac{1}{2}\sum_{j \in J \setminus \{s\}} \Big(-\gamma_j + \frac{d_j\gamma_j}{2}\Big)^2 +$$

$$\frac{4(d_s^2 - 1)k^2 - 4k\gamma_s(2 - d_s^2)(-2 + d_s) - (2 - d_s^2)(4\gamma_s^2 + d_s^2\gamma_s^2 - 4\gamma_s^2 d_s)}{8(2 - d_s^2)},$$

and $\lim_{k \to +\infty} \theta_d(z^k) = +\infty$. Therefore, the function $\theta_d(z^k)$ is upper unbounded on $\mathbb{R}^n$.

(ii) Let $s \in J$ be such that $d_s^2 = 1$ and $\gamma_s \neq 0$. If we consider the sequence $(z^k)_{k \in \mathbb{N}^*}$, with

$$z_j^k \; = \begin{cases} -\gamma_j + \dfrac{d_j\gamma_j}{2}, \text{ if } j \in J \setminus \{s\}, \\ -\gamma_j + \dfrac{d_j\gamma_j}{2} + k\text{sgn}(\gamma_j), \text{ if } j = s, \end{cases}$$

then

$$\theta_d(z^k) \; = \; -\frac{1}{2}\sum_{j \in J \setminus \{s\}} \Big(-\gamma_j + \frac{d_j\gamma_j}{2}\Big)^2 + \frac{1}{8}\Big(4k|\gamma_s|(2 - d_s) - 5\gamma_s^2 + 4d_s\gamma_s^2\Big).$$

Therefore, $\lim_{k \to +\infty} \theta_d(z^k) = +\infty$. It results that the function $\theta_d(z^k)$ is upper unbounded on $\mathbb{R}^n$. ∎

Remark 6.3.7 *From Proposition 6.3.6 it follows that*

$$T^* \subseteq \{d \in T \,|\, d_j^2 < 1,\ \forall\, j \in J\}.$$

Let $j \in J$. Let us consider $u_j : \mathbb{R} \to \mathbb{R}$ the function given by

$$u_j(t) = (2\gamma_j - d_j\gamma_j + 2t)^2 - 4t^2(2-d_j^2) \;=\; 4t^2(d_j^2-1) + 4\gamma_j(2-d_j)t + \gamma_j^2(2-d_j)^2,\ \forall\, t \in \mathbb{R}.$$

It is easy to see that

$$t^* = \frac{\gamma_j(2-d_j)}{2(1-d_j^2)} \tag{6.9}$$

is a maximum point of u_j on $\mathbb{R}$ and

$$u_j(t^*) = \frac{\gamma_j^2(2-d_j)^2(2-d_j^2)}{1-d_j^2} \geq 0. \tag{6.10}$$

As $d_j^2 < 1$, $\forall\, j \in J$, it results that $u_j(t^*) = 0$ if and only if $\gamma_j = 0$, $\forall\, j \in J$.

Proposition 6.3.8 (Tuns (Bode) O.R. [*112*]). *If $d \in T$ and $d_j^2 < 1$, $\forall\, j \in J$, then the function θ_d has an unique maximum point $z^* = (z_1^*, z_2^*, ..., z_n^*)$, where*

$$z_j^* \;=\; \frac{\gamma_j(2-d_j)}{2(1-d_j^2)},\ \forall\, j \in J.$$

Proof. Based on (6.8) and (6.10) it results that the function θ_d will have the same maximum point as its restriction to the set

$$\Lambda = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{R}^n | z_j \geq \frac{\gamma_j(d_j-2)}{2},\ \forall\, j \in J \right\}.$$

For each $z \in \Lambda$ we have

$$\theta_d(z) = \frac{1}{8}\sum_{j\in J} \frac{(2\gamma_j - d_j\gamma_j + 2z_j)^2 - 4z_j^2(2-d_j^2)}{2-d_j^2}.$$

On the set Λ the function θ_d is twice differentiable. We get that

$$\frac{\partial\theta_d}{\partial z_j}(z) = \frac{2z_j(d_j^2-1) + 2\gamma_j - d_j\gamma_j}{2(2-d_j^2)},\ \forall\, z \in \Lambda$$

and

$$\nabla^2\theta_d(z) = \begin{pmatrix} \dfrac{d_1^2-1}{2-d_1^2} & 0 & \cdots & 0 \\ 0 & \dfrac{d_2^2-1}{2-d_2^2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \dfrac{d_n^2-1}{2-d_n^2} \end{pmatrix},\ \forall\, z \in \mathbb{R}^n.$$

As $d_j^2 < 1, \forall\, j \in J$, it results that the function θ_d is strictly concave on Λ. Then, it has an unique maximum point. As $\nabla\theta_d(z^*) = 0_n$ for $z^* = (z_1^*, z_2^*, ..., z_n^*)$, where

$$z_j^* = \frac{\gamma_j(2 - d_j)}{2(1 - d_j^2)},\ \forall\, j \in J,$$

and $z^* \in \Lambda$, it follows that z^* is the maximum point of θ_d on Λ, hence the maximum point of θ_d on $\mathbb{R}^n$. ■

Remark 6.3.9 *From Proposition 6.3.8 one gets that* $\{d \in T \,|\, d_j^2 < 1,\ \forall\, j \in J\} \subseteq T^*$. *Then, in view of Remark 6.3.6 we obtain that* $T^* = \{d \in T \,|\, d_j^2 < 1,\ \forall\, j \in J\}$.

Remark 6.3.10 *The originality of the three-level optimization problem studied above is that it depends on the parameters* d *and* γ. *For the particular case when* $n = 1$, $d \in]0, 1[$ *and* $\gamma \in]0, 1[$, *the optimal solution of the problem coincide with the optimal solution obtained from the economic point of view by* FERREIRA F. *and* BODE O.R. *[31]. More, the result that the absolute value of parameter* d *cannot exceed* 1 *has an important economic significance. Since* d *denotes the degree of differentiation of goods, the result justifies the condition* $d \in]0, 1[$, *which is frequently used in the economic literature.*

In the next section we present all the economic results obtained by FERREIRA F. and BODE O.R. [31]. We note that furthermore, within this chapter, we consider all the economic parameters (i.e. the innovation size, the royalty rate, the fixed-fee, the firms' outputs, the firms' profits, the consumer surplus and the social welfare) as mathematical variables or functions.

6.3.2 Benchmark: No-licensing Case for One Differentiated Product

We begin our exposure by recalling the economic problem studied in [31]. Then, we present all the results obtained by FERREIRA F. and BODE O.R.. We note that these results can be obtained for the particular case of the parametric optimization problem studied above when $n = 1$.

We consider a duopoly model where two firms, denoted by F_1 and F_2, produce a differentiated good. The inverse demand functions $p_i : \mathbb{R}_+^2 \times]0,\, 1[\to \mathbb{R}$, $i \in \{1, 2\}$, are given by $p_i(q_i, q_j, d) = 1 - q_i - dq_j$, for all $(q_i, q_j, d) \in \mathbb{R}_+^2 \times]0,\, 1[$, where:

- p_i represents the price of the firm F_i, $i = 1, 2$;
- q_i and q_j represent the outputs of firms F_i and F_j, $i, j = 1, 2$, $i \neq j$;

- d represents the degree of the differentiation of goods, $d \in (0, 1)$.

The duopoly market is modeled as a Stackelberg competition. Initially, both firms have identical unit production cost $c_i = c$, with $i = 1, 2$ and $0 < c < 1$. We consider that only the leader firm F_1 can engages itself in a R&D process in order to improve its technology. The cost-reducing innovation creates a new technology that reduces innovating firm's unit cost by the amount of k, while the amount invested in R&D is $k^2/2$. So, the innovation size is endogenous. We also analyse the consumer surplus CS and the social welfare W that are, respectively, defined by

$$CS = \frac{q_1^2 + 2dq_1q_2 + q_2^2}{2} \quad \text{and} \quad W = \pi_1 + \pi_2 + CS.$$

In the pre-licensing situation, firm F_1 owns a cost advantage on the market compared with firm F_2: $c_1 = c - k_{nl}$[1] and $c_2 = c$, where k_{nl} denotes the cost reduction due to the R&D process. Depending on the value of the differentiated parameter d, two cases can occur. Based on the economic restrictions, we consider the equation

$$d^2 + 2d - 2 = 0. \tag{6.11}$$

We easy can see that this equation has an unique solution belonging to interval $]0, 1[$. Let us denote by d_1, $0 < d_1 < 1$, this solution. Hence, we have the following:[2]

(A) if $0 < d < d_1$, then firm F_2 competes with firm F_1 using its old technology and gets positive profit (non-drastic innovation);

(B) if $d_1 \leq d < 1$, then firm F_2 finds unprofitable to produce any positive output (drastic innovation). In this case, firm F_1 gains the monopoly.

The profit functions of firm F_1 and firm F_2 are, respectively, given by

$$\pi_{1,nl} = (1 - q_{1,nl} - dq_{2,nl} - c + k_{nl})q_{1,nl} - (k_{nl})^2/2 \tag{6.12}$$

and

$$\pi_{2,nl} = (1 - q_{2,nl} - dq_{1,nl} - c)q_{2,nl}.$$

Using the backward induction, we start by computing the optimal output $q_{2,nl}$. Choosing $q_{2,nl}$ to maximize $\pi_{2,nl}$ yields firm F_2's quantity reaction function given by

$$q_{2,nl} = -\frac{dq_{1,nl} + c - 1}{2}. \tag{6.13}$$

Now, making a variable substitution of $q_{2,nl}$ given by (6.13) in (6.12) and choosing $q_{1,nl}$ to maximize $\pi_{1,nl}$ yields firm F_1's output given by

$$q_{1,nl} = \frac{(c-1)(d-2) + 2k_{nl}}{2(2-d^2)}. \tag{6.14}$$

[1] Throughout this chapter we use the notation subscript nl to refer to the benchmark case.

[2] We note that $d_1 \simeq 0.732$.

Furthermore, making a variable substitution of $q_{1,nl}$ given by (6.14) in (6.13) we get firm F_2's output given by

$$q_{2,nl} = \frac{(d^2+2d-4)(c-1)-2dk_{nl}}{4(2-d^2)}. \tag{6.15}$$

If the expression given by (6.15) is positive, i.e. the innovation is non-drastic, then making the variable substitution of (6.14) and (6.15) in (6.12) and choosing k_{nl} to maximize $\pi_{1,nl}$, yields that the firm's optimal cost reduction is given by the function

$$k_{nl} :]0,1[\times]0,1[\to \mathbb{R},\ k_{nl}(c,d) = \frac{(2-d)(1-c)}{2(1-d^2)}. \tag{6.16}$$

Under these circumstances, firms' optimal outputs are, respectively, given by the functions

$$q_{1,nl} :]0,1[\times]0,1[\to \mathbb{R},\ q_{1,nl}(c,d) = \frac{(2-d)(1-c)}{2(1-d^2)}$$

and

$$q_{2,nl} :]0,1[\times]0,1[\to \mathbb{R},\ q_{2,nl}(c,d) = \frac{(2-2d-d^2)(1-c)}{4(1-d^2)}. \tag{6.17}$$

Therefore, firms' profits, consumer surplus and social welfare are, respectively, given by the functions

$$\pi_{1,nl} :]0,1[\times]0,1[\to \mathbb{R},\ \pi_{1,nl}(c,d) = \frac{(1-c)^2(2-d)^2}{8(1-d^2)}, \tag{6.18}$$

$$\pi_{2,nl} :]0,1[\times]0,1[\to \mathbb{R},\ \pi_{2,nl}(c,d) = \frac{(1-c)^2(2-2d-d^2)^2}{16(1-d^2)^2}, \tag{6.19}$$

$$CS_{nl} :]0,1[\times]0,1[\to \mathbb{R},\ CS_{nl}(c,d) = \frac{(1-c)^2(5d^4+4d^3-20d^2-8d+20)}{32(1-d^2)^2} \tag{6.20}$$

and

$$W_{nl} :]0,1[\times]0,1[\to \mathbb{R},\ W_{nl}(c,d) = \frac{(1-c)^2(3d^4+28d^3-32d^2-40d+44)}{32(1-d^2)^2}. \tag{6.21}$$

From (6.17) we conclude that for $0 < d < d_1$ the innovation is non-drastic, and for $d \geq d_1$ the innovation is drastic, where d_1 is the solution of the equation (6.11).

If the expression given by (6.15) is non-positive, i.e. the innovation is drastic, then firm F_1's monopoly arises. Hence, we have firm F_2's output $\tilde{q}_{2,nl} = 0$[3] and so $\tilde{\pi}_{2,nl} = 0$.

[3]Throughout the chapter we add a $\sim$ to identify the values we get in the drastic innovation case.

Furthermore, based on (6.15), we obtain that the optimal innovation size is given by the function

$$\tilde{k}_{nl} :]0,1[\times]0,1[\to \mathrm{I\!R},\ \tilde{k}_{nl}(c,d) = \frac{(1-c)(4-2d-d^2)}{2d}. \tag{6.22}$$

Therefore, we get that the innovators' optimal output, profit, consumer surplus and social welfare, respectively, are given by the functions

$$\tilde{q}_{1,nl} :]0,1[\times]0,1[\to \mathrm{I\!R},\ \tilde{q}_{1,nl}(c,d) = \frac{1-c}{d},$$

$$\tilde{\pi}_{1,nl} :]0,1[\times]0,1[\to \mathrm{I\!R},\ \tilde{\pi}_{1,nl}(c,d) = \frac{(-8+16d-4d^3-d^4)(1-c)^2}{8d^2}, \tag{6.23}$$

$$\tilde{CS}_{nl} :]0,1[\times]0,1[\to \mathrm{I\!R},\ \tilde{CS}_{nl}(c,d) = \frac{(1-c)^2}{2d^2} \tag{6.24}$$

and

$$\tilde{W}_{nl} :]0,1[\times]0,1[\to \mathrm{I\!R},\ \tilde{W}_{nl}(c,d) = \frac{(1-c)^2(-d^4-4d^3+16d-4)}{8d^2}. \tag{6.25}$$

In what follows, we evaluate the effects of the degree d of the differentiation of goods over: the amount that reduces the leader's unit cost, the profits of both firms (leader and follower), the consumer surplus and the social welfare.

This analysis leads us, based on the economic restrictions, to the following equations:

$$d^2 - 4d + 1 = 0, \tag{6.26}$$

$$d^4 + 2d^3 + 8d - 8 = 0, \tag{6.27}$$

$$d^4 - 5d^3 - 3d^2 + 10d - 2 = 0 \tag{6.28}$$

and

$$7d^4 - 13d^3 - 9d^2 + 28d - 10 = 0. \tag{6.29}$$

We easy can prove that all these equations have an unique solution belonging to interval $]0,1[$. Let d_2, $0 < d_2 < 1$, be the solution of the equation (6.26), d_3, $0 < d_3 < 1$, be the solution of the equation (6.27), d_4, $0 < d_4 < 1$, be the solution of the equation (6.28) and d_5, $0 < d_5 < 1$, be the solution of the equation (6.29).[4]

Hence, we state the followings.

[4] We note that $d_2 \simeq 0.268$, $d_3 \simeq 0.812$, $d_4 \simeq 0.219$ and $d_5 \simeq 0.458$.

Theorem 6.3.11 (FERREIRA F. *and* BODE O.R. [*31*]). *If there exists no technology transfer, then:*

(i) For $d \in (d_2, d_1)$ (respectively, $d \in (0, d_2) \cup [d_1, 1)$), the optimal innovation size decreases (respectively, increases) with the differentiation of the goods;

(ii) For $d \in (0, 0.5) \cup (d_3, 1)$ (respectively, $d \in (0.5, d_3)$), the profit of the innovator firm increases (respectively, decreases) with the differentiation of the goods;

(iii) For $d \in (0, d_4) \cup [d_1, 1)$ (respectively, $d \in (d_4, d_1)$), the consumer surplus increases (respectively, decreases) with the differentiation of the goods;

(iv) For $d \in (0, d_5) \cup [d_1, 1)$ (respectively, $d \in (d_5, d_1)$), the social welfare increases (respectively, decreases) with the differentiation of the goods.

Proof. (i) From (6.16) and (6.22) one gets that

$$\frac{\partial k_{nl}}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in (0,d_2), \quad \frac{\partial k_{nl}}{\partial d}(c,d) > 0,\ \forall\, c \in (0,1),\ \forall\, d \in (d_2,d_1)$$

and

$$\frac{\partial \tilde{k}_{nl}}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in [d_1,1).$$

(ii) From (6.18) and (6.23) we get, respectively,

$$\frac{\partial \pi_{1,nl}}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in (0,0.5), \quad \frac{\partial \pi_{1,nl}}{\partial d}(c,d) > 0,\ \forall\, c \in (0,1),\ \forall\, d \in (0.5,d_1)$$

and

$$\frac{\partial \tilde{\pi}_{1,nl}}{\partial d}(c,d) > 0,\ \forall\, c \in (0,1),\ \forall\, d \in [d_1,d_3), \quad \frac{\partial \tilde{\pi}_{1,nl}}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in (d_3,1).$$

(iii)-(iv) Based on (6.20), (6.21), (6.24) and (6.25) we obtain that

$$\frac{\partial CS_{nl}}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in (0,d_4), \quad \frac{\partial CS_{nl}}{\partial d}(c,d) > 0,\ \forall\, c \in (0,1),\ \forall\, d \in (d_4,d_1),$$

$\frac{\partial \tilde{CS}_{nl}}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in [d_1,1)$ and

$$\frac{\partial W_{nl}}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in (0,d_5), \quad \frac{\partial W_{nl}}{\partial d}(c,d) > 0,\ \forall\, c \in (0,1),\ \forall\, d \in (d_5,d_1),$$

$\frac{\partial \tilde{W}_{nl}}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in [d_1,1)$. ■

Remark 6.3.12 (FERREIRA F. *and* BODE O.R. [*31*]). *We note that if there exists no technology transfer and the innovation is non-drastic (i.e. $d \in (0, d_1)$), the profit of the non-innovator firm increases with the differentiation of the goods.*

Proof. From (6.19) it follows that $\frac{\partial \pi_{2,nl}}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in (0,d_1)$. ■

In the next section we analyse the case when there exists a technology transfer from the leader firm to the follower firm based on a per-unit licensing contract.

6.4 Per-unit royalty Licensing Case

6.4.1 Modeling and Solving the Economic Problem

Let $n \in \mathbb{N}^*$, $J = \{1, 2, ..., n\}$, $r = (r_1, r_2, ..., r_n) \in \mathrm{I\!R}^n$ and $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n) \in \mathrm{I\!R}^n$, with $\gamma_j \in]0, 1[, \forall\, j \in J$.

Let $T \subseteq]0, 1[^n$ be the set of variation of the parameter $d = (d_1, d_2, ..., d_n) \in T$.

Using d we can set the diagonal matrix $D \in \mathrm{I\!R}^n \times \mathrm{I\!R}^n$ such that

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}.$$

For each $d \in T$, let $F_d : \mathrm{I\!R}^n \times \mathrm{I\!R}^n \times \mathrm{I\!R}^n \times \mathrm{I\!R}^n \to \mathrm{I\!R}$, $f_d : \mathrm{I\!R}^n \times \mathrm{I\!R}^n \times \mathrm{I\!R}^n \times \mathrm{I\!R}^n \to \mathrm{I\!R}$ and $g_d : \mathrm{I\!R}^n \times \mathrm{I\!R}^n \times \mathrm{I\!R}^n \times \mathrm{I\!R}^n \to \mathrm{I\!R}$ be the functions given, respectively, by

$$F_d(x, y, z, r) = <\gamma - x - Dy + z, x> -\frac{1}{2}\|z\|^2 + <r, y>, \tag{6.30}$$

$$f_d(x, y, z, r) = <\gamma - x - Dy + z, x> + <r, y>, \tag{6.31}$$

$$g_d(x, y, z, r) = <\gamma - y - Dx + z - r, y>, \tag{6.32}$$

$\forall\, (x, y, z, r) \in \mathrm{I\!R}^n \times \mathrm{I\!R}^n \times \mathrm{I\!R}^n \times \mathrm{I\!R}^n$.

For each $z \in \mathrm{I\!R}^n$ and $x, r \in \mathrm{I\!R}^n_+$, let $S^*_d(x, z, r)$ be the set defined as

$$S^*_d(x, z, r) = argmax\{g_d(x, y, z, r)\,|\, y \in \mathrm{I\!R}^n\}.$$

The elements of the set $S^*_d(x, z, r)$ will be generically denoted by $y^{x,z,r}$.

For each $z \in \mathrm{I\!R}^n$, $y \in S^*_d(x, z, r)$ and $r \in \mathrm{I\!R}^n_+$, let $S^*_d(z, r)$ be the set defined as

$$S^*_d(z, r) = argmax\{f_d(x, y^{x,z,r}, z, r)\,|\, x \in \mathrm{I\!R}^n_+\}.$$

The elements of the set $S^*_d(z, r)$ will be generically denoted by $x^{z,r}$.

For each $z \in \mathrm{I\!R}^n$, $x \in S^*_d(z, r)$ and $y \in S^*_d(x, z, r)$, let $S^*_d(z)$ be the set defined as

$$S^*_d(z) = argmax\{F_d(x^{z,r}, y^{x,z,r}, z, r)\,|\, r \in \mathrm{I\!R}^n_+\}.$$

The elements of the set $S^*_d(z)$ will be generically denoted by r^z.

Let us consider the four-level parametric optimization problem

$$(P_{royalty}) \quad \begin{cases} F_d(x, y, z, r) \to \max \\ y \in S^*_d(x, z, r) \\ x \in S^*_d(z, r) \\ r \in S^*_d(z) \\ z \in \mathrm{I\!R}^n \end{cases}, \quad d \in T.$$

For each $d \in T$, by $(P_{d,royalty})$ we denote the four-level optimization problem obtained from $(P_{royalty})$ if the parameter is fixed to d.

Remark 6.4.1 *If $T =]0,1[^n$, then the problem $(P_{royalty})$ is the mathematical model attached to the basic economic problem described in Section 6.3.2 in case the technology transfer occurs by means of a per-unit royalty.*

Determining the Set $S_d^*(x,z,r)$

Let $d \in T$, $x, r \in \mathbb{R}_+^n$ and $z \in \mathbb{R}^n$. We consider the problem

$$(P^1_{d,x,z,r}) \quad \begin{cases} \varphi_{d,x,z,r}(y) \to \max, \\ y \in \mathbb{R}^n, \end{cases}$$

where $\varphi_{d,x,z,r}(y) = g_d(x,y,z,r) = -\|y\|^2 + < \gamma - Dx + z - r, y >$, $\forall\, y \in \mathbb{R}^n$.

One gets that $\nabla \varphi_{d,x,z,r}(y) = -2y + \gamma - Dx + z - r$, $\forall\, y \in \mathbb{R}^n$. Since

$$\nabla^2 \varphi_{d,x,z,r}(y) = \begin{pmatrix} -2 & 0 & \cdots & 0 \\ 0 & -2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -2 \end{pmatrix}, \ \forall\, y \in \mathbb{R}^n,$$

it results that the function $\varphi_{d,x,z,r}$ is strictly concave. It follows that $y^{x,z,r} = \frac{\gamma - Dx + z - r}{2}$ is the maximum point of $\varphi_{d,x,z,r}$.. Therefore, recalling the problem $(P_{d,royalty})$, $d \in T$, it results that for all $z \in \mathbb{R}^n$, if $r \in S_d^*(z)$ and $x \in S_d^*(z,r)$, then the set $S_d^*(x,z,r)$ has just one element, i.e.

$$S_d^*(x,z,r) = \{y^{x,z,r}\} = \Big\{\frac{\gamma - Dx + z - r}{2}\Big\}. \tag{6.33}$$

Determining the Set $S_d^*(z,r)$

Now, let $d \in T$, $z \in \mathbb{R}^n$ and $r \in \mathbb{R}_+^n$. We consider the following optimization problem

$$(P^2_{d,z,r}) \quad \begin{cases} \phi_{d,z,r}(x) \to \max, \\ x \in \mathbb{R}_+^n, \end{cases}$$

where

$$\phi_{d,z,r}(x) = f_d(x, y^{x,z,r}, z, r) =< \gamma - \frac{1}{2}D\gamma + z, x > + < (D^2 - I_n)x, x > + < r, y^{x,z,r} >,$$

for all $x \in \mathbb{R}_+^n$, I_n being the identity matrix in n dimensions.

We remark that

$$\phi_{d,z,r}(x) = -\sum_{j \in J} x_j^2 + \sum_{j \in J} \Big(\gamma_j - d_j y_j^{x,z,r} + z_j\Big) x_j + \sum_{j \in J} r_j y_j^{x,z,r}, \ \forall\, x \in \mathbb{R}_+^n.$$

We get that $\nabla \phi_{d,z,r}(x) = -2x + \gamma - \frac{D(\gamma+z)}{2} + D^2 x + z$, $\forall\, x \in \mathrm{IR}^n_+$. Since

$$\nabla^2 \phi_{d,z,r}(x) = \begin{pmatrix} -2+d_1^2 & 0 & \cdots & 0 \\ 0 & -2+d_2^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -2+d_n^2 \end{pmatrix}, \forall\, x \in \mathrm{IR}^n_+,$$

it results that the function $\phi_{d,z,r}$ is strictly concave. It follows that $x^{z,r} = (x_1^{z,r}, \ldots, x_n^{z,r})$, with

$$x_j^{z,r} = \begin{cases} \frac{(d_j-2)(\gamma_j+z_j)}{2(d_j^2-2)}, \text{ if } \gamma_j + z_j > 0 \\ \\ 0, \text{ if } \gamma_j + z_j \leq 0 \end{cases}, \forall\, j \in J, \tag{6.34}$$

is the maximum point of $\phi_{d,z,r}$.

Therefore, recalling the problem $(P_{d,royalty})$, $d \in T$, one gets that for all $z \in \mathrm{IR}^n$, if $r \in S_d^*(z)$ then the set $S_d^*(z,r)$ has just one element, i.e.

$$S_d^*(z,r) = \{x^{z,r}\} = \begin{cases} \frac{(d-2)(\gamma+z)}{2(d^2-2)}, \text{ if } \gamma + z > 0, \\ \\ 0, \text{ if } \gamma + z \leq 0. \end{cases}$$

Determining the Set $S_d^*(z)$

Let $d \in T$ and $z \in \mathrm{IR}^n$. We consider the following optimization problem

$$(P^3_{d,z}) \quad \begin{cases} \rho_{d,z}(r) \to \max, \\ z \in \mathrm{IR}^n, \end{cases}$$

where $\rho_{d,z}(r) = F_d(x^{z,r}, y^{x,z,r}, z, r)$.

We remark that

$$\rho_{d,z}(r) = -\sum_{j\in J}(x_j^{z,r})^2 + \sum_{j\in J}\Big(\gamma_j - d_j y_j^{x,z,r} + z_j\Big)x_j^{z,r} + \sum_{j\in J} r_j y_j^{x,z,r} - \frac{\sqrt{\sum_{j\in J} z_j^2}}{2}, \; \forall\, r \in \mathrm{IR}^n_+.$$

We get that $\nabla \rho_{d,z}(r) = \frac{\gamma+z-2r}{2}$, $\forall\, r \in \mathrm{IR}^n_+$. Since

$$\nabla^2 \rho_{d,z}(r) = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}, \forall\, r \in \mathrm{IR}^n_+,$$

it results that the function $\rho_{d,z}(r)$ is strictly concave. It follows that $r^z = (r_1^z, r_2^z, ..., r_n^z)$ with

$$r_j^z = \begin{cases} \frac{\gamma_j + z_j}{2}, \text{ if } \gamma_j + z_j > 0 \\ \\ 0, \text{ if } \gamma_j + z_j \leq 0 \end{cases}, \forall j \in J, \tag{6.35}$$

is the maximum point of $\rho_{d,z}(r)$.

Therefore, recalling the problem $(P_{d,royalty})$, $d \in T$, it results that for all $z \in \mathrm{I\!R}^n$ the set $S_d^*(z)$ has just one element, i.e.

$$S_d^*(z) = \{r^z\} = \begin{cases} \frac{\gamma + z}{2}, \text{ if } \gamma + z > 0, \\ \\ 0, \text{ if } \gamma + z \leq 0. \end{cases}$$

From (6.34) and (6.35) one gets that

$$y_j^{x,z,r} = \begin{cases} \frac{(\gamma_j + z_j)(2d_j - 2)}{4(d_j^2 - 2)}, \text{ if } \gamma_j + z_j > 0, \\ \\ 0, \text{ if } \gamma_j + z_j \leq 0. \end{cases} \tag{6.36}$$

The Optimal Solution of $(P_{d,royalty})$ *Problem*

Under these circumstances, to solve the initial problem $(P_{d,royalty})$ is equivalent to:

$$\text{determine the set } \ argmax\{F_d(x^{z,r}, y^{x,z,r}, z, r^z) \,|\, z \in \mathrm{I\!R}^n\}.$$

Therefore, now we solve the problem

$$(P^4) \quad \begin{cases} \theta_d(z) \to \max, \\ z \in \mathrm{I\!R}^n, \end{cases}$$

where

$$\theta_d(z) = F_d(x^{z,r}, y^{x,z,r}, z, r^z) \ = < \gamma - x^{z,r} - Dy^{x,z,r} + z, x^{z,r} > -\frac{1}{2}\|z\|^2 + < r^z, y^{x,z,r} > .$$

Proposition 6.4.2 (Tuns (Bode) O.R. *and* Ferreira F. [*114*]). *If* $d \in T$, *then the function* θ_d *is strictly concave and it has an unique maximum point* $z^* = (z_1^*, z_2^*, ..., z_n^*)$ *with*

$$z_j^* = \frac{\gamma_j(2d_j - 3)}{2d_j^2 - 2d_j - 1}, \forall j \in J.$$

6.4.2 Per-unit royalty Licensing Case for One Differentiated Product

In the present paragraph, recalling the economic problem introduced in Section 6.3.2, we present all the results obtained by Ferreira F. and Tuns (Bode) O.R. in [32]. This section deals with the case of licensing by means of a per-unit royalty and yields the main results. We note that these results can be obtained for the particular case of the four-level parametric optimization problem studied above when $n = 1$.

In case of the royalty licensing, the production costs of firm F_1 and firm F_2 are, respectively, given by $c - k_r$ and $c - k_r + r_1$,[5] where r_1 denotes the per-unit royalty. It is obvious that if $r_1 \geq k_r$ then it is not convenient for firm F_2 to accept the licensing. Hence, the following restriction is imposed: $r_1 < k_r$.

In this situation, the profits of the firms F_1 and F_2 are, respectively, given by

$$\pi_{1,r} = (1 - q_{1,r} - dq_{2,r} - c + k_r)q_{1,r} - (k_r)^2/2 + r_1 q_{2,r}$$

and

$$\pi_{2,r} = (1 - q_{2,r} - dq_{1,r} - c + k_r - r_1)q_{2,r}.$$

Using the backward induction, we start by computing the optimal output $q_{2,r}$ and, then, $q_{1,r}$. By substituting these outputs in $\pi_{1,r}$ and choosing r_1 to maximize it, we get that $r_1 = \frac{1-c+k_r}{2}$. Based on the restriction $r_1 < k_r$, it is necessary that $k_r > 1-c$. Furthermore, based on the above values and choosing k_r to maximize $\pi_{1,r}$ we obtain the optimal cost reduction given by the function

$$k_r :]0,1[\times]0,1[\to \mathbb{R},\ k_r(c,d) = \frac{(3-2d)(1-c)}{-2d^2+2d+1}. \tag{6.37}$$

The function $r_1 :]0,1[\times]0,1[\to \mathbb{R}$ given by

$$r_1(c,d) = \frac{(2-d^2)(1-c)}{-2d^2+2d+1} \tag{6.38}$$

returns the optimal royalty rate.

Standard computations yield the corresponding outputs given by the functions

$$q_{1,r} :]0,1[\times]0,1[\to \mathbb{R},\ q_{1,r}(c,d) = \frac{(2-d)(1-c)}{1+2d-2d^2},$$

$$q_{2,r} :]0,1[\times]0,1[\to \mathbb{R},\ q_{2,r}(c,d) = \frac{(1-d)(1-c)}{1+2d-2d^2},$$

[5] Throughout this chapter we use the notation subscript r to refer to the per-unit royalty licensing case.

profits by the functions

$$\pi_{1,r} :]0,1[\times]0,1[\to \mathbb{R},\ \pi_{1,r}(c,d) = \frac{(1-c)^2(3-2d)}{2(1+2d-2d^2)}, \tag{6.39}$$

$$\pi_{2,r} :]0,1[\times]0,1[\to \mathbb{R},\ \pi_{2,r}(c,d) = \frac{(1-c)^2(1-d)^2}{(1+2d-2d^2)^2}, \tag{6.40}$$

consumer surplus by the function

$$CS_r :]0,1[\times]0,1[\to \mathbb{R},\ CS_r(c,d) = \frac{(1-c)^2(2d^3-4d^2-2d+5)}{2(2d^2-2d-1)^2} \tag{6.41}$$

and social welfare by the function

$$W_r :]0,1[\times]0,1[\to \mathbb{R},\ W_r(c,d) = \frac{(1-c)^2(3d^3-6d^2-d+5)}{(2d^2-2d-1)^2}. \tag{6.42}$$

By comparing the total profit of the innovator firm F_1 obtained by a royalty licensing with the profit obtained when it does not license, using standard computations, it follows that,[6] for all $c \in (0,1)$, we have

$$\pi_{1,r}(c,d) - \pi_{1,nl}(c,d) = \frac{(1-c)^2(2d^4-2d^3+3d^2-12d+8)}{8(1-d^2)(1+2d-2d^2)} > 0,\ \forall\, d \in (0,d_1)$$

and

$$\tilde{\pi}_{1,r}(c,d) - \tilde{\pi}_{1,nl}(c,d) = \frac{(1-c)^2(2d^6+6d^5-9d^4-28d^3+36d^2-8)}{8d^2(2d^2-2d-1)} > 0,\ \forall\, d \in [d_1,1).$$

By comparing the total profit of the licensee firm F_2 obtained if accepts the license by paying a royalty to the innovator firm with the profit obtained when it does not accept the license, for all $c \in (0,1)$ one gets that

$$\pi_{2,r}(c,d) - \pi_{2,nl}(c,d) = \frac{(1-c)^2 h(d)}{16(1-d^2)^2(1+2d-2d^2)^2} > 0,\ \forall\, d \in (0,d_1)$$

and

$$\tilde{\pi}_{2,r}(c,d) - \tilde{\pi}_{2,nl}(c,d) = \frac{(1-c)^2(1-d)^2}{(1+2d-2d^2)^2} > 0,\ \forall\, d \in [d_1,1),$$

where $h(d) = -4d^8 - 8d^7 + 48d^6 - 4d^5 - 113d^4 + 92d^3 + 16d^2 - 40d + 12$.

Hence, we can conclude that a royalty licensing strictly dominates no-licensing. For the non-innovator firm, a royalty licensing is also always better than no-licensing.

In what follows, we evaluate the effects of the degree d of the differentiation of goods over: the amount that reduces the leader's unit cost, the optimal royalty rate, the consumer surplus, the social welfare and the profits of both firms (leader and follower).

We can state the following.

[6] We note that in the royalty licensing case the innovation is non-drastic for all $d \in (0,1)$. Throughout this chapter, to be consistent in the notations, we use the $\sim$ notation for this case also when we do comparison with the other licensing contracts in the drastic case, i.e. when $d \in [d_1,1)$.

Theorem 6.4.3 (Ferreira F. *and* Tuns (Bode) O.R. [*32*]). *If there exists a technology transfer based on a royalty licensing, then:*

(i) The optimal innovation size, the optimal royalty rate, the consumer surplus and the social welfare increase with the differentiation of the goods;

(ii) In the non-drastic innovation case ($d \in (0, d_1)$)*, if the goods are sufficiently differentiated, then the interest of the innovator firm in licensing its technology increases with the differentiation of the goods;*

(iii) In the drastic innovation case ($d \in [d_1, 1)$)*, if the goods are neither sufficiently differentiated nor sufficiently homogenous* ($d \in [d_1, d_6)$) (*respectively, sufficiently homogenous* ($d \in (d_6, 1)$)*, then the interest of the innovator firm in licensing its technology increases* (*respectively, decreases*) *with the differentiation of the goods.*

Proof. (i) From (6.37) it follows that $\frac{\partial k_r}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in (0,1)$. Also, based on (6.38) we get that $\frac{\partial r_1}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in (0,1)$.

From (6.41) and (6.42) we obtain that

$$\frac{\partial CS_r}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in (0,1), \quad \text{and} \quad \frac{\partial W_r}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in (0,1).$$

(ii) Furthermore, for the leader firm, (6.18) and (6.39) implies that

$$\frac{\partial(\pi_{1,r} - \pi_{1,nl})}{\partial d}(c,d) = \frac{(c-1)^2 i(d)}{4(d^2-1)^2(2d^2-2d-1)^2},$$

where $i(d) = -12d^5 + 48d^4 - 56d^3 + 6d^2 + 27d - 14$. By looking at the plot of the function $i(d)$, we can conclude that

$$\frac{\partial(\pi_{1,r} - \pi_{1,nl})}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in (0, d_1).$$

From (6.23) and (6.39) standard computations yield that for all $c \in (0,1)$ we have

$$\frac{\partial(\tilde{\pi}_{1,r} - \tilde{\pi}_{1,nl})}{\partial d}(c,d) = \frac{(c-1)^2 j(d)}{4d^3(2d^2-2d-1)^2},$$

where $j(d) = 4d^8 - 16d^6 + 28d^5 - 63d^4 + 50d^3 + 32d^2 - 24d - 8$. We easy can see that the equation $j(d) = 0$ has an unique solution belonging to interval $]0.1[$. Let d_6, $0 < d_6 < 1$, be this solution.[7] We can conclude that $\frac{\partial(\tilde{\pi}_{1,r} - \tilde{\pi}_{1,nl})}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in [d_1, d_6)$ and $\frac{\partial(\tilde{\pi}_{1,r} - \tilde{\pi}_{1,nl})}{\partial d}(c,d) > 0,\ \forall\, c \in (0,1),\ \forall\, d \in (d_6, 1)$. ■

Remark 6.4.4 (Ferreira F. *and* Tuns (Bode) O.R. [*32*]). *In the non-drastic innovation case, the interest of the non-innovator firm in accepting the new technology by paying a per-unit royalty increases with the differentiation of the goods.*

[7] We note that $d_6 \simeq 0.849$.

Proof. For the follower firm, based on (6.19) and (6.40), we have $\frac{\partial(\pi_{2,r}-\pi_{2,nl})}{\partial d}(c,d) \leq 0$, $\forall\, c \in (0,1)$, $\forall\, d \in (0, d_1)$. ∎

In the following two sections, recalling the economic problem introduced in Section 6.3.2, we study the cases when there exists a technology transfer between the two firms based either on a fixed-fee licensing contract or on a two-part tariff licensing contract.

6.5 Fixed-fee Licensing Case for One Differentiated Product

In this section we consider licensing by means of a fixed-fee only. We note that the results within this paragraph belong to the author and can be found in Ferreira F. and Tuns (Bode) O.R. [32].

Let us suppose that the firm F_2 accepts the licensing contract by paying a fixed-fee, denoted by f_1^L. This entitles it to produce by using the new technology innovation, which generates the same cost reduction as firm F_1.

The profit functions for the leader and follower firms are, respectively, given by

$$\pi_{1,f}^L = (1 - q_{1,f}^L - dq_{2,f}^L - c + k_f^L)q_{1,f}^L - (k_f^L)^2/2 + f_1^L$$

and

$$\pi_{2,f}^L = (1 - q_{2,f}^L - dq_{1,f}^L - c + k_f^L)q_{2,f}^L - f_1^L.$$

Again we use the backward induction in order to determine the maximum fixed-fee that the leader firm can charge. For the beginning, we determine the optimal output of each firm. We get that the corresponding optimal output for the leader firm is

$$q_{1,f}^L = \frac{(1 - c + k_f^L)(2 - d)}{2(2 - d^2)}$$

and for the follower firm is

$$q_{2,f}^L = \frac{(1 - c + k_f^L)(4 - 2d - d^2)}{4(2 - d^2)}.$$

Thus, based on the optimal firms' output, we determine the maximum fixed-fee that the leader firm can charge. This fee is such that the follower's profit equals its no-licensing profit.

(A) *Non-drastic innovation* (i.e. $d \in (0, d_1)$)

In the non-drastic innovation case, if the follower's profit equals its no-licensing profit, i.e. $\pi^L_{2,f} = \pi^L_{2,nl}$, then the corresponding cost reduction is given by the function

$$k^L_f :]0,1[\times]0,1[\to \mathbb{R},\ k^L_f(c,d) = \frac{(c-1)(d^4 - 12d^3 + 8d^2 + 32d - 32)}{d(9d^3 - 12d^2 - 24d + 32)} \tag{6.43}$$

and the maximum fixed-fee that the leader firm can charge by the function

$$f^L_1 :]0,1[\times]0,1[\to \mathbb{R},\ f^L_1(c,d) = \frac{(1-c)^2 m(d)}{16d^2(1-d^2)^2(9d^3 - 12d^2 - 24d + 32)^2}, \tag{6.44}$$

where $m(d) = -17d^{12} + 148d^{11} + 512d^{10} - 1912d^9 - 3076d^8 + 10336d^7 + 5248d^6 - 26624d^5 + 5120d^4 + 27648d^3 - 17408d^2 - 4096d + 4096$. Hence, the firms' optimal outputs in this case are, respectively, given by the functions

$$q^L_{1,f} :]0,1[\times]0,1[\to \mathbb{R},\ q^L_{1,f}(c,d) = \frac{4(2-d)(1-c)(2-d^2)}{d(9d^3 - 12d^2 - 24d + 32)}$$

and

$$q^L_{2,f} :]0,1[\times]0,1[\to \mathbb{R},\ q^L_{2,f}(c,d) = \frac{2(2-d^2)(1-c)(4-2d-d^2)}{d(9d^3 - 12d^2 - 24d + 32)}.$$

Therefore, the functions

$$\pi^L_{1,f} :]0,1[\times]0,1[\to \mathbb{R},\ \pi^L_{1,f}(c,d) = \frac{(1-c)^2 n(d)}{16d(1-d^2)^2(9d^3 - 12d^2 - 24d + 32)} \tag{6.45}$$

and

$$\pi^L_{2,f} :]0,1[\times]0,1[\to \mathbb{R},\ \pi^L_{2,f}(c,d) = \frac{(1-c)^2(d^4 + 4d^3 - 8d + 4)}{16(1-d^2)^2}, \tag{6.46}$$

where $n(d) = -17d^8 + 72d^7 + 24d^6 - 312d^5 + 116d^4 + 464d^3 - 224d^2 - 384d + 256$, returns the firms' profits.

Hence, the consumer surplus and social welfare are, respectively, given by the functions

$$CS^L_f :]0,1[\times]0,1[\to \mathbb{R},\ CS^L_f(c,d) = \frac{2(1-c)^2(2-d^2)^2(5d^4 + 4d^3 - 32d^2 + 32)}{d^2(9d^3 - 12d^2 - 24d + 32)^2} \tag{6.47}$$

and

$$W^L_f :]0,1[\times]0,1[\to \mathbb{R},\ W^L_f(c,d) = \frac{(1-c)^2 p(d)}{2d^2(9d^3 - 12d^2 - 24d + 32)^2}, \tag{6.48}$$

where $p(d) = 11d^8 + 136d^7 - 400d^6 - 576d^5 + 1968d^4 + 192d^3 - 2816d^2 + 1024d + 512$.

Then, for the leader firm it is imposed one restrictive condition: it will license its technology if and only if its total profit (i.e. market profit + fixed-fee) will exceed the profit it makes with no-licensing, i.e. $\pi^L_{1,f} > \pi^L_{1,nl}$. Standard computations yield that this happens for all $d \in (0, d_1)$.

(B) *Drastic innovation* (i.e. $d \in [d_1, 1)$)

In the drastic innovation case, if the follower's profit equals its no-licensing profit, i.e. $\tilde{\pi}^L_{2,f} = \tilde{\pi}^L_{2,nl}$, then the optimal cost reduction is the same as in the non-drastic innovation case, i.e. $\tilde{k}^L_f = k^L_f$, and the maximum fixed-fee that the leader firm can charge is given by the function

$$\tilde{f}^L_1 :]0,1[\times]0,1[\to \mathbb{R},\ \tilde{f}^L_1(c,d) = \frac{4(1-c)^2(2-d^2)^2(d^2+2d-4)^2}{d^2(9d^3-12d^2-24d+32)^2}. \tag{6.49}$$

We note that the optimal output for the firm F_1 in the drastic innovation case is the same as in the non-drastic innovation case, i.e. $\tilde{q}^L_{1,f} = q^L_{1,f}$. Obviously, $\tilde{q}^L_{2,f} = 0$. Hence, the function $\tilde{\pi}^L_{1,f} :]0,1[\times]0,1[\to \mathbb{R}$, given by

$$\tilde{\pi}^L_{1,f}(c,d) = \frac{(1-c)^2(-d^4+12d^3-8d^2-32d+32)}{2d(9d^3-12d^2-24d+32)}, \tag{6.50}$$

returns the leader's profit in the drastic innovation case. Obviously, $\tilde{\pi}^L_{2,f} = 0$. Therefore, consumer surplus and social welfare are, respectively, given by the functions

$$\tilde{CS}^L_f :]0,1[\times]0,1[\to \mathbb{R},\ \tilde{CS}^L_f(c,d) = \frac{8(1-c)^2(2-d)^2(2-d^2)^2}{d^2(9d^3-12d^2-24d+32)^2} \tag{6.51}$$

and

$$\tilde{W}^L_f :]0,1[\times]0,1[\to \mathbb{R},\ \tilde{W}^L_f(c,d) = \frac{(1-c)^2 q(d)}{2d^2(9d^3-12d^2-24d+32)^2}, \tag{6.52}$$

where $q(d) = -9d^8 + 120d^7 - 176d^6 - 576d^5 + 1248d^4 + 384d^3 - 1984d^2 + 768d + 256$.

Again, for the leader firm it is imposed the restrictive condition that it will license its technology if and only if its total profit (i.e. market profit + fixed-fee) will exceed the profit it makes with no-licensing, i.e. $\tilde{\pi}^L_{1,f} > \tilde{\pi}^L_{1,nl}$.

Let us consider the equation

$$9d^7 + 24d^6 - 76d^5 - 160d^4 + 360d^3 + 160d^2 - 576d + 256 = 0. \tag{6.53}$$

We easy can prove that equation (6.53) has an unique solution belonging to interval $]0.1[$. Let us denote by d_7, $0 < d_7 < 1$, this solution.[8]

Standard computations yield that the leader firm will license its technology for all $d \in [d_1, d_7)$ and does not license for any $d \in [d_7, 1)$. Hence, in this case it is not always better for the leader firm to license its technology. Therefore, we can state the following result.

[8]We note that $d_7 \simeq 0.793$.

Remark 6.5.1 (Ferreira F. *and* Tuns (Bode) O.R. [*32*]).

(i) If the goods are sufficiently differentiated ($d \in (0, d_7)$)*, then a fixed-fee licensing strictly dominates no-licensing;*

(ii) If the goods are sufficiently homogenous ($d \in [d_7, 1)$)*, then the innovator firm never license its technology by a fixed-fee only.*

In what follows, we evaluate the effects of the degree d of the differentiation of goods over: the amount that reduces the leader's unit cost, the maximum fixed-fee that can be charged by the leader firm, the profits of both firms (leader and follower), the consumer surplus and the social welfare.

We have the following result.

Theorem 6.5.2 (Ferreira F. *and* Tuns (Bode) O.R. [*32*]).

If there exists a technology transfer based on a fixed-fee licensing contract (*i.e.* $d \in (0, d_7)$)*, then:*

(i) The optimal innovation size, the maximum fixed-fee that can be charged by the innovator firm, the consumer surplus and the social welfare increase with the differentiation of the goods;

(ii) The interest of the innovator firm in licensing its technology increases with the differentiation of the goods.

Proof. (i) Let us consider first the non-drastic innovation case (i.e. $d \in (0, d_1)$):

From (6.43) it follows that $\frac{\partial k_f^L}{\partial d}(c, d) < 0, \forall\, c \in (0, 1), \forall\, d \in (0, d_1)$. Also, from (6.44) we get that $\frac{\partial f_1^L}{\partial d}(c, d) < 0, \forall\, c \in (0, 1), \forall\, d \in (0, d_1)$.

Based on (6.47) and (6.48) we obtain that $\frac{\partial CS_f^L}{\partial d}(c, d) < 0, \forall\, c \in (0, 1), \forall\, d \in (0, d_1)$, and $\frac{\partial W_f^L}{\partial d}(c, d) < 0, \forall\, c \in (0, 1), \forall\, d \in (0, d_1)$.

Now, let us consider the drastic innovation case (i.e. $d \in [d_1, 1)$):

In the drastic innovation case we saw that the licensor firm will license its technology only for $d \in [d_1, d_7)$. So, we make the analysis only for $d \in [d_1, d_7)$. From (6.43) and the fact that $\tilde{k}_f^L = k_f^L, \forall\, d \in [d_1, d_7)$ we easily get that $\frac{\partial \tilde{k}_f^L}{\partial d}(c, d) < 0, \forall\, c \in (0, 1), \forall\, d \in [d_1, d_7)$.

Based on (6.49) one gets that $\frac{\partial \tilde{f}_1^L}{\partial d}(c, d) < 0, \forall\, c \in (0, 1), \forall\, d \in [d_1, d_7)$.

From (6.51) and (6.52) it follows that $\frac{\partial \tilde{CS}_f^L}{\partial d}(c, d) < 0, \forall\, c \in (0, 1), \forall\, d \in [d_1, d_7)$ and $\frac{\partial \tilde{W}_f^L}{\partial d}(c, d) < 0, \forall\, c \in (0, 1), \forall\, d \in [d_1, d_7)$.

(ii) In the non-drastic innovation case (i.e. $d \in (0, d_1)$), for the leader firm, based on (6.18) and (6.45), standard computations yield that

$$\frac{\partial(\pi_{1,f}^L - \pi_{1,nl}^L)}{\partial d}(c, d) < 0, \forall\, c \in (0, 1), \forall\, d \in (0, d_1).$$

On the other hand, when the innovation is drastic (i.e. $d \in [d_1, d_7)$), based on (6.23) and (6.50), for the leader firm it follows that $\frac{\partial(\tilde{\pi}^L_{1,f} - \tilde{\pi}^L_{1,nl})}{\partial d}(c, d) < 0$, $\forall\, c \in (0, 1)$, $\forall\, d \in [d_1, d_7)$. ■

6.6 Two-part tariff Licensing Case for One Differentiated Product

In the following analyses we are going to consider the situation when there can exist a technology transfer from the leader firm (the innovator) to the follower firm based on a two-part tariff licensing contract, i.e. both fixed-fee and a royalty per-unit of output.[9] Hence, both firms F_1 and F_2, i.e. the licensor (the leader) and the licensee (the follower), use the technology innovation that reduces the production cost, and the licensee agrees to pay the two-part tariff. We note that the results within this paragraph belong to the author and can be found in Ferreira F. and Bode O.R. [31].

Firm F_1's total profit in this case will be its own profit in the product market due to competition plus the fixed-fee it charges and the royalties it receives, i.e.

$$\pi^L_{1,rf} = (1 - q^L_{1,rf} - dq^L_{2,rf} - c + k^L_{rf})q^L_{1,rf} - (k^L_{rf})^2/2 + f^L_2 + r^L_2 q^L_{2,rf}.$$

Firm F_2's profit function is given by

$$\pi^L_{2,rf} = (1 - q^L_{2,rf} - dq^L_{1,rf} - c + k^L_{rf} - r^L_2)q^L_{2,rf} - f^L_2.$$

By using backward induction, standard computations yield that profits of firms F_1 and F_2 are, respectively, given by

$$\pi^L_{1,rf} = \frac{2(r^L_2)^2(1 + 2d - 2d^2) - 4r^L_2(1 - c)(2 - d^2) - (1 - c)^2(2 - d)^2}{2(5d^2 - 4d - 4)} + f^L_2$$

and

$$\pi^L_{2,rf} = \frac{\big((1 - c)(4 - 2d - d^2) - r^L_2 d(3 - 2d)\big)^2}{(4 + 4d - 5d^2)^2} - f^L_2.$$

Now, in order to determine the maximum fixed-fee that the leader can charge, we have to consider both non-drastic and drastic innovation cases.

[9] Throughout this chapter we use the notation subscript rf to refer to the two-part tariff licensing case.

(A) *Non-drastic innovation case* (i.e. $d \in (0, d_1)$)

For the case of non-drastic innovation, the maximum fixed-fee that the leader firm can charge is such that the follower's profit equals its no-licensing profit, i.e $\pi^L_{2,rf} = \pi^L_{2,nl}$. It results that the optimal royalty and the optimal cost reduction are, respectively, given by the functions

$$r^L_2 :]0,1[\times]0,1[\to \mathbb{R},\ r^L_2(c,d) = \frac{(1-c)(3d^4-5d^3-4d+8)}{2(3d^4-3d^3-7d^2+6d+2)} \tag{6.54}$$

and

$$k^L_{rf} :]0,1[\times]0,1[\to \mathbb{R},\ k^L_{rf}(c,d) = \frac{(1-c)(2d-3)}{3d^2-3d-1}. \tag{6.55}$$

Hence, the function $f^L_2 :]0,1[\times]0,1[\to \mathbb{R}$, given by

$$f^L_2(c,d) = \frac{(1-c)^2 r(d)}{16(d^2-1)^2(3d^4-3d^3-7d^2+6d+2)^2}, \tag{6.56}$$

where $r(d) = -9d^{12} - 18d^{11} + 205d^{10} - 234d^9 - 393d^8 + 684d^7 + 296d^6 - 848d^5 - 4d^4 + 592d^3 - 256d^2 - 64d + 48$, returns the maximum fixed-fee that the leader firm can charge.

Under the above circumstances, we get at the end that the optimal outputs and profits for the leader and follower firms, and the consumer surplus and social welfare, in the non-drastic innovation case, are, respectively, given by the functions

$$q^L_{1,rf} :]0,1[\times]0,1[\to \mathbb{R},\ q^L_{1,rf}(c,d) = \frac{(1-c)(2-d)(3d^2-d-4)}{2(2-d^2)(3d^2-3d-1)},$$

$$q^L_{2,rf} :]0,1[\times]0,1[\to \mathbb{R},\ q^L_{2,rf}(c,d) = \frac{(c-1)(5d^3-9d^2+4)}{2(2-d^2)(3d^2-3d-1)},$$

$$\pi^L_{1,rf} :]0,1[\times]0,1[\to \mathbb{R},\ \pi^L_{1,rf}(c,d) = \frac{(1-c)^2 s(d)}{16(d+1)^2(d-1)^2(3d^4-3d^3-7d^2+6d+2)}, \tag{6.57}$$

$$\pi^L_{2,rf} :]0,1[\times]0,1[\to \mathbb{R},\ \pi^L_{2,rf}(c,d) = \frac{(1-c)^2(d^4+4d^3-8d+4)}{16(d^2-1)^2}, \tag{6.58}$$

$$CS^L_{rf} :]0,1[\times]0,1[\to \mathbb{R},\ CS^L_{rf}(c,d) = \frac{(1-c)^2(15d^5-45d^4+17d^3+39d^2-8d-20)}{4(d^2-2)(3d^2-3d-1)^2} \tag{6.59}$$

and

$$W^L_{rf} :]0,1[\times]0,1[\to \mathbb{R},\ W^L_{rf}(c,d) = \frac{(1-c)^2(33d^5-75d^4-37d^3+163d^2-48d-40)}{4(d^2-2)(3d^2-3d-1)^2}, \tag{6.60}$$

where $s(d) = -3d^8 + 15d^7 + 3d^6 - 82d^5 + 50d^4 + 132d^3 - 100d^2 - 88d + 72$.

We observe that the profit of the follower firm is equal to the profit that it gets by a fixed-fee contract, i.e. $\pi^L_{2,rf} = \pi^L_{2,f}$.

Standard computations yield that the leader firm can license its technology based on a two-part tariff in the non-drastic innovation case, because its total profit (i.e. market profit + fixed-fee + royalties) exceeds the profit it makes with no-licensing, i.e. $\pi^L_{1,rf} > \pi^L_{1,nl}$, $\forall\, d \in (0, d_1)$.

(B) *Drastic innovation case* (i.e. $d \in [d_1, 1)$)

For the case of drastic innovation, the maximum fixed-fee that the leader firm can charge is such that the follower's profit equals its no-licensing profit, i.e. $\tilde{\pi}^L_{2,rf} = \tilde{\pi}^L_{2,nl}$. It results that the function $\tilde{f}^L_2 :]0,1[\times]0,1[\to \mathrm{I\!R}$, given by

$$\tilde{f}^L_2(c,d) = \frac{(1-c)^2(5d^3 - 9d^2 + 4)^2}{4(3d^4 - 3d^3 - 7d^2 + 6d + 2)^2}, \tag{6.61}$$

returns the maximum fixed-fee that the leader firm can charge in the drastic innovation case.

One gets that the optimal royalty, optimal cost reduction and optimal leader's output are the same as in the non-drastic innovation case, i.e. $\tilde{r}^L_2 = r^L_2$, $\tilde{k}^L_{rf} = k^L_{rf}$ and $\tilde{q}^L_{1,rf} = q^L_{1,rf}$, $\forall\, c \in (0,1)$, $\forall\, d \in [d_1, 1)$. Furthermore, we get that the leader's profit is given by the function

$$\tilde{\pi}^L_{1,rf} :]0,1[\times]0,1[\to \mathrm{I\!R},\ \tilde{\pi}^L_{1,rf}(c,d) = \frac{(1-c)^2(3d^3 - 2d^2 - 10d + 10)}{2(3d^4 - 3d^3 - 7d^2 + 6d + 2)}. \tag{6.62}$$

Obviously, $\tilde{q}^L_{2,rf} = 0$ and $\tilde{\pi}^L_{2,rf} = 0$.

Therefore, the functions $\tilde{CS}^L_{rf} :]0,1[\times]0,1[\to \mathrm{I\!R}$ and $\tilde{W}^L_{rf} :]0,1[\times]0,1[\to \mathrm{I\!R}$, given by

$$\tilde{CS}^L_{rf}(c,d) = \frac{(1-c)^2(2-d)^2(3d^2 - d - 4)^2}{8(d^2-2)^2(3d^2 - 3d - 1)^2} \tag{6.63}$$

and

$$\tilde{W}^L_{rf}(c,d) = \frac{(1-c)^2 t(d)}{8(d^2-2)^2(3d^2 - 3d - 1)^2}, \tag{6.64}$$

where $t(d) = 36d^7 - 51d^6 - 222d^5 + 405d^4 + 212d^3 - 644d^2 + 128d + 144$, returns the consumer surplus and social welfare, respectively.

We note that in this case the leader firm can license its technology based on a two-part tariff, since its total profit (i.e. market profit + fixed-fee + royalties) exceeds the profit it makes with no-licensing, i.e. $\tilde{\pi}^L_{1,rf} > \tilde{\pi}^L_{1,nl}$, $\forall\, d \in [d_1, 1)$. We have the following result.

Remark 6.6.1 (Ferreira F. *and* Bode O.R. [*31*]). *A two-part tariff licensing strictly dominates no-licensing.*

From the above results we remark that even the innovation is drastic, it is always better for the innovator firm to license its technology either by a per-unit royalty or by a two-part tariff. But this is not true if the innovation is licensed by a fixed-fee contract.

Now, we evaluate the effects of the degree d of the differentiation of goods over: the optimal innovation size, the optimal royalty rate, the maximum fixed-fee that can be charged by the leader firm, the profits of both firms (leader and follower), the consumer surplus and the social welfare. We analyse this in both non-drastic and drastic innovation cases.

Based on economic restrictions, let us consider the equation

$$6d^6 - 42d^5 + 125d^4 - 156d^3 + 14d^2 + 112d - 56 = 0. \tag{6.65}$$

It is easy to prove that this equation has an unique solution belonging to interval]0.1[. Let us denote by d_8, $0 < d_8 < 1$, this solution.[10]

For the non-drastic innovation case (i.e. $d \in (0, d_1)$), we have the following result.

Theorem 6.6.2 (Ferreira F. *and* Bode O.R. [*31*]).

If the innovation is non-drastic $(d \in (0, d_1))$ *and the technology is licensed based on a two-part licensing contract, then:*

(i) The optimal innovation size, the maximum fixed-fee that the innovator firm can charge, the consumer surplus and the social welfare increase with the differentiation of the goods;

(ii) If the goods are sufficiently differentiated $(d \in (0, d_8))$ (*respectively, neither sufficiently differentiated nor sufficiently homogenous* $(d \in (d_8, d_1))$), *then the optimal royalty rate increases* (*respectively, decreases*) *with the differentiation of the goods.*

Proof. (i) Based on (6.55) one gets that

$$\frac{\partial k_{rf}^L}{\partial d}(c, d) < 0, \, \forall\, c \in (0, 1), \, \forall\, d \in (0, d_1).$$

From (6.56) standard computations yield that

$$\frac{\partial f_2^L}{\partial d}(c, d) < 0, \, \forall\, c \in (0, 1), \, \forall\, d \in (0, d_1).$$

[10] We note that $d_8 \simeq 0.721$.

Furthermore, based on (6.59) and (6.60), we get that

$$\frac{\partial CS_{rf}^{L}}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in (0,d_1),$$

and

$$\frac{\partial W_{rf}^{L}}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in (0,d_1).$$

(ii) From (6.54) it follows that

$$\frac{\partial r_2^{L}}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in (0,d_8),$$

and

$$\frac{\partial r_2^{L}}{\partial d}(c,d) > 0,\ \forall\, c \in (0,1),\ \forall\, d \in (d_8,d_1).$$

■

Now, let us consider the following equations

$$6d^2 - 18d + 11 = 0 \tag{6.66}$$

and

$$54d^{10} - 99d^9 - 621d^8 + 1866d^7 - 42d^6 - 4446d^5 + 3146d^4 + 3020d^3 - 3276d^2 - 344d + 736 = 0. \tag{6.67}$$

Again, we easy can prove that these equations have an unique solution belonging to interval $]0,1[$. Let us denote by d_9, $0 < d_9 < 1$, this solution of the equation (6.66) and by d_{10}, $0 < d_{10} < 1$, this solution of the equation (6.67).[11]

For the drastic innovation case (i.e. $d \in [d_1, 1)$), we have the following result.

Theorem 6.6.3 (Ferreira F. *and* Bode O.R. [*31*]).

If the innovation is drastic ($d \in [d_1,1)$) *and the technology is licensed based on a two-part tariff licensing contract, then:*

(i) If the goods are neither sufficiently differentiated nor sufficiently homogenous ($d \in [d_1, d_9)$) (*respectively, sufficiently homogenous* ($d \in (d_9, 1)$)), *then the optimal innovation size increases* (*respectively, decreases*) *with the differentiation of the goods;*

(ii) The optimal royalty rate and the consumer surplus decrease with the differentiation of the goods;

(iii) The maximum fixed-fee that the innovator firm can charge increases with the differentiation of the goods;

(iv) If the goods are neither sufficiently differentiated nor sufficiently homogenous ($d \in [d_1, d_{10})$) (*respectively, sufficiently homogenous* ($d \in (d_{10}, 1)$)), *then the social welfare increases* (*respectively, decreases*) *with the differentiation of the goods.*

[11] We note that $d_9 \simeq 0.855$ and $d_{10} \simeq 0.863$.

Proof. (i) Based on (6.55) we have that

$$\frac{\partial \tilde{k}_{rf}^{L}}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in [d_1, d_9), \quad \text{and} \quad \frac{\partial \tilde{k}_{rf}^{L}}{\partial d}(c,d) > 0,\ \forall\, c \in (0,1),\ \forall\, d \in (d_9, 1).$$

(ii) Also, (6.54) implies that

$$\frac{\partial \tilde{r}_{rf}^{L}}{\partial d}(c,d) > 0,\ \forall\, c \in (0,1),\ \forall\, d \in [d_1, 1).$$

From (6.63) we obtain that

$$\frac{\partial \tilde{CS}_{rf}^{L}}{\partial d}(c,d) > 0,\ \forall\, c \in (0,1),\ \forall\, d \in [d_1, 1).$$

(iii) Now, based on (6.61), we have that

$$\frac{\partial \tilde{f}_{2}^{L}}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in [d_1, 1).$$

(iv) From (6.64) it follows that

$$\frac{\partial \tilde{W}_{rf}^{L}}{\partial d}(c,d) < 0,\ \forall\, c \in (0,1),\ \forall\, d \in [d_1, d_{10})$$

and

$$\frac{\partial \tilde{W}_{rf}^{L}}{\partial d}(c,d) > 0,\ \forall\, c \in (0,1),\ \forall\, d \in (d_{10}, 1).$$

■

The present chapter studied the licensing, one of the most used methods for technology transfer between firms. We analyzed different licensing contracts in a differentiated Stackelberg model, when one of the firms engages itself in a R&D process that gives an endogenous cost-reducing innovation. Depending on the degree of the differentiation of goods, we saw that the innovation can be either non-drastic or drastic.

We computed explicitly the optimal outputs, the profits, the optimal innovation size, the consumer surplus and the social welfare, in both non-drastic and drastic innovation cases. Furthermore, we did a comparative static analysis and conclude that the degree of the differentiation of goods represents a great importance in the results.

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