Tamás László

Poincaré series and polynomials for links of normal surface singularities

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To my children

Emma and Ábel

Preface

The present monograph is based on the series of articles and works [39, 40, 46, 44, 45, 42] of the author and his collaborators investigating Poincaré series associated with the topology of normal surface singularities. The aim of this book was to summarize the important results in this subject we have developed in the last decade, which highlighted the importance of the appearance of Poincaré series in singularity theory.

The book consists of 8 chapters including an introductory part which is offered for non-specialists as well, as an introduction to local singularity theory, in particular to the theory of normal surface singularities as an extremely active area of current research. Then the next chapter focuses on this theory and presents, from our viewpoint, the needed 'packages', classical and breakthrough results in the study of normal surface singularities. This is followed by the main parts of the book, where we motivate and substantiate in detail the importance of the appearance of the topological Poincaré series and present our main results in chronological order, divided into five chapters.

Taking this opportunity, the author would like to express his gratitude to his colleagues - András Némethi, Zsolt Szilágyi and János Nagy - for the fruitful collaborations and works during the developement of this topic.

The book is recommended to all interested readers in the hope that it will become a useful source for both senior researchers and for the younger generations.

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Contents

Chapter 1
Introduction

The subject of this book can be placed in *local singularity theory*, which is a meeting point in mathematics, where many areas come together, such as algebra, geometry, topology and combinatorics, just to mention some of them.

Before we start to describe this subject, we would like to offer this chapter for non–specialists as well, as a survey of this extremely active area of current research, with challenging problems. Since a lot of results and directions were developed in the last decades and the presentation of all of them would be too long, we have to pick some pieces to present here, which make the overall clear and they are also important from our point of view. We hope that this chapter will give the frame of the whole picture drawn by this monograph.

In algebraic geometry, the research on the smooth complex algebraic surfaces has a history of more than a hundred years. It started with the classification of Enriques and the Italian school. Then in the 60's, a 'modern' classification was provided by Kodaira, which uses the new techniques of algebraic geometry and topology – eg. sheaves, cohomologies and characteristic classes – with paying particular attention to the relationships of the analytic structures with topological invariants of the underlying smooth 4–manifolds. Typical examples are the topological characterization of rational surfaces or of the K3 surfaces. Later, the works of Donaldson and Witten (on 4–dimensional Seiberg–Witten theories) gave powerful tools for this comparison research.

In parallel with these theories, the study of singular surfaces started, giving birth to the local singularity theory. This theory investigates the local behavior of the singularities and has to solve new problems in the shadow of the old question:

what is the relation between the analytic and topological structures?

This is the guiding question of our research too, targeting *normal surface singularities*.

Definition Let $f_1, \ldots, f_m : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be germs of analytic functions. Then the germ of the common zero set

$$(X, 0) = (\{f_1 = \cdots = f_m = 0\}, 0) \subset (\mathbb{C}^n, 0)$$

is called a *complex surface singularity*, if the rank of the Jacobian matrix $J(x) := (\partial f_i/\partial z_j(x))_{i,j}$ is $n-2$ for any smooth point $x \in X$. Moreover, if $\operatorname{rank} J(0) < n-2$, but $\operatorname{rank} J(x) = n-2$ for any point $x \in X \setminus 0$, we say that $(X,0)$ has an *isolated singularity* at the origin. □

In particular, if $m = 1$ we talk about *2–dimensional hypersurface singularities* and if $m = n-2$, then our object is called a *complete intersection surface singularity*. Notice that in general, m can be higher then $n-2$ (cf. [75, 1.2]).

The local ring $\mathcal{O}_{(X,0)}$ of analytic functions on $(X,0)$ is defined as the quotient of the ring $\mathcal{O}_{(\mathbb{C}^n,0)}$ of power series, convergent in a small neighbourhood of 0, by the ideal $(f_1,\dots,f_m)$. Its unique maximal ideal is $\mathfrak{m}_{(X,0)} = (z_1,\dots,z_n)$. This ring determines the singularity up to a local analytic isomorphism. Let us provide the following example: assume that $\mathbb{Z}_p$ acts on $\mathbb{C}^2$ by $\xi * (z_1,z_2) := (\xi z_1, \xi^{-1} z_2)$. This induces an action on $\mathcal{O}_{(\mathbb{C}^2,0)} = \mathbb{C}\{z_1,z_2\}$ for which the ring of invariants is $\langle z_1^p, z_1 z_2, z_2^p\rangle$. This is isomorphic with $\mathbb{C}\{u,v,w\}/\langle uw = v^p\rangle$, hence the geometric quotient $(\mathbb{C}^2,0)/\mathbb{Z}_p \simeq \{(u,v,w) \in (\mathbb{C}^3,0) \ : \ uw = v^p\}$ defines a surface singularity.

In any dimension, the *normality* condition means that we require $\mathcal{O}_{(X,0)}$ to be integrally closed in its quotient field, or equivalently, a bounded holomorphic function defined on $X \setminus 0$ can be extended to a holomorphic function defined on X. In the case of surface singularities, this condition implies that $(X,0)$ has at most an isolated singularity at the origin (see [47, §3]).

One can define several invariants from the local ring in order to encode the type of the singularity. For example, we mention the *Hilbert–Samuel function*, or, in particular, the *embedding dimension* and the *multiplicity* of $(X,0)$. For their definitions and properties we refer to [75].

One may think of a normal surface singularity as an abstract geometric object $(X,0)$ with its local ring $\mathcal{O}_{(X,0)}$ and maximal ideal $\mathfrak{m}_{(X,0)}$, which encode the local analytic type. Then the main approach to analyze $(X,0)$ is a 'good' resolution $\pi : (\widetilde{X},E) \to (X,0)$. Thus, $\widetilde{X}$ is a smooth surface, π is proper and maps $\widetilde{X} \setminus E$ isomorphically onto $X \setminus 0$, where the exceptional divisor $E = \pi^{-1}(0)$ is a normal crossing divisor. This means that the irreducible components E_i are smooth projective curves, intersect each other transversally and $E_i \cap E_j \cap E_k = \emptyset$ for distinct indices i,j,k.

The numerical analytic invariants of this description might come from two different directions. They can be ranks of sheaf–cohomologies of analytic vector bundles on $\widetilde{X}$. The most important in this category, from our viewpoint, is the *geometric genus*, which can be defined by the following formula

$$p_g := \dim H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}),$$

cf. 2.2.1. Notice that p_g can be expressed on the level of X as well, using holomorphic 2–forms ([48], [75, 1.4]).

The other direction is based on the *Hilbert–Poincaré series* of $\mathcal{O}_{(X,0)}$ associated with $\pi^{-1}(0)$–divisorial multi–filtration. We will give the definition of this invariants in section 3.1, since they serve a motivation for the topological counterpart, which will be one of the main objects in this book.

The resolution makes a bridge with the topological investigation of the normal surface singularity, which, as we will see in 2.1.1, is equivalent with the description of its *link*. This is a special 3–manifold which can be constructed using the *dual resolution graph* too, via the configuration of the exceptional irreducible curves E_i of the resolution (see Section 2.1.1).

It raises the following natural questions:

Is it possible to recover some of the analytic invariants from the link, or equivalently, from the resolution graph? What kind of statements can be made about the analytic type of a singularity with a given topology?

Before we start to discuss the main questions, which motivated a huge amount of work in the last decades, we stop for a moment and motivate the reason why we choose the study of surface singularities.

If $(X, 0)$ is a curve singularity, then its link consists of as many disjoint copies of the circle S^1, as the number of irreducible components of the curve at its singular point. Hence, it contains no other information about the analytic type of $(X, 0)$.

We have the same situation in higher dimension too: from the point of view of the main questions, the topological information encoded by the link is rather poor. To justify this sentence, consider the example of a Brieskorn singularity $(X, 0) = \{x_1^2 + x_2^2 + x_3^2 + x_4^3 = 0\} \subset (\mathbb{C}^4, 0)$. Brieskorn proved in [16], that the link is diffeomorphic to S^5, but X is far from being smooth. More examples can be found also in [17].

It turned out that the case of surfaces is much more interesting and complicated: *one can have many analytic types with a given topology*. This suggests that we have to assume some conditions on the analytic side in order to investigate the connections.

H. Laufer in [49] (completing the work of Grauert, Brieskorn, Tjurina and Wagreich) gives the complete list of those resolution graphs which have a unique analytic structure. These are the so–called *taut* singularities. He classified even those resolution graphs, which support only a finitely many analytic structures, they are called *pseudo–taut* singularities. This class is very restrictive, in the sense that almost all of them are rational. Hence, in order to understand the case of more general resolution graphs, first we intend to characterize topologically some of the discrete invariants of $(X, 0)$.

Summerizing the above discussion, Artin and Laufer in 60's and 70's started to determine some of the analytic invariants from the graph. They characterized topologically the *rational* and *minimally elliptic* singularities. Laufer believed that this program, nowadays called the *Artin–Laufer program* (Section 2.2) can not be continued for more general cases. After twenty years, Némethi [64] clarified the elliptic case completely, and proposed the continuation of the program with some extra assumptions. This means that if we pose some analytical and topological conditions on the singularities, there is a hope to understand the connection between the analytical and the topological data.

One believes that for the continuation of the Artin–Laufer program, ie. to make topological characterizations of some special analytic types or their invariants, one has to find and understand first the topological counterparts of the analytic invariants.

In 2002, the work of Némethi and Nicolaescu ([77, 78, 79]) suggested a new approach. They formulated the so–called *Seiberg–Witten invariant conjecture* (see Subsection 2.4.2), which relates the geometric genus of the normal surface singularity with the Seiberg–Witten invariant of its link. This generalizes the conjecture of Neumann and Wahl ([87]), formulated for complete intersection singularities with integral homology sphere links.

They proved the relation for some 'nice' analytic structures, but later Luengo-Velasco, Melle-Hernandez and Némethi showed that it fails in general (cf. section 2.4.2). However, the corrected versions transfer us into the world of low dimensional topology, and create tools to understand the Seiberg–Witten invariants via some homology theories. For example, the Seiberg–Witten invariant appears as the normalized Euler characteristic of the Seiberg–Witten Floer homology of Kronheimer and Mrowka, and of the Heegaard–Floer homology introduced by Ozsváth and Szabó. These theories had an extreme influence on modern mathematics of the last decade, and solved a series of open problems and conjectures related to the classification of the smooth 4–manifolds and the theory of knots.

Motivated by the work of Ozsváth and Szabó and the Seiberg–Witten invariant conjecture, Némethi opened a new channel towards the continuation of the Artin–Laufer program.

He constructed a new invariant, the *graded root*, which is a special tree–graph with vertices labeled by integers. The main idea is that the set of topological types, sharing the same graded root, form a family with uniform analytic behavior too. Némethi conjectures that each family, identified by a root, can be uniformly treated at least from the point of view of some analytic invariants associated with the corresponding analytic types. The graded roots describe and give a model for the Heegaard–Floer homology in the rational and almost rational cases, using the *computation sequences* of Laufer. This concept gave birth to the *theory of lattice cohomology*, see [69, 72, 74, 41, 39].

It is a cohomological theory attached to a lattice defined by the resolution graph of the singularity. The lattice cohomology is a topological invariant of the singularity link, which has a strong relation with the geometry of the exceptional divisors in the resolution. This has even more structures than the Heegaard–Floer homology. Nevertheless, by disregarding these extra data, they are isomorphic proven by the recent work of Zemke [112] (see [72] for the the conjecture).

Moreover, the normalized Euler characteristic of the lattice cohomology equals to the Seiberg–Witten invariant, and the non–vanishing of higher cohomology modules explains the failure and corrects the Seiberg–Witten invariant conjecture in the pathological cases.

There is another concept which is strongly related to the Seiberg–Witten invariant conjecture and connects the geometry with the topology, and serves the base of this monograph. This is the *theory of Hilbert–Poincaré series associated with the singularities*.

Campillo, Delgado and Gusein-Zade studied Hilbert–Poincaré series associated with a divisorial multi–index filtration on $\mathcal{O}_{(X,0)}$ (3.1). Then, Némethi [66] unified and generalized the formulae of this concept and defined the topological counterpart, the *multivariable topological Poincaré series*, showing their coincidence in some 'nice' cases.

It was proven that the *constant term* of a quadratic polynomial associated with the topological series equals the Seiberg–Witten invariant. This shows a strong analogy with the analytic side, where the geometric genus can be interpreted in this way. This analogy, together with the interactions between the analytical and topological series, places the Seiberg–Witten invariant conjecture in a new framework by highlighting the importance of Poincaré series.

In the theory of Poincaré series it is natural to ask the following question:

what kind of informations can be decoded from the Poincaré series?

In the case of normal surface singularity links, this problem targets those informations/invariants which are encoded by the topological Poincaré series and aims to develop methods which extract these informations.

The first step is to recover the Seiberg–Witten invariants from the topological Poincaré series. This subject contributes the chapters 4, 5 and 6 of this book and has very interesting final outputs.

It turnes out that the Seiberg–Witten invariant is the *multivariable periodic constant* of the Poincaré series. Moreover, the Ehrhart theory we discuss in chapter 4 identifies the Seiberg–Witten invariant with a certain coefficient of a multivariable equivariant Ehrhart quasipolynomial. This approach gives rise some precise algorithms for the calculation of the periodic constants, or equivalently, for the Seiberg–Witten invariants. For example, one of them will be presented in chapter 7 which uses a surprising relation with the theory of modules over semigroups and affine monoids.

We emphasize that this approach is a rather complex regularization procedure. Usually it is hard to find the quasipolynomial, one needs to know all the coefficients of the topological Poincaré series and to understand their asymptotic behaviour. Nevertheless, there exists another reformulation which will be the main subject of the last chapter 8.

Particular examples and families suggest that there must exists an object which is even more general and guides the above periodic constant computation as well: one predicts a unique canonical decomposition of the topological Poincaré series into a sum consisting of a rational function 'with negative degree' (or, with zero periodic constant) and a finite polynomial, such that the sum of the coefficients of this polynomial gives exactly the periodic constant of the series. In this way, one finds a multivariable polynomial generalization of the Seiberg–Witten invariants.

As a complete answer to this problem, chapter 8 will provide from the Poincaré series a simple expression for the periodic constant, or for the Seiberg–Witten invariants of the link, which is somewhat different than the ones given before. This will be done using combination of two dualities: one of them is the topological trace of a Gorenstein type duality, the second is the Ehrhart–Macdonald–Stanley duality of Ehrhart theory. It turns out that the periodic constant is an easy precise sum of coefficients of the 'dual' series. On the other hand, the second goal of 8 is to make the connection with Ehrhart theory deeper: in this way we show that the afforementioned ‘polynomial - negative degree part’ decomposition exists indeed, and we determine the precise algorithm for its calculation.

Chapter 2
Preliminaries

This chapter is devoted to the presentation of the classical definitions and concrete results regarding to the topology of normal surface singularities. It provides the first interactions of the geometrical and topological settings and presents a conjecture on the relationship between the corresponging invariants of these settings.

Besides the references given in the body of the chapter, we recommend the following classical books and lecture notes as well [61, 103, 28, 85, 75, 68].

2.1 Topology of normal surface singularities

In this section we give an introduction to the topology of normal surface singularities with the definition of the main object, the *link* of the singularity. Using its key properties, we show how one can encode its topological data with combinatorial objects.

2.1.1 The link of $(X, 0)$

The topological approach of singularities, which will be presented in the sequel, was started with a breakthrough result of Milnor [61], regarding complex hypersurfaces. Nevertheless, his argument works not only in the hypersurface case, but also in general, when we consider an arbitrary complex analytic singularity, see [52]. In the case of surfaces, the idea is the following.

We consider a normal surface singularity $(X, 0)$ embedded into $(\mathbb{C}^n, 0)$. Then, if ϵ is small enough, the $(2n-1)$–dimensional sphere S^{2n-1}_ϵ intersects $(X, 0)$ transversally and the intersection

$$M := X \cap S^{2n-1}_\epsilon$$

is a closed, oriented 3–manifold, which does not depend on the embedding and on ϵ. M is called the *link* of $(X, 0)$. Moreover, if B^{2n}_ϵ is the $2n$–dimensional ball of radius ϵ around 0, then one shows that $X \cap B^{2n}_{\epsilon}$ is homeomorphic to the cone over M, hence the *link characterizes completely the local topological type of the singularity*.

An important discovery of Mumford [63] was that if M is simply connected, then X is smooth at 0. Neumann [85] extended this fact as follows: the link of a normal surface singularity can be recovered from its fundamental group except two cases, which are completely understood. These exceptions are the Hirzebruch–Jung (or cyclic quotient) and the cusp singularities.

The first connection between the analytical and topological properties of $(X, 0)$ is realized by the *resolution of the singular point*. The resolution of $(X, 0)$ is a holomorphic map $\pi : (\widetilde{X}, E) \to (X, 0)$ such that $\widetilde{X}$ is smooth, π is proper and maps $\widetilde{X} \setminus E$ isomorphically onto $X \setminus 0$. $E := \pi^{-1}(0)$ is called the *exceptional divisor* with irreducible components $\{E_i\}_i$. Moreover, if we assume that E is a normal crossing divisor, namely the irreducible components E_i are smooth projective curves, intersect each other transversally and $E_i \cap E_j \cap E_u = \emptyset$ for distinct indices i, j, u, then we talk about *good* resolution. One can define *minimal* (not necessarily good) resolutions as well, if there is no rational smooth irreducible component E_i with self–intersection number $b_i := (E_i, E_i) = -1$. But in almost all the cases in our discussions we use a good resolution.

To encode the combinatorial data of a good resolution, one can associate with it the *dual resolution graph* $\Gamma(\pi)$ (usually we omit π from the notation). In this graph the vertices correspond to the irreducible components E_i and the edges represent their intersection points. Moreover, we add two weights for every vertex of Γ: the self–intersection number b_i and the genus g_i of E_i. In this way we may also associate an intersection form I whose matrix is $(E_i, E_j)_{i,j}$, where (E_i, E_j) is the number of edges connecting the two corresponding vertices for $i \neq j$.

The first result, originated from DuVal and Mumford, says that Γ is connected and I is negative definite. Then a crucial work of Grauert [32] shows that every connected negative definite weighted dual graph does arise from resolving some normal surface singularity $(X, 0)$.

π identifies $\partial\widetilde{X}$, the boundary of $\widetilde{X}$, with M. Hence, the graph Γ can be regarded as a plumbing graph which makes M into an S^1–plumbed 3–manifold. Using the plumbing construction (see e.g. [103, 1.1.9]), any resolution graph Γ determines M completely. Conversely, we have to consider the equivalence class of plumbing graphs defined by finite sequences of blow–ups and/or blow–downs along rational (-1)–curves, since the resolution π and its graph are not unique. But different resolutions provide equivalent graphs in the aforementioned sense. Then a result of Neumann [85] shows that the oriented diffeomorphism type of M determines completely the equivalence class of Γ.

Finally, we consider two families of 3–manifolds, which will be our working objects throughout this book. M is called *rational homology sphere* (in short: $\mathbb{Q}HS$) if $H_1(M, \mathbb{Q}) = 0$. In particular, we say that it is an *integral homology sphere* ($\mathbb{Z}HS$) if $H_1(M, \mathbb{Z}) = 0$. Note that $H_1(M, \mathbb{Q})$ vanishes if and only if the free part of $H_1(M, \mathbb{Z})$ vanishes. The plumbing construction says that the first Betti number $b_1(M)$ is equal to $c(\Gamma) + 2\sum_i g_i$, where $c(\Gamma)$ is the number of independent cycles of the graph Γ. Hence, the final conclusion is that

$$M \text{ is } \mathbb{Q}HS \text{ if and only if } \Gamma \text{ is a tree and } g_i = 0 \text{ for all } i.$$

2.1.2 Combinatorics of the resolution/plumbing graphs

Let Γ be a connected negative definite plumbing graph and denote the set of vertices by $\mathcal{V}$. In the sequel we will alter the indices of the vertices as follows: whenever we index the irreducible exceptional divisor associated with a vertex of Γ, we will use the symbols i, j. However, sometimes it will be more convenient to use the standard notation $v \in \mathcal{V}$.

As described in the previous section, Γ can be realized as the resolution graph of some normal surface singularity $(X, 0)$, and the link M can be considered as the plumbed 3–manifold associated with Γ.

In the sequel **we assume that M is a $\mathbb{Q}HS$.**

Let $\widetilde{X}$ be the smooth 4-manifold with boundary M obtained either by the resolution $\pi : \widetilde{X} \to X$ of $(X, 0)$ with resolution graph Γ, or via plumbing disc bundles associated with the vertices of Γ with Euler number b_i (for more details on plumbings we refer to [36, §8] or [103, 1.1.9]). Since $\widetilde{X}$ has a deformation retract to the bouquet of $|\mathcal{V}|$ copies of 2–spheres $S^2 \vee \cdots \vee S^2$, the only non–vanishing homologies are $H_0(\widetilde{X}, \mathbb{Z}) = \mathbb{Z}$ and $H_2(\widetilde{X}, \mathbb{Z}) = \mathbb{Z}^{|\mathcal{V}|}$. Moreover, there is an intersection form I on $H_2(\widetilde{X}, \mathbb{Z})$. Since we identify the homology classes of the zero sections with $\{E_i\}_{i\in\mathcal{V}}$, the matrix of I with respect to the basis $\{E_i\}_{i\in\mathcal{V}}$ is given by

$$I_{ij} = \begin{cases} b_j & \text{if } i = j \\ 1 & \text{if } i \neq j \text{ and the corresponding vertices are connected by an edge} \\ 0 & \text{if } i \neq j \text{ otherwise.} \end{cases}$$

We know that in our case I is non–degenerate, negative definite and makes $L := H_2(\widetilde{X}, \mathbb{Z})$ into a lattice generated by $\{E_j\}_{j\in\mathcal{V}}$. Let $L' := \mathrm{Hom}(L, \mathbb{Z})$ be the dual of L. The fact that the homology of $\widetilde{X}$ has no torsion part and the Poincaré–Lefschetz duality imply that $L' \cong H^2(\widetilde{X}, \mathbb{Z}) \cong H_2(\widetilde{X}, M, \mathbb{Z})$. Then the begining of the relative long exact homology sequence for the pair $(\widetilde{X}, M)$ splits into the short exact sequence

$$0 \longrightarrow L \stackrel{\iota}{\longrightarrow} L' \longrightarrow H \longrightarrow 0,$$

where $H := L'/L = H_1(M, \mathbb{Z})$. The morphism $\iota : L \to L'$ can be identified with $L \to \mathrm{Hom}(L, \mathbb{Z})$ given by $l \mapsto (l, \cdot)$. The intersection form has a natural extension to $L_{\mathbb{Q}} = L \otimes \mathbb{Q}$ and we can regard L' as a sublattice of $L_{\mathbb{Q}}$ in a way that $L' = \{l' \in L \otimes \mathbb{Q} : (l', L) \subseteq \mathbb{Z}\}$. For conventional reason, one may choose the generators of L' to be the (anti)dual elements E_j^* defined via $(E_j^*, E_i) = -\delta_{ji}$ (the negative of the Kronecker symbol). Clearly, the coefficients of E_j^* are the columns of $-I^{-1}$, ie. $(I^{-1})_{ij} = (E_i^*, E_j^*)$ and the negative definiteness of I guarantees that

$$\text{all the entries of } E_j^* \text{ are strict positive.} \tag{2.1}$$

We will also set $\det(\Gamma) := \det(-I)$ to be the determinant associated with the graph Γ. Furthermore, by a result of [31, page 83 and §20],

$$\begin{aligned} -|H| \cdot (E_v^*, E_w^*) &\text{ equals the determinant of the subgraph obtained} \\ &\text{from } \Gamma \text{ by eliminating the shortest path connecting } v \text{ and } w. \end{aligned} \tag{2.2}$$

We denote by δ_v the valency of the vertex v. Furthermore, we distinguish the following subsets of vertices: the set of *nodes* $\mathcal{N} = \{v \in \mathcal{V} : \delta_v \geq 3\}$, and the set of *ends* $\mathcal{E} = \{v \in \mathcal{V} : \delta_v = 1\}$. If we delete from Γ the nodes and their adjacent edges we get the collection of (maximal) *chains* of the graph. A *leg* is a chain which is connected by only one node. $|\mathcal{V}|$ or s stay for the number of vertices, while $|\mathcal{N}|$ and $|\mathcal{E}|$ for the number of nodes and ends.

2.1.2.1 Cycles The elements of $L_{\mathbb{Q}}$ are called *rational cycles*. There is a *natural partial ordering* on the cycles: $l'_1 \leq l'_2$ if $l'_{1j} \leq l'_{2j}$ for all $j \in \mathcal{V}$. Moreover, we say that l' is *effective* if $l' \geq 0$. If $l'_i = \sum_j l'_{ij} E_j$ for $i \in \{1, 2\}$, then we write $\min\{l'_1, l'_2\} := \sum_j \min\{l'_{1j}, l'_{2j}\} E_j$ and analogously $\min\{F\}$ for a finite subset $F \subset L_{\mathbb{Q}}$. We will also use the notation $l'_1 \prec l'_2$ if $l'_{1,j} < l'_{2,j}$ for all $j \in \mathcal{V}$. (Note that $\ngeq$ differs from $\prec$.) Furthermore, if $l' = \sum_j l'_j E_j$ then we set $|l'| := \{j \in \mathcal{J} : l'_j \neq 0\}$ for the *support* of l'.

2.1.2.2 Characteristic elements and $spin^c$–structures of M We define the set of characteristic elements in L' by

$$Char := \{k \in L' : (k, x) + (x, x) \in 2\mathbb{Z} \text{ for any } x \in L\}.$$

There is a unique rational cycle $K \in L'$ which satisfies the system of *adjunction relations*

$$(K, E_j) = -b_j - 2 \text{ for all } j \in \mathcal{J}, \tag{2.3}$$

and it is called the *canonical cycle*. Then $Char = K + 2L'$ and there is a natural action of L on $Char$ by $l * k := k + 2l$, whose orbits are of type $k + 2L$. Then $H = L'/L$ acts freely and transitively on the set of orbits by $[l'] * (k + 2L) := k + 2l' + 2L$.

In many cases, it is more convenient to use the anti-canonical cycle $Z_K := -K \in L'$. Hence, by adjunction formulae we have the following expression

$$Z_K - E = \sum_{v \in \mathcal{V}} (\delta_v - 2) E_v^*, \tag{2.4}$$

where we denote $E := \sum_{v \in \mathcal{V}} E_v$.

Consider the tangent bundle $T\widetilde{X}$ of the oriented 4–manifold $\widetilde{X}$ (we can pick a Riemannian metric as well). Then $T\widetilde{X}$ determines an orthonormal frame bundle (principal $O(4)$–bundle) which we denote by $F_O(\widetilde{X})$. It is well known that the orientability of $\widetilde{X}$ means that this bundle can be reduced to an $SO(4)$–bundle, making the fibers connected. This can be thought in a way that any trivialization of the bundle over the disconnected 0–skeleton of $\widetilde{X}$ can be extended to a trivialization over the connected 1–skeleton.

In the previous sense, $spin$ and $spin^c$–structures are generalizations of the orientation as follows. A $spin$–structure on $\widetilde{X}$ (more precisely on $T\widetilde{X}$) means that the trivialization of the tangent bundle can be extended to the 2–skeleton. Then the $spin^c$–structure is a 'complexified' version of that: we say that $\widetilde{X}$ has a $spin^c$–structure if there exists a complex line bundle $\mathcal{L}$ such that $T\widetilde{X} \oplus \mathcal{L}$ has a $spin$–structure. This $\mathcal{L}$ is called the *determinantal line bundle* of the $spin^c$–structure. If $\widetilde{X}$ admits a $spin$–structure, then using the fiber product one can construct a *canonical* $spin^c$–structure as well. This can be done also when an almost complex structure is given. (More details regarding of these definitions and constructions can be found in [34, 56].)

By [34, Proposition 2.4.16], the fact that in our case $L' = H^2(\widetilde{X}, \mathbb{Z})$ has no 2–torsion implies that $\mathcal{L}$ determines the $spin^c$–structure, and the first Chern class (of $\mathcal{L}$) realizes an identification between the set of $spin^c$–structures $Spin^c(\widetilde{X})$ on $\widetilde{X}$ and $Char \subseteq L'$. Moreover, $Spin^c(\widetilde{X})$ is an L' torsor compatible with the above action of L' on $Char$.

If we look at the boundary, the image of the restriction $Spin^c(\widetilde{X}) \to Spin^c(M)$ consists of exactly those $spin^c$–structures on M, whose Chern classes are the restrictions $L' \to H^2(M, \mathbb{Z}) \cong H_1(M, \mathbb{Z})$, i.e. are the torsion elements in H.
Therefore, in our situation, all the $spin^c$–structures on M are obtained by restriction, $Spin^c(M)$ is an H torsor, and the actions are compatible with the factorization $L' \to H$. Hence, one has an identification of $Spin^c(M)$ with the set of L–orbits of $Char$, and this identification is compatible with the action of H on both sets. In this way, any $spin^c$-structure of M will be represented by an orbit $[k] := k + 2L \subseteq Char$.

On the other hand, $Spin^c(M)$ is equipped with a natural involution $\sigma \longleftrightarrow \overline{\sigma}$ such that the Chern class of $\overline{\sigma}$ is the negative of the Chern class of σ and $\overline{h * \sigma} = (-h) * \overline{\sigma}$.

The canonical $spin^c$–structure σ_{can} corresponds to $[-K]$, moreover every orbit $[k]$ has the form of $K + 2(l' + L)$ for some $l' \in L'$. Therefore, any $spin^c$-structure, or characteristic orbit, can be expressed in the form $h * (\overline{\sigma_{can}})$ for a fixed $h \in H$. In our discussions, we will mostly use only this group element h to represent the corresponding $spin^c$–structure on M. We emphasize that there is a twist in the above representation, in particular $h = 0$ will represent the 'anti-canonical' $spin^c$–structure corresponding to the orbit $[K + 2L]$.

2.1.2.3 Distinguished cycles associated with $spin^c$-structures Note that if we look the $spin^c$–structures as characteristic orbits, in the anti-canonical $spin^c$–structure $[K]$ there is a special element, namely the canonical cycle K. In the following, we generalize this fact for all orbits $[k]$: among all the characteristic elements in $[k]$ one can choose a very special one which plays a special role. Nevertheless, as we pointed out in the previous section, in the forthcoming discussion we will always represent the $spin^c$–structures of M with the group elements of H.

First of all, we observe that the lattice L' admits a natural partition parametrized by the group H, where for any $h \in H$ one sets

$$L'_h = \{l' \in L' \mid [l'] = h\} \subset L'. \tag{2.5}$$

Note that $L'_0 = L$. Given an $h \in H$ one can define

$$r_h := \sum_v l'_v E_v \in L'_h \tag{2.6}$$

as the unique element of L'_h such that $0 \le l'_v < 1$.

Then, we define the rational Lipman (or anti-nef) cone by

$$\mathcal{S}_{\mathbb{Q}} := \{l' \in L_{\mathbb{Q}} \mid (l', E_v) \le 0 \text{ for all } v \in \mathcal{V}\},$$

which is a cone generated over $\mathbb{Q}_{\ge 0}$ by E^*_v. Define $\mathcal{S}' := \mathcal{S}_{\mathbb{Q}} \cap L'$ as the semigroup (monoid) of anti-nef cycles of L'; it is generated over $\mathbb{Z}_{\ge 0}$ by the cycles E^*_v. Since $\{E^*_v\}_v$ have positive entries, $\mathcal{S}_{\mathbb{Q}} \setminus \{0\}$ is in the open first quadrant. Moreover, for any fixed $a \in L'$ one has:

$$\{l' \in \mathcal{S}' \ : \ l' \not\geq a\} \ \text{ is finite.} \tag{2.7}$$

The Lipman cone $\mathcal{S}'$ also admits a natural equivariant partition indexed by H by $\mathcal{S}'_h = \mathcal{S}' \cap L'_h$. Furthermore, we have the following properties:

(a) if $l'_1, l'_2 \in \mathcal{S}'_h$ then $l'_2 - l'_1 \in L$ and $\min\{l'_1, l'_2\} \in \mathcal{S}'_h$;
(b) for any $s \in L'$ the set $\{l' \in \mathcal{S}' \mid l' \not\geq s\}$ is finite;
(c) for any $h \in H$ there exists a unique *minimal cycle* $s_h := \min\{\mathcal{S}'_h\} \in \mathcal{S}'_h$ (cf. [69, 5.4]);

In fact $r_h \le s_h$ (see eg. [69, Lemma 7.4]), however, in general $r_h \ne s_h$. Note that this fact does not contradict the minimality of s_h in $\mathcal{S}'_h$ since r_h might not sit in $\mathcal{S}'_h$.

2.2 The Artin–Laufer program. Case of rational singularities

2.2.1 Algebro–geometric definitions and preliminaries

The aim of this section is to introduce some tools from the analytical (algebro-geometric) point of view for the study of the normal surface singularity $(X, 0)$. Since our discussions concentrates mostly to the topology of $(X, 0)$, this description will be rather sketchy: we need just those parts, which motivate the names and notations in 2.1.2 and create the main tools connecting the geometry and topology of $(X, 0)$. For more details regarding this section, we recommend some general references such as [75] and [5].

2.2.1.1 We start with a resolution $\pi : \widetilde{X} \to X$. The group of divisors $\mathrm{Div}(\widetilde{X})$ of $\widetilde{X}$ consists of formal finite sums $D = \sum_i m_i D_i$, where D_i is an irreducible curve on $\widetilde{X}$ and $m_i \in \mathbb{Z}$. For any divisor D, one can say that it is supported on $|D| = \cup_{m_i \ne 0} D_i$. If we pick a meromorphic function f defined on $\widetilde{X}$, then its divisor $\mathrm{div}(f) = \sum_i m_i D_i$ is a *principal divisor*, where

D_i's are irreducible components of the zeros and the poles of f, and m_i is the multiplicity (order of zero, resp. pole) of f along D_i. Divisors supported on the exceptional divisor E are called cycles, already defined in 2.1.2. We have seen already that one can define a natural partial ordering, the effectiveness and the intersection of cycles, which is determined by the resolution graph Γ.

The pullback $f\circ\pi$ of a given analytic function $f:(X,0)\to(\mathbb{C},0)$ determines an effective principal divisor $\mathrm{div}(f\circ\pi)$. Let $m_{E_j}(f)$ be the multiplicity of $\mathrm{div}(f\circ\pi)$ along E_j, then $\mathrm{div}(f\circ\pi)=\sum_{j\in\mathcal{J}} m_{E_j}(f)E_j+St(f)$, where $St(f)$ is supported on the strict transform $\overline{\pi^{-1}(f^{-1}(0)\setminus 0)}$ (closure of $\pi^{-1}(f^{-1}(0)\setminus 0)$) of the set $f^{-1}(0)$. Then, for such an f and a resolution π (encoded by its resolution graph Γ) one can associate the cycle

$$(f)_\Gamma=\sum_{j\in\mathcal{V}} m_{E_j}(f)E_j.$$

In order to get some information on the local ring $\mathcal{O}_{(X,0)}$ (i.e. about the structure of analytic functions $f:(X,0)\to(\mathbb{C},0)$) from the resolution, we may define the set of cycles

$$\mathcal{S}_{an}:=\{(f)_\Gamma\ :\ f\in\mathfrak{m}_{(X,0)}\}. \tag{2.8}$$

Then $\mathcal{S}_{an}$ is a monoid and if $l_1,l_2\in\mathcal{S}_{an}$, then $l=\min\{l_1,l_2\}$ (defined in 2.1.2.1) is an element of $\mathcal{S}_{an}$ too. This fact guarantees the existence of a unique non–zero minimal element in $\mathcal{S}_{an}$, which, according to S.S.-T. Yau, is called the *maximal ideal cycle* of the singularity and it is denoted by Z_{max}. One can show that Z_{max} (or the whole $\mathcal{S}_{an}$) depends on the analytic structure of $(X,0)$. In general, it can not be recovered from the topology. However, there are some cases, when this situation might happen.

It can be proven, that for any $f\in\mathfrak{m}_{(X,0)}$ one has $(\mathrm{div}(f\circ\pi),E_j)=0$ for all $j\in\mathcal{V}$. This, together with $(St(f),E_j)\geq 0$ imply that $((f)_\Gamma,E_j)\leq 0$ for every $j\in\mathcal{V}$. This motivates the definition of the 'topological candidate' for $\mathcal{S}_{an}$, namely

$$\mathcal{S}_{top}:=\{l\in L\ :\ (l,E_j)\leq 0\ \forall j\in\mathcal{J}\},$$

which is the Lipman cone defined in section 2.1.2.3.

$\mathcal{S}_{top}$ shares the same properties as its analytic counterpart. Hence, it has a unique non–zero minimal element Z_{min}, which was introduced by Artin [3, 4] and it is called the *minimal cycle* or *Artin's (fundamental) cycle*. Notice that, since $\mathcal{S}_{an}\subseteq\mathcal{S}_{top}$, we have $Z_{min}\leq Z_{max}$, where in general strict inequality appears.

It turns out that Z_{min} can be calculated easily by an algorithm on the graph Γ, established by Laufer [48], as follows.

Laufer algorithm 2.2.2 One constructs a sequence $\{z_n\}_{n=1}^t$ of cycles in the following way.

1. Start with a cycle $z_1=E_j$ for some $j\in\mathcal{V}$.
2. If z_n is already constructed for some $n>0$ and there exists some $E_{j(n)}$ for which $(z_n,E_{j(n)})>0$, then set $z_{n+1}=z_n+E_{j(n)}$.
3. If $(z_n,E_j)\leq 0$ for all j, then stop and z_n gives Z_{min}. □

2.2.2.1 Some invariants of the geometry can be deduced from the cohomology of sheaves on $\widetilde{X}$. For example, consider $\mathcal{O}_{\widetilde{X}}$, the sheaf of holomorphic functions on $\widetilde{X}$ and $\mathcal{O}^*_{\widetilde{X}}$, the subsheaf of invertible functions. We may also consider the group $Pic(\widetilde{X})$ of holomorphic line bundles on $\widetilde{X}$ (modulo isomorphism), which is naturally isomorphic to $H^1(\widetilde{X}, \mathcal{O}^*_{\widetilde{X}})$. Note that, in fact, the groups $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$, or $H^1(\widetilde{X}, \mathcal{O}^*_{\widetilde{X}})$ does not depend on the resolution $\pi : \widetilde{X} \to X$.

The analytic invariant $h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) := \dim H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ is called the *geometric genus* of the singularity and it is denoted by p_g.

For any integral cycle $l = \sum_j m_j E_j$ we associate the line bundle $\mathcal{O}(-l)$, defined by the invertible sheaf of holomorphic functions on $\widetilde{X}$, which vanish of order m_j on E_j. Then one defines the quotient $\mathcal{O}_l := \mathcal{O}_{\widetilde{X}}/\mathcal{O}(-l)$ as well.

According to [5, §6], the short exact exponential sequence

$$0 \longrightarrow \mathbb{Z}_{\widetilde{X}} \longrightarrow \mathcal{O}_{\widetilde{X}} \stackrel{exp}{\longrightarrow} \mathcal{O}^*_{\widetilde{X}} \longrightarrow 0$$

gives rise to the long exact exponential cohomology sequence, which in our case ($\widetilde{X}$ is a smooth complex surface, M is $\mathbb{Q}HS$) splits into the short exact sequence

$$0 \longrightarrow \mathbb{C}^{p_g} \longrightarrow Pic(\widetilde{X}) \stackrel{c_1}{\longrightarrow} L' \longrightarrow 0, \tag{2.9}$$

where $c_1(\mathcal{L})$ is the first Chern class of $\mathcal{L} \in Pic(\widetilde{X})$. Notice that $c_1(\mathcal{O}(-l)) = l$, hence c_1 admits a group–section $L \to Pic(\widetilde{X})$ above the subgroup L of L' which, in fact, can be extended naturally to $L' \to Pic(\widetilde{X})$ (see [76, 3.6]), defining the line bundles $\mathcal{O}(-l')$ for any $l' \in L'$.

As an example, we denote by $\Omega^2_{\widetilde{X}}$ the sheaf of holomorphic 2–forms on $\widetilde{X}$. It is an element of $Pic(\widetilde{X})$, hence it corresponds to a class of divisors. Modulo the principal divisors, this class defines the *canonical divisor* $K_{\widetilde{X}}$. Its intersection with the exceptional divisor can be calculated by the adjunction formulae $(K_{\widetilde{X}}, E_j) = -b_j - 2$ for all j. $K_{\widetilde{X}}$ is analytic, but one can associate with it the canonical cycle via $c_1(\Omega^2_{\widetilde{X}}) = K \in L'$.

Definition 2.1 We say that $(X, 0)$ is *Gorenstein* if we can find a section $\widetilde{\omega}$ of $\Omega^2_{\widetilde{X}}$ whose divisor is supported on E. It is *numerically Gorenstein* if the coefficients of K are integers.
Note that the first definition is equivalent with the fact that there is a global section of $\Omega^2_{X\setminus 0}$ which is nowhere vanishing on $X \setminus 0$, i.e. $\Omega^2_{X\setminus 0}$ is holomorphically trivial. On the other hand, numerical Gorenstein property means that $\Omega^2_{X\setminus 0}$ is a topologically trivial line bundle. Therefore, if $(X, 0)$ is Gorenstein, then it is numerically Gorenstein as well. The general theory says that the numerical Gorenstein property is equivalent with the fact that the first Chern class of $\Omega^2_{\widetilde{X}}$ projected to $H^2(M, \mathbb{Z}) = H^2(X \setminus 0, \mathbb{Z})$ is zero. In the sense of 2.1.2.2, this means that the class of K in H is zero, hence $K \in L$.

As a generalization, one can define the $\mathbb{Q}$–*Gorenstein* property as well, which requires that some power of $\Omega^2_{X\setminus 0}$ should be holomorphically trivial.

The formal neighbourhood theorem implies that $p_g = \dim_{\mathbb{C}} \varprojlim_{l>0} H^1(\widetilde{X}, \mathcal{O}_l)$, hence if one wishes to compute p_g, one has to understand $\dim_{\mathbb{C}} H^1(\widetilde{X}, \mathcal{O}_l)$ for $l \in L$ and $l > 0$. Then, by Riemann–Roch theorem, it is known that although $\dim_{\mathbb{C}} H^0(\widetilde{X}, \mathcal{O}_l)$ and $\dim_{\mathbb{C}} H^1(\widetilde{X}, \mathcal{O}_l)$ are analytic,

$$\chi(l) := \chi(\mathcal{O}_l) = \dim_{\mathbb{C}} H^0(\widetilde{X}, \mathcal{O}_l) - \dim_{\mathbb{C}} H^1(\widetilde{X}, \mathcal{O}_l) \tag{2.10}$$

is topological and equals to $-(l, l+K)/2$. One defines also the 'twisted' version of the Riemann–Roch formula, namely we fix an $\mathcal{L} \in Pic(\widetilde{X})$ and write $c_1(\mathcal{L}) = l' \in L'$ for its Chern class. If we set $k := K - 2l'$, then $\chi(\mathcal{L} \otimes \mathcal{O}_l) = -(l, l+k)/2$.

In this way, for any characteristic element $k \in Char$ one defines a Riemann–Roch function

$$\chi_k : L \to \mathbb{Z} \quad \text{by} \quad \chi_k(l) = -\frac{1}{2}(l, l+k). \tag{2.11}$$

2.2.3 Artin–Laufer program

As we mentioned in the introductory part, it is interesting to investigate special families of normal surface singularities, where some of the analytic invariants (coming from $\mathcal{O}_{(X,0)}$) are topological. Since one of the most important numerical analytic invariants of $(X, 0)$ is the geometric genus p_g, we will restrict our discussion to it. However, at some point we will mention what is happening with some other analytic invariants (defined in 2.2.1) as well.

The *Artin–Laufer program* has a long history, started with the work of Artin in the 60's. In [3, 4] he showed that the *rational singularities* can be characterized completely from the graph (see also 2.2.4). He computed even the multiplicity and the embedding dimension of these singularities from the topological data.

Then Laufer [47, 48] developed further the theory. Among others, he found an algorithm for finding the Artin's cycle Z_{min}, which is now called the *Laufer algorithm*, see 2.2.2, and extended the topological characterization of rational singularities to *minimally elliptic singularities* ([50]). He also noticed that for more complicated singularities the program can not be continued.

However, Némethi's work in [64] pointed out and conjectured that if we pose some analytical and topological conditions, e.g. the Gorenstein and $\mathbb{Q}HS$ conditions, then some numerical analytic invariants (including p_g) are topological. This was carried out explicitly for elliptic singularities.

In order to achieve results in the topological characterization of some analytic invariants, one has to find their 'good' topological candidate. Eg., for a fixed topological type one has to find a topological upper bound for the p_g of any analytic structure supported by this topological type, which is optimal in the sense that for some 'nice' singularities it yields exactly p_g. A good example for this phenomenon is the length of the elliptic sequence in the case of elliptic singularities, introduced and intensively studied by Laufer [50] and S.S.-T. Yau [111]. In Section 2.4, we will expose another candidate for p_g and give some details on the development of results of the last ten years. Another example can be found in Chapter 5, where we study the topological counterpart of the Hilbert–Poincaré series associated with $(X, 0)$.

But first, we recall the Artin–Laufer characterization of rational singularities, since this class is the origo of our research regarding the topology of normal surface singularities.

2.2.4 Rational singularities

In general (without any assumption on the link), a normal surface singularity $(X, 0)$ is called *rational* if $p_g = 0$. The formal neighbourhood theorem immediately implies that this is equivalent with $\dim_{\mathbb{C}} H^1(\widetilde{X}, \mathcal{O}_l) = 0$ for any $l > 0$. In particular, this induces the vanishing of all genera g_j, hence Γ is a tree and the link of a rational singularity is automatically a $\mathbb{Q}HS$.

Notice that somehow the definition of the rational singularity is motivated by the short exact sequence 2.9, since if $p_g = 0$, then $Pic(\widetilde{X})$ is isomorphic to L', hence it is completely topological. Artin [3, 4] proved that in this case $\mathcal{S}_{an} = \mathcal{S}_{top}$ and $Z_{max} = Z_{min}$. These equalities were enough to calculate some analytic invariants, such as the multiplicity, the embedding dimension and the Hilbert–Samuel function in terms of Z_{min}, which shows how this cycle controls most of the geometry of the rational singularities.

Moreover, Artin succeeded to replace the vanishing of $\dim_{\mathbb{C}} H^1(\widetilde{X}, \mathcal{O}_l)$ by a criterion formulated in terms of $\chi(l)$, namely $\chi(l) \geq 1$ for all $l > 0$. However, in general it is difficult to verify this criterion for all positive cycles. Therefore, another breakthrough was that, in fact, it is enough to consider only the Artin's cycle Z_{min}, since it controls the criterion for all the other positive cycles as well. This fact can be formulated also in terms of the Laufer algorithm.

In the next theorem we summarize the results of Artin and Laufer, characterizing topologically the rational singularities.

Theorem 2.1 (Topological characterization of rational singularities) *Let $(X, 0)$ be a normal surface singularity, then the following statements are equivalent:*

1. $p_g = 0$*;*
2. $\chi(l) \geq 1$ *for any* $l > 0$*;*
3. $\chi(Z_{min}) = 1$*;*
4. *In the Laufer algorithm 2.2.2 one has* $(z_n, E_{j(n)}) = 1$ *for every* $n \geq 1$.

Starting from the topological point of view, we may set the following definition:

Definition 2.2 If a resolution graph Γ satisfies one of the last three conditions in the previous theorem, we say that Γ is a *rational graph*.

The class of rational graphs is closed under taking subgraphs and decreasing the self–intersections. We observe that $Z_{min} \geq \sum_{j \in \mathcal{V}} E_j$, a fact which follows from the Laufer algorithm and connectedness of Γ. If we have equality, we say that Z_{min} is a *reduced Artin's cycle*: in this case $(X, 0)$ is called *minimal rational* and Γ is a *minimal rational graph*.

Example 2.3

1. Let Γ be an arbitrary tree (and all the genus decorations are zero). Let

$$b_j = \begin{cases} -\delta_j & \text{if } \delta_j \neq 1 \\ -2 & \text{if } \delta_j = 1 \end{cases} \quad \text{for any } j \in \mathcal{V},$$

   where δ_j is the valency of the vertex j. Then the intersection matrix I is automatically negative definite and with the Laufer algorithm one can show that $Z_{min} = \sum_{j\in\mathcal{V}} E_j$ and $\chi(Z_{min}) = 1$. Hence, any $(X, 0)$ with minimal resolution graph Γ is a minimal rational singularity.
2. Assume that $(X, 0)$ is rational and numerically Gorenstein. We can show that $K = 0$, hence the adjunction formulae 2.3 imply that $b_j = -2$ for all j. This graphs are the minimal resolution graphs of rational double points (or ADE singularities). □

2.4 Seiberg–Witten invariants and a conjecture of Némethi and Nicolaescu

Historically, the *Seiberg–Witten invariants* were defined for compact smooth 4–manifolds. They were introduced by Witten [110] during his investigation with Seiberg on the Seiberg–Witten gauge theory in theoretical physics. They are similar to the invariants defined by the Donaldson theory, and they provide a strong tool in proving key results for smooth 4–manifolds. The advantage is that the main objects which define the numerical data, the moduli spaces of solutions of the Seiberg–Witten equations, are mostly compact, hence the problems coming from the compactification of the moduli spaces as in Donaldson theory can be avoided.

Besides the original work of Witten, detailed presentation of the theory can be found in the book of Nicolaescu [89], see also the book of Morgan [62].

2.4.1 Seiberg–Witten invariants for closed 3–manifolds

In our case, we analyze the Seiberg–Witten invariants for closed 3–manifolds. Considering an additional geometric data (g, η) on M, where g is a Riemannian metric and η is a closed 2–form, one can define the *Seiberg–Witten equations* (we refer to [53, 91] for precise definitions and details). Then for any $spin^c$–structure σ on M, the space of solutions divided by the gauge group defines the moduli space of (σ, g, η)–monopoles, and the *Seiberg–Witten invariant* $\widetilde{\mathfrak{sw}}_\sigma(M, g, \eta)$ is the signed count of them.

It turns out that, when $b_1(M) = 0$ (i.e. M is a $\mathbb{Q}HS$), the situation is the worst, since $\widetilde{\mathfrak{sw}}_\sigma(M, g, \eta)$ depends on the choice of the parameters g and η, thus it is not an invariant. However, altering by a counter term $KS_M(\sigma, g, \eta)$, called the *Kreck–Stolz* invariant, solves the problem. Therefore one defines the ‘modified’ Seiberg–Witten invariant by

$$\mathfrak{sw}_\sigma(M) := \frac{1}{8}KS_M(\sigma, g, \eta) + \widetilde{\mathfrak{sw}}_\sigma(M, g, \eta).$$

Theorem 2.2 ([53]) *If M is a connected 3–manifold with $b_1(M) = 0$, then*

$$\mathfrak{sw} : Spin^c(M) \longrightarrow \mathbb{Q} \ \ (\textit{more precisely } \mathbb{Z}[1/8|H|])$$

is an oriented diffeomorphism invariant of M.

It is important to emphasize that $\mathfrak{sw}$ in an involution invariant, that is $\mathfrak{sw}_\sigma(M) = \mathfrak{sw}_{\overline{\sigma}}(M)$. Therefore, in order to make our formulae more transparent we will always identify $\mathfrak{sw}_{h*\overline{\sigma_{can}}}$ with $\mathfrak{sw}_{-h*\sigma_{can}}$.

In general, it is extremely difficult to compute $\mathfrak{sw}_\sigma(M)$ using its analytic definition. Therefore, there are some projects which aim to replace this definition with a different one, or, to provide a topological/combinatorial calculation for the invariants:

- Answering a question of Turaev, Nicolaescu's result [91] shows that $\mathfrak{sw}_\sigma(M)$ is the Reidemeister–Turaev torsion normalized by the Casson–Walker invariant. This identification is based on the surgery formula for the monopole count given by Marcolli and Wang [55], and for the Kreck–Stolz invariant contained in the paper of Ozsváth and Szabó [94]. In case of a plumbing graph Γ, combinatorial formula for the Casson–Walker invariant is given by the book of Lescop [51], while the Reidemeister–Turaev torsion is determined by Némethi and Nicolaescu in [77]. This formula for the torsion is based on a Dedekind–Fourier sum which, in most of the cases, is still hard to determine.
  On the other hand, Braun and Némethi [15] provides a cut–and–paste surgery formula for the Seiberg–Witten invariants in the case of negative definite plumbed 3–manifold, motivated by Okuma's formula [93] targeting analytic invariants of splice–quotient singularities.
- Another program is the *categorification* of the invariants. The aim is to construct homological theories whose 'normalized Euler characteristic' gives the Seiberg–Witten invariant (with a suitable normalization). This interpretation also gives several alternative definitions for the $\mathfrak{sw}_\sigma(M)$.
  For examples, with a generalization of the Seiberg–Witten monopoles, Kronheimer and Mrowka [38] constructed the *Seiberg–Witten Floer homology* which, in fact, is isomorphic to the *Heegaard–Floer homology*, developed by Ozsváth and Szabó [96, 97, 95], and they categorify $\mathfrak{sw}_\sigma(M)$. Moreover, as a consequence of exact 'triangles', one also gets further surgery formulae for the Seiberg–Witten invariants.
  In [65], Némethi proved that the normalized Euler characteristic of the lattice cohomology is also the Seiberg–Witten invariant. This proof uses the surgery formula of [15].

One of the main achievement discussed by this monograph will be found in Chapter 5, which provides an Ehrhart theoretical interpretation of the Seiberg–Witten invariants and calculates them by using the *topological Poincaré series*.

2.4.2 The Seiberg–Witten invariant conjecture

In the spirit of the Artin–Laufer program, the article [77] of Némethi and Nicolaescu formulates the following conjecture, giving a possible topological counterpart for the geometric genus. It is an extension of the Casson invariant conjecture of Neumann and Wahl [87].

SWI Conjecture ([77]) Assume that $(X, 0)$ is a normal surface singularity whose link M is a $\mathbb{Q}HS$. Then the following facts hold:

1. There is a topological upper bound for p_g, given by

$$p_g \leq -\mathfrak{sw}_{\sigma_{can}}(M) - \frac{k_{can}^2 + |\mathcal{J}|}{8}.$$

2. If $(X, 0)$ is $\mathbb{Q}$–Gorenstein, then in part 1 one has equality. □

This can be generalized in the following way:

GSWI Conjecture ([76]) We consider a cycle $l' \in L'$. Then

1. For any line bundle $\mathcal{L} \in Pic(\widetilde{X})$ with $c_1(\mathcal{L}) = l'$ one has

$$\dim_{\mathbb{C}} H^1(\widetilde{X}, \mathcal{L}) \leq -\mathfrak{sw}_{-h*\sigma_{can}}(M) - \frac{k^2 + |\mathcal{J}|}{8}.$$

2. If $\mathcal{L} = \mathcal{O}_{\widetilde{X}}(l')$ and $(X, 0)$ is $\mathbb{Q}$–Gorenstein then in part 1 one has equality. □

In particular, if $\mathcal{L} = \mathcal{O}_{\widetilde{X}}$, then we get back SWI. The conjecture was verified first in [77] for some families of rational, elliptic and hypersurface singularities. It was proved also for singularities with good $\mathbb{C}^*$–action [78] and for suspension singularities (of type $\{f(x, y) + z^n = 0\}$ with f irreducible) [79]. Then [67] proves the validity of the conjecture for *splice–quotient singularities*, a class which was defined by Neumann and Wahl [88] and contains most of the other classes above. Furthermore, using the Heegaard–Floer/lattice homological interpretation of $\mathfrak{sw}$, Némethi verified GSWI for all *almost rational singularities* (see [69, 72]).

Unfortunately, the conjecture at this generality is **not true**. A paper of Luengo-Velasco, Melle-Hernández and Némethi on *superisolated singularities* [54] gives counterexamples even for the SWI case.

Chapter 3
Poincaré series: definitions and motivations

In this chapter we start with some useful notations and facts which will be used throughout the chapter. Then we present definitions and results regarding the *analytic Hilbert–Poincaré series* of normal surface singularities, which serve as a motivation for the topological side. After this part, we continue with the definition and immediate properties of the topological Poincaré series. A discussion regarding the statement of Theorem 3.1 will serve as a motivation and it provides a short summary for the connections between the three numerical datas: the Seiberg–Witten invariant, the periodic constant and the Ehrhart coefficient.

3.1 Equivariant multivariable Hilbert series of divisorial filtrations

We fix a resolution π of $(X,0)$ with resolution graph Γ. The lattice L defines a *divisorial multi–index filtration* on $\mathcal{O}_{(X,0)}$ by associating the ideal

$$\mathcal{F}(l) := \{f \in \mathcal{O}_{(X,0)} \ : \ (f)_\Gamma \geq l\}$$

for any $l = \sum_j l_j E_j \in L$. The ususal way to describe this multi–index filtration is taking the *Hilbert function* $\mathfrak{h}(l) := \dim \mathcal{O}_{(X,0)}/\mathcal{F}(l)$ and its corresponding generating series, called the *multivariable Hilbert series*

$$\mathcal{H}(\mathbf{t}) = \sum_{l=\sum l_j E_j \in L} \mathfrak{h}(l)\mathbf{t}^l \in \mathbb{Z}[[L]], \tag{3.1}$$

where $\mathbf{t}^l = t_1^{l_1}\cdots t_s^{l_s}$ and $\mathbb{Z}[[L]]$ denotes the $\mathbb{Z}[L]$–submodule of formal power series $\mathbb{Z}[[t_1^{\pm 1/\det(I)},\dots,t_s^{\pm 1/\det(I)}]]$, generated by the monomials $\mathbf{t}^l$. More details regarding these definitions can be read from [26, 23].

Using the Hilbert series we also define a multivariable Poincaré series, which is more close to the topology of $(X,0)$. But first, let us present a more general setting defined in [24, 66], which gives the equivariant version of this concept.

Consider $c : (Y, 0) \to (X, 0)$, the *universal abelian covering* of $(X, 0)$ with Galois group $H = H_1(M, \mathbb{Z})$, let $\pi_Y : \widetilde{Y} \to Y$ the normalized pullback of π by c, and $\widetilde{c} : \widetilde{Y} \to \widetilde{X}$ the morphism which covers c, i.e. the induced finite map which makes the diagram commutative.

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{c}} & \tilde{X} \\ {\scriptstyle \pi_Y}\downarrow & \# & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow[c]{} & X \end{array} \tag{3.2}$$

If we denote the pullback of the cycle $l' \in L'$ by $\widetilde{c}$ with $\widetilde{c}^*(l')$, then [76, 3.3] proves that $\widetilde{c}^*(l')$ is an integral cycle (an element of the lattice L_Y associated with $\widetilde{Y}$ which is, in fact, a partial resolution of $(Y, 0)$ with Hirzebruch–Jung singularities, cf. [76, 3.2]).
Then $\mathcal{O}_{(Y,0)}$ inherits the divisorial multi–index filtration given by

$$\mathcal{F}(l') := \{ f \in \mathcal{O}_{Y,o} \ : \ \mathrm{div}(f \circ \pi_Y) \geq \widetilde{c}^*(l') \}.$$

The natural action of H on Y induces an action on $\mathcal{O}_Y$ as follows: $h \cdot g(y) = g(h \cdot y)$, $g \in \mathcal{O}_Y$, $h \in H$. This action decomposes $\mathcal{O}_Y$ as $\oplus_{\lambda \in \hat{H}} (\mathcal{O}_Y)_\lambda$ according to the characters $\lambda \in \hat{H} := \mathrm{Hom}(H, \mathbb{C}^*)$, where

$$(\mathcal{O}_Y)_\lambda := \{ g \in \mathcal{O}_Y \mid g(h \cdot y) = \lambda(h) g(y), \ \forall y \in Y, h \in H \}. \tag{3.3}$$

Note that there exists a natural isomorphism $\theta : H \to \hat{H}$ which is defined as $h \mapsto \exp(2\pi\sqrt{-1}(l', \cdot)) \in \mathrm{Hom}(H, \mathbb{C}^*)$, where l' is any element of L' with $h = [l']$. In order to simplify our notations we will use the notation $(\mathcal{O}_Y)_h$ for $(\mathcal{O}_Y)_{\theta(h)}$ (and similarly for any linear H-representation).

The subspace $\mathcal{F}(l')$ is invariant under this action and $\mathcal{F}(l')_h = \mathcal{F}(l') \cap (\mathcal{O}_Y)_h$. Hence, H acts on $\mathcal{O}_{(Y,0)}/\mathcal{F}(l')$ and one can define the *Hilbert function* $\mathfrak{h}(l')$ for any $l' \in L'$ as the dimension of the $\theta([l'])$-eigenspace $(\mathcal{O}_Y/\mathcal{F}(l'))_{[l']}$. Then the corresponding *equivariant multivariable Hilbert series* is

$$\mathcal{H}(\mathbf{t}) = \sum_{l' \in L'} \mathfrak{h}(l') \mathbf{t}^{l'} \in \mathbb{Z}[[L']].$$

In $\mathcal{H}(\mathbf{t})$ the exponents l' of the terms $\mathbf{t}^{l'}$ reflect the H eigenspace decomposition too. E.g., $\sum_{l \in L} \mathfrak{h}(l) \mathbf{t}^l$ corresponds to the H–invariants, hence it is the Hilbert series defined at the beginning of this section. Moreover, the H-eigenspace decomposition of $\tilde{c}_*(\mathcal{O}_{\tilde{Y}})$ is given by (see [66, 92])

$$\tilde{c}_*(\mathcal{O}_{\tilde{Y}}) = \bigoplus_{h \in H} \mathcal{O}_{\tilde{X}}(-r_h) \ \text{ with } \mathcal{O}_{\tilde{X}}(-r_h) = (\tilde{c}_*(\mathcal{O}_{\tilde{Y}}))_h, \tag{3.4}$$

where $\mathcal{O}_{\tilde{X}}(l')$ is the only line bundle $\mathcal{L}$ on $\tilde{X}$ satisfying $\tilde{c}^*\mathcal{L} = \mathcal{O}_{\tilde{Y}}(\tilde{c}^*(l'))$ (see [67, 3.5]) and r_h is the representative of h defined in (2.6).

If l' is in the special 'vanishing zone' $-K + \mathcal{S}'$, then by vanishing (of a certain first cohomology), and by the Riemann–Roch formula, one obtains (see [67]) that the expression

$$\mathfrak{h}(l') + \frac{(K+2l')^2 + |\mathcal{V}|}{8} \tag{3.5}$$

depends only on the class $[l'] \in L'/L$ of l'.

The key bridge connecting $\mathcal{H}(\mathbf{t})$ with the topology of the link and with the graph Γ is realized by defining the *equivariant multivariable Poincaré series* from $\mathcal{H}(\mathbf{t})$ as follows (cf. [23, 24, 66, 67]):

$$\mathcal{P}(\mathbf{t}) := -\mathcal{H}(\mathbf{t}) \cdot \prod_v (1 - t_v^{-1}) \in \mathbb{Z}[[L']].$$

Notice that apparently $\mathcal{P}$ loses some analytic information of $\mathcal{H}$. However, [67, (3.2.6)] shows explicitly that the identity can be 'inverted'. Namely, if we write $\mathcal{P}(\mathbf{t}) = \sum_{l'} p_{l'}\mathbf{t}^{l'}$, then

$$\mathfrak{h}(l') = \sum_{l \in L,\, l \ngeq 0} p_{l'+l}.$$

This is well–defined, since by [67, (3.2.2)] one has that $\mathcal{P}$ is supported on $\mathcal{S}'$, therefore the sum in the formula is finite via 2.7. In particular, cf. (3.5),

$$\mathfrak{h}(l') = \sum_{l \in L,\, l \ngeq 0} p_{l'+l} = -\mathrm{const}_{[-l']} - \frac{(K+2l')^2 + |\mathcal{V}|}{8} \tag{3.6}$$

for any $l' \in -K + \mathcal{S}'$, where $\mathrm{const}_{[-l']}$ depends only on the class $[-l']$ of $-l'$. The right hand side can be seen as a 'multivariable Hilbert polynomial' of degree 2 associated with the series $\mathcal{H}(\mathbf{t})$ (or with $\mathcal{P}(\mathbf{t})$). The point is that its constant term equals to the *normalized equivariant geometric genus* of the universal abelian cover Y (see details in [67]), that is

$$-\mathrm{const}_{[-r_h]} = \dim(H^1(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}})_{\theta(h)}) + \frac{(K+2r_h)^2 + |\mathcal{V}|}{8}. \tag{3.7}$$

The main point is that $\mathcal{P}(\mathbf{t})$ has a *topological candidate*, which is defined purely from the graph Γ and will be the subject of the next section. The two series agree for several singularities, see for example [24, 66, 67].

It turns out that, in particular, the identification of their constant terms (for 'nice' analytic structures) is the subject of the Seiberg–Witten Invariant Conjecture 2.4.2, since the constant term of the topological candidate will realize the Seiberg–Witten invariant (cf. 3.1) as it will be presented in the sequel.

3.2 The topological Poincaré series

Definition 3.1 Consider the following rational 'zeta-function'

$$f(\mathbf{t}) = \prod_{v \in \mathcal{V}} (1 - \mathbf{t}^{E_v^*})^{\delta_v - 2}. \tag{3.8}$$

Then its multivariable Taylor expansion $Z(\mathbf{t}) = \sum z(l')\mathbf{t}^{l'}$ at the origin is called the *topological (combinatorial) Poincaré series* associated with the plumbing graph Γ.

Since the Lipman cone $\mathcal{S}'$ is generated by the elements E_v^* over $\mathbb{Z}_{\geq 0}$, $Z(\mathbf{t})$ is supported on $\mathcal{S}'$ (i.e. $z(l') = 0$ for every $l' \notin \mathcal{S}'$). Therefore, if we apply the same *special truncation* as in the analytic case (3.6), shown by the following picture

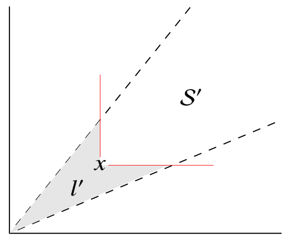


Fig. 3.1 The truncation $l' \ngeq x$

then we get a finite sum

$$\sum_{l\in L,\, l\ngeq 0} z(l'+l)\ . \tag{3.9}$$

One has a natural decomposition $Z(\mathbf{t}) = \sum_{h\in H} Z_h(\mathbf{t})$, where $Z_h(\mathbf{t}) = \sum_{[l']=h} z(l')\mathbf{t}^{l'}$. Then the sum (3.9) involves only the part $Z_{[l']}$ (sometimes we also write $Z_{l'}$ for $Z_{[l']}$).

As we already mentioned at the end of 3.1, $Z(\mathbf{t})$ is the topological candidate for $\mathcal{P}(\mathbf{t})$, since they agree for 'nice' analytic structures. This fact motivated the birth of the next theorem as the topological analog of (3.6) , which also explains how the $Z(\mathbf{t})$ encodes the Seiberg–Witten invariants of the link M. Moreover, this was the starting point of the research presented in this monograph.

Theorem 3.1 ([65]) *For any* $l' \in -K + \mathrm{int}(\mathcal{S}')$ *(where* $\mathrm{int}(\mathcal{S}') = \mathbb{Z}_{>0}\langle E_v^*\rangle_{v\in\mathcal{V}}$*)*

$$\sum_{l\in L,\, l\ngeq 0} z(l'+l) = -\mathfrak{sw}_{[-l']*\sigma_{can}}(M) - \frac{(K+2l')^2+|\mathcal{V}|}{8}, \tag{3.10}$$

where $*$ *denotes the torsor action of* H *on* $\mathrm{Spin}^c(M)$.

If we fix $h \in H$ and we write $l' = l + r_h$ with $l \in L$, then the finite sum on the left hand side appears as a *counting function* of the coefficients of Z_h associated with the special truncation, while the right hand can be seen as a *multivariable quadratic (Hilbert) polynomial* in variable $l \in L$ whose *constant term* is called the r_h-normalized Seiberg–Witten invariant. For simplicity, we introduce the following notation

$$\mathfrak{sw}_h^{norm}(M) := -\mathfrak{sw}_{-h*\sigma_{can}}(M) - \frac{(K+2r_h)^2+|\mathcal{V}|}{8}. \tag{3.11}$$

Note that in order to guarantee the validity of the formula (3.10), the vector l' should sit in a special *chamber*. This, after we establish the necessary bridges, will read as follows: *'the third degree' coefficient of a multivariable Ehrhart quasipolynomial associated with a certain polytope and specific chamber can be identified with the Seiberg–Witten invariant*.

In the followings, we will further motivate and summarize the results of the forthcomming chapter, which explains the above highlighted sentence. The way how one recovers the needed information from the topological Poincaré series $Z(\mathbf{t})$ can be done at several levels:

I. *The first one is entirely at the level of the series.* We develop a theory which associates with any series the counting function of its coefficients (given by the truncation of the monomials) — like the right hand side of (3.10). This is usually a *piecewise quasipolynomial*. Once we fix a chamber, the free term of the counting function is the so–called *periodic constant* (denoted by pc). In this terminology, the Seiberg–Witten invariant can be interpreted as the *multivariable periodic constant* (cf. 4.4) of the series $Z(\mathbf{t})$, where the chosen chamber is described by the inequalities of the assumption (a part of the Lipman cone $\mathcal{S}'$). The 'periodicity' is related with the quasipolynomial behavior of the counting function.
The motivation for the multivariable periodic constant goes back to the periodic constant of one–variable series, which was introduced by Némethi and Okuma. Its idea, cf. [81, 93], will be detailed in 4.2.1. (For applications see e.g. [80, 81, 65, 15].)
We create the general theory, which carries necessarily several difficult technical ingredients. For example, one has to choose the 'right' truncation and summation procedure of the coefficients, which, in the context of general series, is not automatically motivated, and also it depends on the chamber decomposition of the space of exponents. The theory has some similarities with the theory of vector partition functions as well.

II. On the other hand, there is a more sophisticated way to generalize the identity (3.10). From any Taylor expansion of a multivariable rational function with denominator of type $\prod_i(1-\mathbf{t}^{a_i})$ we construct a *polytope* situated in a lattice which carries also a representation of a finite abelian group H. Associated with these data, we consider the *equivariant multivariable Ehrhart piecewise quasipolynomials*, whose existence, main properties (like the *Ehrhart–MacDonald–Stanley type reciprocity law* or *chamber decompositions*) will also be established in 4.1. This applied to the topological Poincaré series, and to the quasipolynomial of those chambers which belong to the Lipman cone shows that the first three top–degree *Ehrhart coefficients* (at least) will carry geometrical/topological meaning, including the Seiberg–Witten invariants of the link M.

In Figure 3.2 (cf. [40]) we summarize these two points with a schematic picture showing the connections and areas we target.

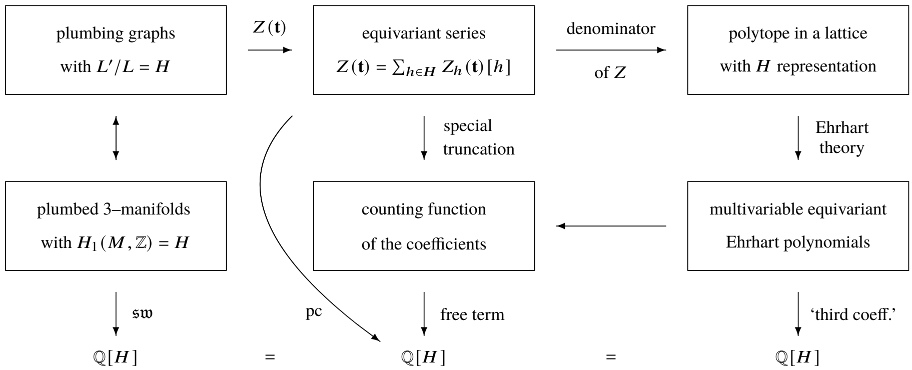


Fig. 3.2 The theories associated with Γ.

3.3 'Classical' connections between polytopes and gauge invariants

The coefficient identification (5.6), and in fact (3.10) too, supply an additional addendum to the intimate relationship between lattice point counting and the Riemann–Roch formula, exploited in global algebraic geometry by toric geometry.

In the literature of normal surface singularities there is a sequence of results which connect the topology of the link with the number of lattice points in a certain polytope. Here we list some historical details on this subject.

The first is based on the theory of *Newton non–degenerate hypersurface singularities*, see e.g. the second volume of the monograph of Arnold, Gussein–Zade and Varchenko [2]. According to this, for such a germ one defines the *Newton polytope* Γ_N^- using the non–trivial monomials of the defining equation of the germ. Then one can prove that several invariants of the germ can be recovered from Γ_N^-. For example, by a result of Merle and Teissier [59], the geometric genus p_g equals the *number of lattice points* in $((\mathbb{Z}_{>0})^3 \cap \Gamma_N^-)$, see also the work of Braun and Némethi [14] into this direction. On the other hand, when the link is a rational homology sphere, then by [104] in this case the (non-equivariant) Seiberg–Witten invariant conjecture is true, ie. the normalized canonical Seiberg–Witten invariant equals to the p_g, hence it can be expressed as a number of lattice points.

The second is provided by the Laufer–Durfee formula, which determines the signature of the Milnor fiber σ as $-8p_g - K^2 - |\mathcal{V}|$ ([30]). Finally, there is a conjecture of Neumann and Wahl [87], formulated for hypersurfaces with integral homology sphere links, and proved for Brieskorn, suspension [87] and splice–quotient [81] singularities, according to which $\sigma/8 = \lambda(M)$, the Casson invariant of the link. Therefore, if all these steps run, for example as in the Brieskorn case, then the Casson invariant of the link, normalized by $K^2 + |\mathcal{V}|$, can be expressed as the number of lattice points of a polytope associated with the equation of the germ.

The next chapter defines another polytope, which carries an action of the group H, and its *Ehrhart invariants determine the Seiberg–Witten invariant in any case*. It is not described from the equations of the germ, but from its topological Poincaré series $Z(\mathbf{t})$.

Chapter 4

Ehrhart theory of rational functions and the multivariable periodic constant

4.1 Equivariant multivariable Ehrhart theory

In this section we generalize the classical Ehrhart theory to the equivariant multivariable version, involving non–convex polytopes, which will fit with our comparison with the equivariant multivariable series provided by plumbing graphs.

Let us start with a d–dimensional *rational lattice* $\mathcal{X} \subset \mathbb{Q}^d$ and a group homomorphism $\rho : \mathcal{X} \to \mathfrak{H}$ to a finite abelian group $\mathfrak{H}$. We consider a *rational vector–dilated polytope* with parameter $\mathbf{l} = (\mathbf{l}_1, \dots, \mathbf{l}_r)$, $\mathbf{l}_v \in \mathbb{Z}^{m_v}$,

$$\mathcal{P}^{(\mathbf{l})} = \bigcup_{v=1}^{r} \mathcal{P}_v^{(\mathbf{l}_v)}, \text{ where } \mathcal{P}_v^{(\mathbf{l}_v)} = \{\mathbf{x} \in \mathbb{R}^d \ : \ \mathbf{A}_v \mathbf{x} \leq \mathbf{l}_v\}, \tag{4.1}$$

where $\mathbf{A}_v$ is an integral $m_v \times d$ matrix. If $\{A_{v,\lambda i}\}_{\lambda i}$ and $\{l_{v,\lambda}\}_\lambda$ are the entries of $\mathbf{A}_v$ and $\mathbf{l}_v$, then the inequality $\mathbf{A}_v\mathbf{x} \leq \mathbf{l}_v$ in (4.1) reads as $\sum_{i=1}^d x_i A_{v,\lambda i} \leq l_{v,\lambda}$ for any $\lambda = 1, \dots, m_v$.

We will vary the parameter $\mathbf{l}$ in some 'chambers' (described below for the needed cases) such that the polytopes $\mathcal{P}^{(\mathbf{l})}$ remain *combinatorially stable* (or preserve their *combinatorial type*) when $\mathbf{l}$ runs in the same chamber. This means that their face lattices are isomorphic. (This implies that they are connected by homeomorphisms, which preserve the stratification of the faces.) We also suppose that $\mathcal{P}^{(\mathbf{l})}$ is homeomorphic to a d–dimensional manifold. Denote the set of all closed facets of $\mathcal{P}^{(\mathbf{l})}$ by $\mathcal{F}$ and let $\mathcal{T}$ be a subset of $\mathcal{F}$, such that $\cup_{F^{(\mathbf{l})} \in \mathcal{T}} F^{(\mathbf{l})}$ is homeomorphic to a $(d-1)$–manifold.

Then we have the following generalization to the *equivariant version* of results of Stanley [105], McMullen [60] and Beck [8, 9].

Theorem 4.1 *For any $h \in \mathfrak{H}$ and $\mathcal{T} \subset \mathcal{F}$ let*

$$\mathcal{L}_h(\mathbf{A}, \mathcal{T}, \mathbf{l}) := \textit{cardinality of } \Big(\big(\mathcal{P}^{(\mathbf{l})} \setminus \cup_{F^{(\mathbf{l})} \in \mathcal{T}} F^{(\mathbf{l})} \big) \cap \rho^{-1}(h) \Big). \tag{4.2}$$

(a) If $\mathbf{l}$ moves in some region in such a way that $\mathcal{P}^{(\mathbf{l})}$ stays combinatorially stable then the expression $\mathcal{L}_h(\mathbf{A}, \mathcal{T}, \mathbf{l})$ is a quasipolynomial in $\mathbf{l} \in \mathbb{Z}^{\sum m_v}$.

(b) For a fixed combinatorial type of $\mathcal{P}^{(\mathbf{l})}$ and for a fixed $\mathcal{T}$, the quasipolynomials $\mathcal{L}_h(\mathbf{A},\mathcal{T},\mathbf{l})$ and $\mathcal{L}_{-h}(\mathbf{A},\mathcal{F}\setminus\mathcal{T},\mathbf{l})$ satisfy the Ehrhart–MacDonald–Stanley reciprocity law

$$\mathcal{L}_h(\mathbf{A},\mathcal{T},\mathbf{l}) = (-1)^d\cdot\mathcal{L}_{-h}(\mathbf{A},\mathcal{F}\setminus\mathcal{T},\mathbf{l})|_{\text{replace } \mathbf{l} \text{ by } -\mathbf{l}}. \tag{4.3}$$

To avoid any confusion regarding the expression of (4.3) we note: the two quasipolynomials in (4.3) are associated with that domain of definition (chamber) which corresponds to the fixed combinatorial type. Usually for $-\mathbf{l}$ the combinatorial type of $\mathcal{P}^{(\mathbf{l})}$ is different, hence the right hand side of (4.3) *need not be equal to* $(-1)^d\cdot\mathcal{L}_{-h}(\mathbf{A},\mathcal{F}\setminus\mathcal{T},-\mathbf{l})$. This last expression is the value at $-\mathbf{l}$ of the quasipolynomial associated with the chamber which contains $-\mathbf{l}$.

For a reformulation of the identity (4.3) in terms of the fixed chamber see Theorem 4.2(c).

Proof The statements for $\mathfrak{H}=0$ are identical with those of Beck from [9]. Part (a) above for arbitrary $\mathfrak{H}$ can be proved identically as in [9] applied for the situation when the parameters $\mathbf{l}$ run in an overlattice of $\mathbb{Z}^{\sum m_v}$, instead of $\mathbb{Z}^{\sum m_v}$. Equivalently, one can apply [25], which considers the non–equivariant case, but the integral parameters $\mathbf{l}$ of Beck are replaced by *rational affine parameters*.

For the convenience of the reader we provide the proof. First we notice that via standard additivity formulae, cf. [9, § 2], it is enough to prove the statement for each convex $\mathcal{P}_v^{(\mathbf{l}_v)}$. But, considering $\mathcal{P}_v^{(\mathbf{l}_v)}$ and $K:=\ker(\rho)$, for any $\mathbf{r}\in\mathcal{X}$ one has the isomorphism

$$\{\mathbf{x}\in K+\mathbf{r}\ :\ \mathbf{A}_v\mathbf{x}\leq\mathbf{l}_v\}\simeq\{\mathbf{y}\in K\ :\ \mathbf{A}_v\mathbf{y}\leq\mathbf{l}_v-\mathbf{A}_v\mathbf{r}\}.$$

Hence [25, Theorem 2] (or [9] for an overlattice of $\mathbb{Z}^{\sum m_v}$) can be applied, which shows (a). Next, part (b) can also be reduced to [9]. Indeed, we can reduce the discussion again to $\mathcal{P}_v^{(\mathbf{l}_v)}$. We drop the index v, we choose $\mathbf{r}_h\in\mathcal{X}$ with $\rho(\mathbf{r}_h)=h$, and we fix some $\mathbf{l}_0$. Then for $\mathbf{x}\in K\pm\mathbf{r}_h$ with $\mathbf{A}\mathbf{x}\leq\mathbf{l}_0$ we take $\mathbf{y}:=\mathbf{x}\mp\mathbf{r}_y$ and $\mathbf{k}:=\mathbf{l}_0\mp\mathbf{A}\mathbf{r}_h$, which satisfy $\mathbf{y}\in K$ and $\mathbf{A}\mathbf{y}\leq\mathbf{k}$. Therefore, using [9] for this polytope, we obtain

$$\begin{aligned}\mathcal{L}_h(\mathbf{A},\mathcal{T},\mathbf{l}_0)=\mathcal{L}_0(\mathbf{A},\mathcal{T},\mathbf{k}) &= (-1)^d\cdot\mathcal{L}_0(\mathbf{A},\mathcal{F}\setminus\mathcal{T},\mathbf{k})|_{\text{replace } \mathbf{k} \text{ by } -\mathbf{k}}\\ &= (-1)^d\cdot\mathcal{L}_{-h}(\mathbf{A},\mathcal{F}\setminus\mathcal{T},\mathbf{l}_0)|_{\text{replace } \mathbf{l_0} \text{ by } -\mathbf{l_0}},\end{aligned}$$

where the second and the third term is associated with the lattice K. □

Definition 4.1 The quasipolynomial $\mathcal{L}_h(\mathbf{A},\mathcal{T},\mathbf{l})$ considered in Theorem 4.1, associated with a fixed combinatorial type of $\mathcal{P}^{(\mathbf{l})}$, is called the *equivariant multivariable quasipolynomial* associated with the corresponding data.

If we vary $\mathbf{l}$ in $\mathbb{Z}^{\sum m_v}$ (hence we allow the variation of the combinatorial type) we obtain the *equivariant multivariable piecewise quasipolynomial* $\mathcal{L}_h(\mathbf{A},\mathcal{T},\mathbf{l})$ (see also Theorem 4.2 and Corollary 4.1 below).

Remark 4.1 Parallel to the collection $\{\mathcal{L}_h\}_h$ defined in (4.2) one can consider their Fourier transforms as well: for any character $\xi\in\widehat{\mathfrak{H}}=\mathrm{Hom}(\mathfrak{H},S^1)$, one defines

$$\mathcal{L}_{\xi}(\mathbf{A},\mathcal{T},\mathbf{l}) := \sum_{\mathbf{x}\in\mathcal{P}^{(\mathbf{l})}\setminus\cup_{F^{(\mathbf{l})}\in\mathcal{T}}F^{(\mathbf{l})}} \xi^{-1}(\rho(\mathbf{x})), \tag{4.4}$$

which satisfies $\mathcal{L}_{\xi} = \sum_h \mathcal{L}_h \cdot \xi^{-1}(h)$, and $|\mathfrak{H}| \cdot \mathcal{L}_h = \sum_{\xi} \mathcal{L}_{\xi} \cdot \xi(h)$. Hence, the above properties of $\mathcal{L}_h$ can be obtained from similar properties of $\mathcal{L}_{\xi}$ as well. Hence, Theorem 4.1 can be deduced from [19, § 4.3] too.

Remark 4.2 In the sequel we will not consider polytopes with this high generality: our polytopes will be special ones associated with the denominators of type $\prod_i(1-\mathbf{t}^{a_i})$ of multivariable rational functions, or their Taylor series. In order to avoid unnecessary technical details, the stability of the combinatorial type of $\mathcal{P}^{(\mathbf{l})}$, and the corresponding chamber decomposition of $\mathbb{R}^{\sum m_v}$ will also be treated for this special polytopes, see 4.3.2.

4.2 Multivariable rational functions and their periodic constants

4.2.1 The one–variable case

The periodic constant for one–variable series was introduced by Némethi and Okuma. One can find the details in [81, 3.9] and [93, 4.8(1)], however, we present it in the sequel.

Let $S(t) = \sum_{l\geq 0} c_l t^l \in \mathbb{Z}[[t]]$ be a formal power series. Suppose that for some positive integer p, the expression $\sum_{l=0}^{pn-1} c_l$ is a polynomial $P_p(n)$ in the variable n. Then the constant term $P_p(0)$ of $P_p(n)$ is independent of the 'period' p. We call $P_p(0)$ the *periodic constant* of S and denote it by $\mathrm{pc}(S)$. For example, if $l \mapsto Q(l)$ is a quasipolynomial and $S(t) := \sum_{l\geq 0} Q(l)t^l$, then one can take for p the period of Q, and one shows that $\mathrm{pc}(\sum_{l\geq 0} Q(l)t^l) = 0$.

Assume that $S(t)$ is the Hilbert series associated with a graded algebra/vector space $A = \oplus_{l\geq 0} A_l$ (i.e. $c_l = \dim A_l$), and the series S admits a Hilbert quasipolynomial $Q(l)$ (that is, $c_l = Q(l)$ for $l \gg 0$). Since the periodic constant of $\sum_l Q(l)t^l$ is zero, the periodic constant of $S(t)$ measures exactly the difference between $S(t)$ and its 'regularized series' $S_{reg}(t) := \sum_{l\geq 0} Q(l)t^l$. That is: $\mathrm{pc}(S) = (S - S_{reg})(1)$ collecting all the anomalies of the starting elements of S.

Note that $S_{reg}(t)$ can be represented by a rational function of negative degree with denominator of type $A(t) = \prod_i(1 - t^{a_i})$, and $(S - S_{reg})(t)$ is a polynomial.

Conversely, one has the following reinterpretation of the periodic constant [15, 7.0.2]. If $\sum_l c_l t^l$ is a rational function $B(t)/A(t)$ with $A(t) = \prod_i(1 - t^{a_i})$, and one rewrites it as $C(t) + D(t)/A(t)$ with C and D polynomials and $D(t)/A(t)$ of negative degree, then $\mathrm{pc}(S) = C(1)$. From this fact one also gets that $\mathrm{pc}(S(t)) = \mathrm{pc}(S(t^N))$ for any $N \in \mathbb{Z}_{>0}$. We will refer to $C(t)$ as the *polynomial part* of S.

As an example, consider a subset $\mathcal{S} \subset \mathbb{Z}_{\geq 0}$ with finite complement. Then $S(t) = \sum_{s\in\mathcal{S}} t^s$ rewritten is $1/(1-t) - \sum_{s\notin\mathcal{S}} t^s$, hence $\mathrm{pc}(S) = -\#(\mathbb{Z}_{\geq 0} \setminus \mathcal{S})$. In particular, if $\mathcal{S}$ is the semigroup of a local irreducible plane curve singularity, then $-\mathrm{pc}(S)$ is the delta–invariant of that germ. Our study below includes the generalization of this fact to surface singularities.

4.2.2 Multivariable generalization

4.2.2.1 We wish to extend the definition of the periodic constant to the case of Taylor expansions at the origin of multivariable rational functions of type

$$f(\mathbf{t}) = \frac{\sum_{k=1}^{r} \iota_k \mathbf{t}^{b_k}}{\prod_{i=1}^{d}(1-\mathbf{t}^{a_i})} \quad (\iota_k \in \mathbb{Z}). \tag{4.5}$$

Let us explain the notation. Let L be a lattice of rank s with fixed bases $\{E_v\}_{v=1}^{s}$. Let L' be an overlattice of it with same rank, $L \subset L' \subset L \otimes \mathbb{Q}$ with $|L'/L| = \mathfrak{d}$. Then, in (8.1), $\{b_k\}_{k=1}^{r}$, $\{a_i\}_{i=1}^{d} \in L'$ and for any $l' = \sum_v l'_v E_v \in L'$ we write $\mathbf{t}^{l'} = t_1^{l'_1} \dots t_s^{l'_s}$. We also assume that *all the coordinates $a_{i,v}$ of a_i are strict positive*, Hence, in general, the coefficients l'_v are not integral, and the Laurent expansion $Tf(\mathbf{t})$ of $f(\mathbf{t})$ at the origin is

$$Tf(\mathbf{t}) = \sum_{l'} p_{l'}\mathbf{t}^{l'} \in \mathbb{Z}[[t_1^{1/\mathfrak{d}},\dots,t_s^{1/\mathfrak{d}}]][t_1^{-1/\mathfrak{d}},\dots,t_s^{-1/\mathfrak{d}}] := \mathbb{Z}[[\mathbf{t}^{1/\mathfrak{d}}]][\mathbf{t}^{-1/\mathfrak{d}}].$$

We also consider the natural partial ordering of $L \otimes \mathbb{Q}$ (defined as in section 2.1.2.1). If all vectors $b_k \geq 0$ then $Tf(\mathbf{t})$ is in $\sum_{l'} p_{l'}\mathbf{t}^{l'} \in \mathbb{Z}[[\mathbf{t}^{1/\mathfrak{d}}]]$. Sometimes we will not make difference between f and Tf.

4.2.2.2 This will be extended to the following equivariant case. We fix a finite abelian group $\mathcal{G}$, and for each $g \in \mathcal{G}$ a series (or rational function) $Tf_g \in \mathbb{Z}[[\mathbf{t}^{1/\mathfrak{d}}]][\mathbf{t}^{-1/\mathfrak{d}}]$ as in 4.2.2.1, and we set

$$Tf^e(\mathbf{t}) := \sum_{g\in G} Tf_g(\mathbf{t}) \cdot [g] \in \mathbb{Z}[[\mathbf{t}^{1/\mathfrak{d}}]][\mathbf{t}^{-1/\mathfrak{d}}][G].$$

Sometimes this equivariant extension is given automatically in the context of 4.2.2.1. Indeed, if in 4.2.2.1 we set $H := L'/L$, and for

$$Tf = \sum_{l'} p_{l'}\mathbf{t}^{l'} \quad \text{we define} \quad Tf_h := \sum_{[l']=h} p_{l'}\mathbf{t}^{l'}, \tag{4.6}$$

we obtain a decomposition of Tf as a sum $\sum_h Tf_h \in \mathbb{Z}[[\mathbf{t}^{1/\mathfrak{d}}]][\mathbf{t}^{-1/\mathfrak{d}}][H]$ (with $\mathfrak{d} = |H|$).

In our cases we always start with this group $L'/L = H$ (hence f determines its decomposition $\sum_h f_h$). Nevertheless, some alterations will appear. First, we might consider the non–equivariant case, hence we can forget the decomposition over H. Another case appears as follows. In order to simplify the rational function we will eliminate some of its variables (e.g., we substitute $t_i = 1$ for certain indices i), or we restrict f to a linear subspace V. Then, after this substitution, the restricted function $f|_{t_i=1}$ will not determine anymore the restrictions $(f_h)|_{t_i=1}$ of the 'old' components f_h. That is, the new pair of lattices $(L_V, L'_V) = (L \cap V, L' \cap V)$ and the 'old group' $H = L'/L$ become rather independent. In such cases we will keep the old group $H = L'/L$ (and the 'old' decomposition f_h) without asking any compatibility with L'_V/L_V.

4.2.2.3 Since all the coordinates $a_{i,v}$ of a_i are strict positive, for any $Tf(\mathbf{t}) = \sum_{l'} p_{l'}\mathbf{t}^{l'}$ we get a well–defined counting function of the coefficients,

$$l' \mapsto Q(l') := \sum_{l'' \ngeq l'} p_{l''}.$$

If $Tf = \sum_h Tf_h$, then each Tf_h determines a counting function Q_h defined in the same way.

If $H = L'/L$ and Tf decomposes into $\sum_h Tf_h$ under the law from (4.6), then

$$\sum_{l'' \ngeq l'} p_{l''} \cdot [l''] = \sum_{h \in H} Q_h(l')[h]. \tag{4.7}$$

The definitions are motivated by formulae (3.10) and (3.6). The functions $Q_h(l')$ will be studied in the next subsections via Ehrhart theory.

4.3 Ehrhart quasipolynomials associated with denominators of rational functions

First we consider the case $d > 0$, the special case $d = 0$ will be treated in 4.3.3.

4.3.1 The polytope associated with $\{a_i\}_{i=1}^d$

In order to run the Ehrhart theory we have first to fix the lattice $\mathcal{X}$ and the representation $\rho : \mathcal{X} \to \mathfrak{H}$, cf. section 4.1. First, we set $\mathcal{X} = \mathbb{Z}^d$ and $\alpha : \mathcal{X} \to L'$ given by $\alpha(\mathbf{x}) = \sum_{i=1}^d x_i a_i \in L'$. In the sequel we consider two possibilities for $(\mathfrak{H}, \rho)$ which basically will cover all the cases we wish to study (equivariant/non–equivariant cases combined with situations before or after the reduction of variables, see the comment in 4.2.2.2):

(a) $\mathfrak{H} = H = L'/L$ and ρ is the composition $\mathcal{X} \xrightarrow{\alpha} L' \to L'/L$.

(b) $\mathfrak{H} = 0$ and $\rho = 0$.

This choice has an effect on the equivariant decomposition $f^e = \sum_g f_g[g]$ of f too. In case (a) usually we have $\mathcal{G} = H$ and the decomposition is given by 4.6. In case (b) we can take either $\mathcal{G} = 0$ (this can happen e.g. when we forget the decomposition in case (a), and we sum up all the components), or we can take any $\mathcal{G}$ (by specifying each f_g). In this latter case each fixed f_g behaves like a function in the non–equivariant case $\mathcal{G} = 0$, hence can be treated in the same way.

Since the case (b) follows from case (a) (by forgetting the extra information from $\mathfrak{H}$), in the sequel we provide the details for case (a). Hence let us assume $\mathfrak{H} = \mathcal{G} = L'/L$.

Consider the matrix $\mathbf{A}$ with column vectors $|H|a_i$ and write $\mathbf{A}_v$ for its rows. Then the construction of (4.1) can be repeated (eventually completing each $\mathbf{A}_v$ to assure the inequalities $x_i \geq 0$ as well). For $l \in \sum_v l_v E_v \in L$ consider

$$\mathcal{P}_v^{\triangleleft} := \{\mathbf{x} \in (\mathbb{R}_{\geq 0})^d \ : \ |H| \cdot \sum_i x_i a_{i,v} < l_v\} \ \text{ and } \ \mathcal{P}^{\triangleleft} := \bigcup_{v=1}^{s} \mathcal{P}_v^{\triangleleft}. \tag{4.8}$$

The closure $\mathcal{P}_v$ of $\mathcal{P}_v^{\triangleleft}$ is a dilated convex (simplicial) polytope depending on the one-dimensional parameter l_v. Moreover, $\mathcal{P}^{\triangleleft}$ is described via the partial ordering of $L \otimes \mathbb{R}$ as the set $\{l \ : \ \sum_i x_i a_i \not\geq l/|H|\}$. Since $L' \subset L/|H|$, we can restrict ourself to the lattice L' (preserving all the general results of section 4.1). Hence for any $l' \in L'$ we set

$$\mathcal{P}^{(l'),\triangleleft} := \{\mathbf{x} \in (\mathbb{R}_{\geq 0})^d \ : \ \sum_i x_i a_i \not\geq l'\}, \quad \mathcal{P}^{(l')} = \text{closure of } (\mathcal{P}^{(l'),\triangleleft}). \tag{4.9}$$

The combinatorial type of $\mathcal{P}^{(l')}$ might vary with l'. Nevertheless, by definition, the facets will be grouped for all different combinatorial types by the same principle: we consider the coordinate facets $F_i := \mathcal{P}^{(l')} \cap \{x_i = 0\}$, $1 \leq i \leq d$, and we denote by $\mathcal{T}$ the collection of all other facets. Hence $\mathcal{P}^{(l'),\triangleleft} = \mathcal{P}^{(l')} \setminus \cup_{F^{(l')} \in \mathcal{T}} F^{(l')}$. The construction is motivated by the summation from (3.10) (although in the general statements the choice of $\mathcal{T}$ is irrelevant).

Then 3.1 and 4.2.1 lead to the next counting function defined in the group ring $\mathbb{Z}[H]$ of H:

$$\mathcal{L}^e(\mathbf{A}, \mathcal{T}, l') := \sum_{h \in H} \mathcal{L}_h(\mathbf{A}, \mathcal{T}, l') \cdot [h] := \sum 1 \cdot [l''] \in \mathbb{Z}[H], \tag{4.10}$$

where the last sum runs over $l'' \in \left(\mathcal{P}^{(l')} \setminus \cup_{F^{(l')} \in \mathcal{T}} F^{(l')}\right) \cap L' = \mathcal{P}^{(l'),\triangleleft} \cap L'$.

The corresponding non–equivariant counting function, corresponding to $\mathcal{G} = 0$ is denoted by

$$\mathcal{L}_{ne}(\mathbf{A}, \mathcal{T}, l') := \sum_{h \in H} \mathcal{L}_h(\mathbf{A}, \mathcal{T}, l') \in \mathbb{Z}.$$

Similarly, we set $\mathcal{L}^e(\mathbf{A}, \mathcal{F} \setminus \mathcal{T}, l')$ too. For both of them Theorem 4.1 applies.

By the very construction, we have the following identity. Consider the equivariant Taylor expansion at the origin of the function determined by the *denominator of* f, namely

$$\widetilde{f}^e(\mathbf{t}) = \frac{1}{\prod_{i=1}^{d}(1 - [a_i]\mathbf{t}^{a_i})} = \sum_{l''} \widetilde{p}_{l''} \mathbf{t}^{l''} \cdot [l''] \in \mathbb{Z}[[\mathbf{t}^{1/|H|}]]\,[H]. \tag{4.11}$$

Note that since all the $\{E_v\}$–coefficients of each a_i are strict positive, for any $l' \in L'$ the set $\{l'' \ : \ \widetilde{p}_{l''} \neq 0, l'' \not\geq l'\}$ is finite. Then, by the above construction,

$$\sum_{l'' \not\geq l'} \widetilde{p}_{l''} \cdot [l''] = \mathcal{L}^e(\mathbf{A}, \mathcal{T}, l'). \tag{4.12}$$

4.3.2 Combinatorial types, chambers

Next, we wish to make precise the *combinatorial stability* condition. The result of Sturmfels [106], Brion–Vergne [19], Clauss–Loechner [25] and Szenes–Vergne [107] implies that $\mathcal{L}^e$ from (4.12) (that is, each $\mathcal{L}_h$) is a *piecewise quasipolynomial* on L': the parameter space $L \otimes \mathbb{R}$

decomposes into several chambers, the restriction of $\mathcal{L}^e$ on each chamber is a quasipolynomial, and $\mathcal{L}^e$ is continuous. The chambers are described as follows.

Notice that the combinatorial type of $\mathcal{P}^{(l')}$ in (4.9) vary in the same way as the closure of its *convex* complement in $\mathbb{R}^d_{\geq 0}$, namely

$$\{\mathbf{x} \in (\mathbb{R}_{\geq 0})^d \ : \ \sum_i x_i a_i \geq l'\}, \tag{4.13}$$

since both are determined by their common boundary $\mathcal{T}$. The inequalities of (4.13) can be viewed as a *vector partition* $\sum_i x_i a_i + \sum_v y_v(-E_v) = l'$, with $x_i \geq 0$ and $y_v \geq 0$. Hence, according to the above references, we have the following chamber decomposition of $L \otimes \mathbb{R}$.

Let $\mathbf{M}$ be the matrix with column vectors $\{a_i\}_{i=1}^d$ and $\{-E_v\}_{v=1}^s$. A subset σ of indices of columns is called *basis* if the corresponding columns form a basis of $L \otimes \mathbb{R}$; in this case we write $Cone(\mathbf{M}_\sigma)$ for the positive closed cone generated by them. Then the chamber decomposition is the polyhedral subdivision of $L \otimes \mathbb{R}$ provided by the common refinement of the cones $Cone(\mathbf{M}_\sigma)$, where σ runs all over the basis. *A chamber is a closed cone of the subdivision whose interior is non–empty.* Usually we denote them by C, let their index set (collection) be $\mathfrak{C}$.

We will need the associated *disjoint* decomposition of $L \otimes \mathbb{R}$ with relative open cones as well. A typical element of this disjoint decomposition is the *relative interior of an intersection of type* $\cap_{C \in \mathfrak{C}'} C$, where $\mathfrak{C}'$ runs over the subsets of $\mathfrak{C}$. For these cones we use the notation C_{op}.

Each chamber C determines an open cone, namely its interior. And, conversely, each top dimensional open cone determines a chamber C, namely its closure.

The next theorem is the direct consequence of [19, 4.4], [107, 0.2] and (4.1) using the additivity of the Ehrhart quasipolynomial on the suitable convex parts of $\mathcal{P}^{(l')}$. (We state it for our specific facet–collection $\mathcal{T}$, the case which will be used later, but it is true for any other facet–decomposition of the boundary whenever $\cup_{F^{(l')} \in \mathcal{T}} F^{(l')}$ is homeomorphic to a $(d-1)$–manifold.)

Theorem 4.2 *(a) For each relative open cone C_{op} of $L \otimes \mathbb{R}$, $\mathcal{P}^{(l')}$ is combinatorially stable, that is, the polytopes $\{\mathcal{P}^{(l')}\}_{l' \in C_{op}}$ have the same combinatorial type. Therefore, for any fixed $h \in H$, the restrictions $\mathcal{L}_h^{C_{op}}(\mathbf{A}, \mathcal{T})$ and $\mathcal{L}_h^{C_{op}}(\mathbf{A}, \mathcal{F} \setminus \mathcal{T})$ to C_{op} of $\mathcal{L}_h(\mathbf{A}, \mathcal{T})$ and $\mathcal{L}_h(\mathbf{A}, \mathcal{F} \setminus \mathcal{T})$ respectively are quasipolynomials.*

(b) These quasipolynomials have a continuous extension to the closure of C_{op}. Namely, if C'_{op} is in the closure of C_{op}, then $\mathcal{L}_h^{C'_{op}}(\mathbf{A}, \mathcal{T})$ is the restriction to C'_{op} of the (abstract) quasipolynomial $\mathcal{L}_h^{C_{op}}(\mathbf{A}, \mathcal{T})$. (Similarly for $\mathcal{L}_h^{C_{op}}(\mathbf{A}, \mathcal{F} \setminus \mathcal{T})$.)

In particular, for any chamber C one has a well–defined quasipolynomial $\mathcal{L}_h^C(\mathbf{A}, \mathcal{T})$, defined as $\mathcal{L}_h^{C_{op}}(\mathbf{A}, \mathcal{T})$, where C_{op} is the interior of C, which equals $\mathcal{L}_h(\mathbf{A}, \mathcal{T})$ for all points of C.

This also shows that for any two chambers C_1 and C_2 one has the continuity property

$$\mathcal{L}_h^{C_1}(\mathbf{A}, \mathcal{T})|_{C_1 \cap C_2} = \mathcal{L}_h^{C_2}(\mathbf{A}, \mathcal{T})|_{C_1 \cap C_2}. \tag{4.14}$$

(c) $\mathcal{L}^C_h(\mathbf{A},\mathcal{T})$ and $\mathcal{L}^C_{-h}(\mathbf{A},\mathcal{F}\setminus\mathcal{T})$, as abstract quasipolynomials associated with a fixed chamber C, satisfy the reciprocity

$$\mathcal{L}^C_h(\mathbf{A},\mathcal{T},l') = (-1)^d\cdot\mathcal{L}^C_{-h}(\mathbf{A},\mathcal{F}\setminus\mathcal{T},-l').$$

We have the following consequences regarding the counting function $l'\mapsto Q_h(l')$ of $f^e(\mathbf{t})$ defined in (4.7):

Corollary 4.1 *(a) Q_h is a piecewise quasipolynomial. Indeed, for any $h\in H$ and $l'\in L'$*

$$Q_h(l') = \sum_k \iota_k\cdot\mathcal{L}_{h-[b_k]}(\mathbf{A},\mathcal{T},l'-b_k). \tag{4.15}$$

In particular, the right hand side of (4.15) is independent of the representation of f as in (8.1) (that is, of the choice of $\{b_k,\,a_i\}_{k,i}$), it depends only on the rational function f.

(b) Fix a chamber C of $L\otimes\mathbb{R}$, cf. 4.2, and for any $h\in H$ define the quasipolynomial

$$Q^C_h(l') := \sum_k \iota_k\cdot\mathcal{L}^C_{h-[b_k]}(\mathbf{A},\mathcal{T},l'-b_k). \tag{4.16}$$

Then the restriction of $Q_h(l')$ to $\cap_k(b_k+C)$ is a quasipolynomial, namely

$$Q_h(l') = Q^C_h(l') \ \ on\ \ \bigcap_k(b_k+C). \tag{4.17}$$

Moreover, there exists $l'_*\in C$ such that $l'_*+C\subset\cap_k(b_k+C)$.

(Warning: $\mathcal{L}^C_{h-[b_k]}(\mathbf{A},\mathcal{T},l'-b_k)\neq\mathcal{L}_{h-[b_k]}(\mathbf{A},\mathcal{T},l'-b_k)$ unless $l'-[b_k]\in C$.)

(c) For any fixed $h\in H$, the quasipolynomial $Q^C_h(l')$ satisfies the following property: for any $l'\in L'$ with $[l']=h$, and any $q\in\square$ (the semi-open unit cube), one has

$$Q^C_h(l') = Q^C_h(l'-q). \tag{4.18}$$

In particular, by taking $l'=q=r_h$:

$$Q^C_h(r_h) = Q^C_h(0). \tag{4.19}$$

Proof For (a) use (4.9) and the fact that $b_k+\sum x_ia_i\not\geq l'$ if and only if $\sum x_ia_i\not\geq l'-b_k$. Since the coefficients of the Taylor expansion depend only on f, the second sentence follows too.

For (b) use part (a) and the fact that $C\cap\bigcap_k(b_k+C)$ contains a set of type l'_*+C.

(c) Consider those values l' in some l'_*+C for which all elements of type $l'-b_k$ and $l'-q-b_k$ are in C. For these values l', (4.18) follows from the identity $\mathcal{P}^{(l'),\triangleleft}\cap\rho^{-1}(h) = \mathcal{P}^{(l'-q),\triangleleft}\cap\rho^{-1}(h)$ whenever $[l']=h$. This is true since for any l'' with $[l'']=[l']$, $l''\geq l'$ is equivalent with $l''\geq l'-q$. Indeed, taking $y=l''-l'$, this reads as follows: for any $y\in L$, $y\geq 0$ if and only if $y\geq -q$.

Now, if two quasipolynomials agree on l'_0+C then they are equal. $\square$

Remark 4.3 Thanks to [107, Theorem 0.2], the continuity property 4.14 has the following extension (coincidence of the quasipolynomials on neighboring strips). Set $\square(\mathbf{A}) := \sum_i[0,1)a_i$.

Then for any two chambers C_1 and C_2, and $S := (-\square(\mathbf{A}) + C_1) \cap (-\square(\mathbf{A}) + C_2)$

$$\mathcal{L}_h^{C_1}(\mathbf{A}, \mathcal{T})|_S = \mathcal{L}_h^{C_2}(\mathbf{A}, \mathcal{T})|_S. \tag{4.20}$$

4.3.3 The $d = 0$ case

All the above properties can be extended for $d = 0$ as well. Although the polytope constructed in 4.9 does not exist, we can look at the polynomial $f(\mathbf{t}) = \sum_k \iota_k \mathbf{t}^{b_k}$ itself. Then using notation of (4.7) we set

$$\sum_{h\in H} Q_h(l')[h] = \sum_{l''\not\geq l'} p_{l''} \cdot [l''] = \sum_{\{k\,:\, b_k \not\geq l'\}} \iota_k [b_k].$$

Moreover, we have the chamber decomposition of $L \otimes \mathbb{R}$ defined by $\{-E_v\}_{v=1}^s$ via the same principle as above. This means two chambers: $C_0 := \mathbb{R}_{\geq 0}\langle -E_v\rangle$ and C_1, the closure of the complement of C_0 in $\mathbb{R}^s$. Then $Q_h(l') = \sum_{\{k\,:\,[b_k]=h\}} \iota_k$ on $\cap_k(b_k + C_1)$ and 0 on $\cap_k(b_k + C_0)$.

4.4 Multivariable equivariant periodic constant

We consider the situation of 4.2.2.1 and 4.3.1(a). For each $h \in H$ consider $r_h \in L'$ as in (2.6).

Definition 4.2 Let $\mathcal{K} \subset L' \otimes \mathbb{R}$ be a closed real cone whose affine closure aff$(\mathcal{K})$ has positive dimension. For any $h \in H$ we assume that there exist

- $l'_* \in \mathcal{K}$
- a sublattice $\widetilde{L} \subset L$ of finite index, and
- a quasipolynomial $l' \mapsto \widetilde{Q}_h(l')$, defined on $\widetilde{L} \cap \mathrm{aff}(\mathcal{K})$ such that

$$Q_h(l') = \widetilde{Q}_h(l') \quad \text{for any } \widetilde{L} \cap (l'_* + \mathcal{K}). \tag{4.21}$$

Then we define the *equivariant periodic constant* of f associated with $\mathcal{K}$ by

$$\mathrm{pc}^{e,\mathcal{K}}(f) = \sum_{h\in H} \mathrm{pc}_h^{\mathcal{K}}(f) \cdot [h] := \sum_{h\in H} \widetilde{Q}_h(0) \cdot [h] \in \mathbb{Z}[H], \tag{4.22}$$

and we say that f *admits a periodic constant in* $\mathcal{K}$. (Sometimes we will use the same notation for the real cone $\mathcal{K}$ and for its lattice points $\mathcal{K} \cap L'$ in L'.)

Remark 4.4 The above definition is independent of the choice of the sublattice $\widetilde{L}$: it can be replaced by any sublattice of finite index. The advantage of such sublattices is that convenient restrictions of Q_h might have nicer forms which are easier to compute. The choice of $\widetilde{L}$ corresponds to the choice of p in 4.2.1, and it is responsible for the name 'periodic' in the name of $\mathrm{pc}^{e,\mathcal{K}}(f)$.

Proposition 4.1 *(a) Consider the chamber decomposition of* $L \otimes \mathbb{R}$ *given by the denominator* $\prod_i (1 - \mathbf{t}^{a_i})$ *of* f *as in Theorem 4.2. Then* f *admits a periodic constant in each chamber* C *and*

$$\mathrm{pc}_h^C(f) = Q_h^C(r_h) = Q_h^C(0). \tag{4.23}$$

(b) If two functions f_1 *and* f_2 *admit periodic constant in some cone* $\mathcal{K}$*, then the same is true for* $\alpha_1 f_1 + \alpha_2 f_2$ *and*

$$\mathrm{pc}^{\mathcal{K}}(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \mathrm{pc}^{\mathcal{K}}(f_1) + \alpha_2 \mathrm{pc}^{\mathcal{K}}(f_2) \quad (\alpha_1, \alpha_2 \in \mathbb{C}).$$

(c) If f *admits periodic constants in two (top dimensional) cones* $\mathcal{K}_1$ *and* $\mathcal{K}_2$*, and the interior int*$(\mathcal{K}_1 \cap \mathcal{K}_2)$ *of the intersection* $\mathcal{K}_1 \cap \mathcal{K}_2$ *is non–empty, then* $\mathrm{pc}^{\mathcal{K}_1}(f) = \mathrm{pc}^{\mathcal{K}_2}(f)$.

In particular, if $\{C_i\}_{i=1,2}$ *are two chambers as in (a), and* f *admits a periodic constant in* $\mathcal{K}$*, and int*$(C_i \cap \mathcal{K}) \neq \emptyset$ $(i = 1, 2)$*, then* $\mathrm{pc}^{C_1}(f) = \mathrm{pc}^{C_2}(f)$.

Proof For (a) use Corollary 4.1; (b) is clear. For (c) we can assume that $\mathcal{K}_2 \subset \mathcal{K}_1$ (by considering $\mathcal{K}_i$ and $\mathcal{K}_1 \cap \mathcal{K}_2$). Then if Q_h is quasipolynomial on $l'_1 + \mathcal{K}_1$ (with $l'_1 \in \mathcal{K}_1$), then $(l'_1 + \mathcal{K}_2) \cap \mathcal{K}_2$ contains a set of type $l'_2 + \mathcal{K}_2$ with $l'_2 \in \mathcal{K}_2$, on which one can take the restriction of the previous quasipolynomial. □

Remark 4.5 Note that in the rational presentation of f we might assume that $a_i \in L$ for all i. Indeed, take $o_i \in \mathbb{Z}_{>0}$ such that $o_i a_i \in L$, and amplify the fraction by $\prod_i (1 - \mathbf{t}^{o_i a_i})/(1 - \mathbf{t}^{a_i})$. Therefore, for each h we can write $f_h(\mathbf{t})$ in the form

$$f_h(\mathbf{t}) = \mathbf{t}^{r_h} \sum_k \iota_k \cdot \frac{\mathbf{t}^{\overline{b}_k}}{\prod_i (1 - \mathbf{t}^{a_i})},$$

where $a_i, \overline{b}_k \in L$, hence $f_h(\mathbf{t})/\mathbf{t}^{r_h} \in \mathbb{Z}[[\mathbf{t}]][\mathbf{t}^{-1}]$. Then if we consider the non–equivariant periodic constant pc^C of $f_h(\mathbf{t})/\mathbf{t}^{r_h}$, 4.7, 4.17 and 4.23 imply that $\mathrm{pc}_h^C(f(\mathbf{t})) = \mathrm{pc}^C(f_h(\mathbf{t})/\mathbf{t}^{r_h})$ for all chambers C associated with $\{a_i\}_i$.

Example 4.5 Assume that $L = L' = \mathbb{Z}$ and $\mathcal{K} = \mathbb{R}_{\geq 0}$, and consider $S(t)$ as in 4.2.1. If $S(t)$ admits a periodic constant in $\mathcal{K}$, then $\mathrm{pc}^{\mathcal{K}}(S) = \mathrm{pc}(S)$, where $\mathrm{pc}(S)$ is the periodic constant defined in 4.2.1. □

Example 4.6 (a) (The $d = 0$ case) Assume that $f(\mathbf{t}) = \sum_{k=1}^r \iota_k \mathbf{t}^{b_k}$. Then, using 4.3.3 (and its notation), $\mathrm{pc}^{e,C_0}(f) = 0$ and $\mathrm{pc}^{e,C_1}(f) = \sum_{k=1}^r \iota_k [b_k] \in \mathbb{Z}[H]$.

(b) Assume that the rank is $s = 2$ and $f(\mathbf{t}) = \mathbf{t}^b/(1 - \mathbf{t}^a)$, with both entries (a_1, a_2) of a positive. We assume that $a \in L$ while $b \in L'$. Again, for $h \neq [b]$ the counting function, hence its periodic constant too, is zero. Assume $h = [b]$, and write $b = (b_1, b_2)$. Then the denominator provides three chambers: $C_0 := \mathbb{Z}_{\geq 0}\langle -E_1, -E_2 \rangle$, $C_1 := \mathbb{Z}_{\geq 0}\langle a, -E_2 \rangle$, $C_2 := \mathbb{Z}_{\geq 0}\langle a, -E_1 \rangle$. Then the three quasipolynomials for $1/(1 - \mathbf{t}^a)$ are $\mathcal{L}_h^{C_0} = 0$, $\mathcal{L}_h^{C_i}(n_1, n_2) = \lceil n_i/a_i \rceil$; hence $\mathrm{pc}_h^{C_0}(f) = 0$, $\mathrm{pc}_h^{C_i}(f) = \lceil -b_i/a_i \rceil$ $(i = 1, 2)$. In particular, $\mathrm{pc}_h^C(f)$, in general, depends on the choice of C.

(c) Assume that $L = L'$ and $f(t) = \frac{t_1^{b_1} t_2^{b_2}}{(1-t_1t_2)(1-t_1^2t_2)}$. Then the chambers associated with the denominator are: $C_0 := \mathbb{R}_{\geq 0}\langle -E_1, -E_2\rangle$, $C_2 := \mathbb{R}_{\geq 0}\langle -E_1, (1,1)\rangle$, $C := \mathbb{R}_{\geq 0}\langle (1,1), (2,1)\rangle$ and $C_1 := \mathbb{R}_{\geq 0}\langle (2,1), -E_2\rangle$. Then, by a computation,

$$\begin{aligned} &\mathcal{L}^{C_0} = 0; && \mathcal{L}^{C_2}(l_1,l_2) = \tfrac{l_2^2}{2} + \tfrac{l_2}{2};\\ &\mathcal{L}^{C}(l_1,l_2) = \tfrac{l_1^2}{2} + l_2^2 + \tfrac{l_1}{2} - l_1 l_2; && \mathcal{L}^{C_1}(l_1,l_2) = \tfrac{l_1^2}{4} + \tfrac{l_1}{2} + \tfrac{1+(-1)^{l_1+1}}{8}. \end{aligned} \tag{4.24}$$

Hence, by Proposition 4.1 and (8.9), one has $\mathrm{pc}^{C_*}(f) = \mathcal{L}^{C_*}(-b_1, -b_2)$. □

Example 4.7 Normal affine monoids. Consider the following objects (cf. 4.2.2.1): a lattice L with fixed bases $\{E_v\}_{v=1}^d$ (hence $s = d$) and with induced partial ordering $\leq$, $L' \subset L \otimes \mathbb{Q}$ an overlattice with finite abelian quotient $H := L'/L$ and projection $\rho : L' \to H$. Furthermore, let $\{a_i\}_{i=1}^d$ be linearly independent vectors in L' with all their $\{E_v\}$–coordinates positive. Let $\mathcal{K}$ be the positive real cone generated by the vectors $\{a_i\}_i$, and consider the Hilbert series of $\mathcal{K}$

$$f(\mathbf{t}) := \sum_{l' \in \mathcal{K} \cap L'} \mathbf{t}^{l'}.$$

Since $\mathcal{K}$ depends only on the rays generated by the vectors a_i, we can assume that $a_i \in L$ for all i.

Set $\square(\mathbf{A}) = \sum_{i=1}^d [0,1)a_i$ as above, and consider the monoid $M := \mathbb{Z}_{\geq 0}\langle a_i\rangle$ (cf. e.g. [21, 2.C]). Then the normal affine monoid $\mathcal{K} \cap L'$ is a module over M and if we set $B := \square(\mathbf{A}) \cap L'$, [21, Prop. 2.43] implies that

$$\mathcal{K} \cap L' = \bigsqcup_{b \in B} b + M.$$

In particular, $f(\mathbf{t})$ equals $\sum_{b\in B} \mathbf{t}^b / \prod_{i=1}^d (1 - \mathbf{t}^{a_i})$ and has the form considered in 4.2.2. If the rank d is ≥ 3 then $\mathcal{K}$ usually is cut in more chambers. Indeed, take e.g. $d = 3$, $a_i = (1,1,1) + E_i$ for $i = 1,2,3$. Then $\mathcal{K}$ is cut in its barycentric subdivision. Nevertheless, if $d = 2$ then $\mathcal{K}$ consists of a unique chamber and f admits a periodic constant in $\mathcal{K}$. Indeed, one has:

Lemma 4.1 *If $d = 2$ then $\mathrm{pc}_h^{\mathcal{K}}(f) = 0$ for all $h \in H$.* □

Proof It is elementary to see that $\mathcal{K}$ is one of the chambers (use the construction from 4.3.2). Take $B = \{b_k\}_k$, and write $f = \sum_k f_k$, where $f_k = \mathbf{t}^{b_k}/(1-\mathbf{t}^{a_1})(1-\mathbf{t}^{a_2})$. The only relevant classes $h \in H$ are given by $\{[b_i] : b_i \in B\}$, otherwise already the Ehrhart quasipolynomials are zero (since $a_i \in L$). Fix such a class $h = [b_i]$. Let $\mathcal{L}_h^{\mathcal{K}}(\mathcal{T})$ be the quasipolynomial associated with the chamber $\mathcal{K}$ and the denominator of f. Then, by (4.23) and (8.9), $\mathrm{pc}_h^{\mathcal{K}}(f_k) = \mathcal{L}^{\mathcal{K}}_{[b_i - b_k]}(\mathcal{T})(-b_k)$. This, by Reciprocity Law 4.2(c) equals $\mathcal{L}^{\mathcal{K}}_{[b_k - b_i]}(\mathcal{F} \setminus \mathcal{T})(b_k)$. Again, since the denominator is a series in L, for $[b_k - b_i] \neq 0$ the series is zero; so we may assume $[b_k - b_i] = 0$. But, since $b_k \in \mathcal{K}$, the value $\mathcal{L}_0^{\mathcal{K}}(\mathcal{F} \setminus \mathcal{T})(b_k)$ of the quasipolynomial carries its geometric meaning, it is the cardinality of the set $\{m = n_1a_1 + n_2a_2 : n_1 > 0,\ n_2 > 0,\ m \not\geq b_k\}$. But since for any such m one has $m \geq a_1 + a_2 > b_k$, contradicting $m \not\geq b_k$, this set is empty. □

Example 4.8 General affine monoids of rank $d = 2$. Consider the situation of Example 4.7 with $d = 2$, and let N be a submonoid of $\widehat{N} = \mathcal{K} \cap L'$ of rank 2, and we also assume that $\widehat{N}$ is the normalization of N. Set

$$f(\mathbf{t}) := \sum_{l' \in N} \mathbf{t}^{l'}.$$

Then $f(\mathbf{t})$ is again of type (8.1). Indeed, by [21, Prop. 2.35], $\widehat{N} \setminus N$ is a union of a finite family of sets of type (I) $b \in \widehat{N}$, or (II) $b + \mathbb{Z}ka_i$, where $b \in \widehat{N}$, $k \in \mathbb{Z}_{\geq 0}$, $i = 1$ or 2. Obviously, two sets of type (II) with different i-values might have an intersection point of type (I). In particular,

$$f(\mathbf{t}) = \sum_{l' \in \widehat{N}} \mathbf{t}^{l'} - \sum_i \frac{\mathbf{t}^{b_{i,1}}}{1 - \mathbf{t}^{k_{i,1}a_1}} - \sum_j \frac{\mathbf{t}^{b_{j,2}}}{1 - \mathbf{t}^{k_{j,2}a_2}} + \sum_k (\pm \mathbf{t}^{b_k}).$$

Note that the periodic constant of the first sum is zero by Lemma 4.1, and the others can easily be computed (even with closed formulae) via Example 4.6, parts (a) and (b).

The computation shows that the periodic constant carries information about the failure of normality of N (compare with the delta-invariant computation from the end of 4.2.1).

The situation is similar when we consider a *semigroup* of $\widehat{N}$, that is, when we eliminate the neutral element of the above N as well. □

Example 4.9 Reduction of variables. The next statement is an example when the number of variables of the function f can be reduced in the procedure of the periodic constant computation. (For another reduction result, see Theorem 5.1.) For simplicity we assume $L' = L$.

Proposition 4.2 *Let* $f(\mathbf{t}) = \frac{\mathbf{t}^b}{\prod_{i=1}^d (1-\mathbf{t}^{a_i})}$ *and assume that* $b = \sum_{v=1}^s b_v E_v \in C$, *where* C *is a chamber associated with the denominator.*

We consider the subset $Pos := \{v : b_v > 0\}$ *with cardinality* p, *and the projection* $\mathbb{R}^s \to \mathbb{R}^p$, *defined by* $(r_v)_{v=1}^s \mapsto (r_v)_{v \in Pos}$ *and denoted by* $v \mapsto v^\dagger$. *Accordingly, we set a new function* $f^\dagger(\mathbf{z}) := \frac{\mathbf{z}^{b^\dagger}}{\prod_{i=1}^d (1-\mathbf{z}^{a_i^\dagger})}$ *in* p *variables, and a new chamber* $C^\dagger := \mathbb{R}_{\geq 0}\langle \{w_j^\dagger\}_j \rangle$, *where* w_j *are the generators of* $C = \mathbb{R}_{\geq 0}\langle \{w_j\}_j \rangle$. *Then* $\mathrm{pc}^C(f) = \mathrm{pc}^{C^\dagger}(f^\dagger)$.

Proof This is a direct application of Theorem 4.1(b). Indeed, by the Ehrhart–MacDonald–Stanley reciprocity law, we get $\mathrm{pc}^C(f) = \mathcal{L}^C(\mathbf{A}, \mathcal{T}, -b) = (-1)^d \cdot \mathcal{L}^C(\mathbf{A}, \mathcal{F} \setminus \mathcal{T}, b)$. Since $b \in C$, by the very definition of $\mathcal{L}^C(\mathbf{A}, \mathcal{F} \setminus \mathcal{T})$, this (modulo the sign) equals the number of integral points of $\mathcal{P}^{(b)} \setminus \cup_{F^{(b)} \in \mathcal{F} \setminus \mathcal{T}} F^{(b)} \subset \mathbb{R}^d$. But, if $v \notin Pos$, i.e., $b_v \leq 0$, then in (4.1) $\mathcal{P}_v^{(b_v)}$ has only non–positive integral points. Therefore we can omit these polytopes without affecting the periodic constant. Then, this fact and $b^\dagger \in C^\dagger$ imply that pc^C can be computed as $(-1)^d \mathcal{L}^{C^\dagger}(\mathbf{A}^\dagger, \mathcal{F}^\dagger \setminus \mathcal{T}^\dagger, b^\dagger)$. □

Remark 4.6 Under the conditions of Proposition 4.2 we have the following application of the statement from Remark 4.3 (based on [107]): *Assume that* $b \in \square(\mathbf{A}) - C$ *and* $b \geq 0$. *Then* $\mathrm{pc}^C(f) = 0$. Indeed, $\mathrm{pc}^C(f) = \mathcal{L}^C(\mathbf{A}, \mathcal{T}, -b) = \mathcal{L}^{C(-b)}(\mathbf{A}, \mathcal{T}, -b)$, where $C(-b)$ is a chamber containing $-b$. But since $-b \leq 0$ one gets $\mathcal{L}^{C(-b)}(\mathbf{A}, \mathcal{T}, -b) = 0$ by 4.2.

One of the key messages of the above examples (starting from 4.6) is the following: 'if b is small compared with the a_i's, then the periodic constant is zero' (compare with 4.2.1 too).

4.10 The polynomial part in the $d = s = 2$ case

In this case $\mathrm{rank}(L) = 2$, and we have two vectors in the denominator of f, namely $a_i = (a_{i,1}, a_{i,2})$, $i = 1, 2$. We will order them in such a way that a_2 sits in the cone of a_1 and E_1, that is, $\det\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2}\end{pmatrix} < 0$. The chamber decomposition will be the following: $C_0 := \mathbb{R}_{\geq 0}\langle -E_1, -E_2\rangle$, $C_2 := \mathbb{R}_{\geq 0}\langle -E_1, a_1\rangle$, $C := \mathbb{R}_{\geq 0}\langle a_1, a_2\rangle$ and $C_1 := \mathbb{R}_{\geq 0}\langle a_2, -E_2\rangle$ (the index choice is motivated by the formulae from 4.6(b)).

Our goal is to write any rational function (with denominator $(1 - \mathbf{t}^{a_1})(1 - \mathbf{t}^{a_2})$) as a sum of $f^+(\mathbf{t})$ and $f^-(\mathbf{t})$, such that $f^+ \in \mathbb{Z}[L']$ (the 'polynomial part of f'), and $\mathrm{pc}^{e,C}(f^-) = 0$. This is a generalization of the decomposition in the one–variable case discussed in 4.2.1, and will be a major tool in the computation of the periodic constant in Section 6.3 for graphs with two nodes. The specific form of the decomposition is motivated by Examples 4.6(b) and 4.7.

As above, we set $\square(\mathbf{A}) = [0, 1)a_1 + [0, 1)a_2$ and for $i = 1, 2$ we also consider the strips

$$\Xi_i := \{b = (b_1, b_2) \in L \otimes \mathbb{R} \mid 0 \leq b_i < a_{i,i}\}.$$

Theorem 4.3 *(1) Any function* $f(\mathbf{t}) = \big(\sum_{k=1}^r \iota_k \mathbf{t}^{b_k}\big)/\prod_{i=1}^2(1 - \mathbf{t}^{a_i})$ *(with* $\iota_k \in \mathbb{Z}$*) can be written as a sum* $f(\mathbf{t}) = f^+(\mathbf{t}) + f^-(\mathbf{t})$*, where*

(a) $f^+(\mathbf{t})$ *is a finite sum* $\sum_\ell \kappa_\ell \mathbf{t}^{\beta_\ell}$*, with* $\kappa_\ell \in \mathbb{Z}$ *and* $\beta_\ell \in L'$*;*

(b) $f^-(\mathbf{t})$ *has the form*

$$f^-(\mathbf{t}) = \frac{\sum_{k=1}^r \iota_k \mathbf{t}^{b'_k}}{\prod_{i=1}^2 (1 - \mathbf{t}^{a_i})} + \frac{\sum_{i=1}^{n_1} \iota_{i,1}\mathbf{t}^{b_{i,1}}}{1 - \mathbf{t}^{a_1}} + \frac{\sum_{i=1}^{n_2} \iota_{i,2}\mathbf{t}^{b_{i,2}}}{1 - \mathbf{t}^{a_2}}, \tag{4.25}$$

with $b'_k \in L' \cap \square(\mathbf{A})$ *for all* k*, and* $b_{i,j} \in L' \cap \Xi_j$ *for any* i *and* $j = 1, 2$*, and* $\iota_k\ \iota_{i,1},\ \iota_{i,2} \in \mathbb{Z}$.

(2) Consider a sum

$$\Sigma(\mathbf{t}) := \frac{Q(\mathbf{t})}{\prod_{i=1}^2 (1 - \mathbf{t}^{a_i})} + \frac{Q_1(\mathbf{t})}{1 - \mathbf{t}^{a_1}} + \frac{Q_2(\mathbf{t})}{1 - \mathbf{t}^{a_2}} + f^+(\mathbf{t}), \tag{4.26}$$

where $Q(\mathbf{t}) := \sum_{k=1}^r \iota_k \mathbf{t}^{b'_k}$ *with* $b'_k \in L' \cap \square(\mathbf{A})$ *for all* k*;* $Q_j(\mathbf{t}) = \sum_{i=1}^{n_1} \iota_{i,j}\mathbf{t}^{b_{i,j}}$ *with* $b_{i,j} \in L' \cap \Xi_j$ *for any* i *and* $j = 1, 2$*; and finally* $f^+ \in \mathbb{Z}[L']$ *is a polynomial as in part (a) above.*

Then, if $\Sigma(\mathbf{t}) = 0$*, then* $Q(\mathbf{t}) = Q_1(\mathbf{t}) = Q_2(\mathbf{t}) = f^+(\mathbf{t}) = 0$.

In particular, the decomposition in part (1) is unique.

(3) The periodic constant of $f^-(\mathbf{t})$ *associated with the chamber* C *is zero. Hence, in the decomposition (1) one also has* $\mathrm{pc}^{e,C}(f) = \mathrm{pc}^{e,C}(f^+) = \sum_\ell \kappa_\ell[\beta_\ell] \in \mathbb{Z}[H]$.

Proof (1) For every $b_k \in L'$ we have a (unique) $b'_k \in L' \cap \square(\mathbf{A})$ such that $b_k - b'_k \in \mathbb{Z}\langle a_1, a_2\rangle$. Set $Q(\mathbf{t}) := \sum_{k=1}^r \iota_k \mathbf{t}^{b'_k}$. Then $f(\mathbf{t}) - Q(\mathbf{t})/\prod_{i=1}^2(1 - \mathbf{t}^{a_i})$ is a sum of terms of type

$\mathbf{t}^{b'}(\mathbf{t}^{k_1a_1+k_2a_2}-1)/\prod_{i=1}^2(1-\mathbf{t}^{a_i})$. This decomposes as a sum with terms of type $\mathbf{t}^c/(1-\mathbf{t}^{a_i})$. Then for every such expression, there exists $c_i \in \Xi_i$ such that $(\mathbf{t}^c-\mathbf{t}^{c_i})/(1-\mathbf{t}^{a_i})$ is as in (a).

Part (2) is again elementary. First we show that $Q(\mathbf{t})=0$. For any $b'\in L'\cap\square(\mathbf{A})$ consider $\Xi_{b'}:=b'+\mathbb{Z}\langle a_1,a_2\rangle$. For any $P(\mathbf{t})=\sum \iota_k\mathbf{t}^{c'_k}$ write $P_{b'}(\mathbf{t})=\sum_{c_k\in\Xi_{b'}}\iota_k\mathbf{t}^{c'_k}$ for its part supported on $\Xi_{b'}$. This decomposition can be done for Q, Q_1, Q_2 and f^+, hence for $\Sigma(\mathbf{t})$. Note that it is enough to prove (2) for such $\Sigma_{b'}(\mathbf{t})$, for a fixed b'. Hence, we can assume that $\Sigma(\mathbf{t})$ is supported on some $\Xi_{b'}$, $b'\in L'\cap\square(\mathbf{A})$. Since $\Xi_{b'}\cap\square(\mathbf{A})=\{b'\}$, in this case $Q(\mathbf{t})=\iota\,\mathbf{t}^{b'}$. Multiplying $\Sigma(\mathbf{t})$ by $\prod_{i=1}^2(1-\mathbf{t}^{a_i})$ and substituting $t_1=t_2=1$ we get $\iota=0$. Hence $Q(\mathbf{t})=0$.

Next, consider the identity $(1-\mathbf{t}^{a_2})Q_1(\mathbf{t})+(1-\mathbf{t}^{a_1})Q_2(\mathbf{t})+\prod_{i=1}^2(1-\mathbf{t}^{a_i})\cdot f^+(\mathbf{t})=0$. Since $\mathbb{Z}[t_1,t_2]$ is UFD and the polynomials $1-\mathbf{t}^{a_1}$ and $1-\mathbf{t}^{a_2}$ are relative primes, we get that $1-\mathbf{t}^{a_i}$ divides $Q_i(\mathbf{t})$. This together with the support assumption of Q_i implies $Q_i=0$.

(3) The vanishing of the periodic constant of the first fraction of f^- follows from the proof of Lemma 4.1. The vanishing of $\mathrm{pc}^{e,C}$ of the other two fractions follows from Example 4.6(b). For the last expression see Example 4.6(a). □

Remark 4.7 (a) Part (a) of the proof provides an algorithm how one finds the decomposition.

(b) Since $\mathrm{pc}^{e,C}(f^-)=0$ by (3), the above decomposition $f=f^++f^-$ is well–suited for computing the periodic constant of f associated with chamber C via f^+.

Chapter 5
Periodic constants, Seiberg–Witten invariants and Ehrhart coefficients

5.1 The case associated with the link of a surface singularity

Consider the topological setup of a surface singularity, as in chapter 2. The lattice L has a canonical basis $\{E_v\}_{v\in\mathcal{V}}$ corresponding to the vertices of the graph Γ. We investigate the periodic constant of the rational function $Z(\mathbf{t})$, defined in 3.2 from Γ. Since $Z(\mathbf{t})$ has the form (8.1), all the results of section 4.2 can be applied. In particular, if $\mathcal{E} = \{v \in \mathcal{V} : \delta_v = 1\}$ denotes the set of *ends* of the graph, then $\mathbf{A}$ has column vectors $a_v = E^*_v$ for $v \in \mathcal{E}$. Hence, the rank of the lattice/space where the polytopes $\mathcal{P}^{(l')} = \cup_v \mathcal{P}_v$ sit is $d = |\mathcal{E}|$, and the convex polytopes $\{\mathcal{P}_v\}$ are indexed by $\mathcal{V}$. Furthermore, the dilation parameter l' of the polytopes runs in a $|\mathcal{V}|$–dimensional space. In the sequel we will drop the symbol $\mathbf{A}$ from $\mathcal{L}^C_h(\mathbf{A}, \mathcal{T}, l')$.

(The construction has some analogies with the construction of the splice–quotient singularities [88]: in that case the equations of the universal abelian cover of the singularity are written in $\mathbb{C}^d$, together with an action of H. Nevertheless, in the present situation, we are not obstructed with the semigroup and congruence relations present in that theory.)

In this new construction, a crucial additional ingredient comes from singularity theory, namely Theorem 3.1 (in fact, this is the main starting point and motivation of the whole approach). This combined with facts from Section 4.2 give:

Corollary 5.1 *Let $\mathcal{S} = \mathcal{S}_{\mathbb{R}}$ be the (real) Lipman cone $\{x \in \mathbb{R}^{|\mathcal{V}|} : (x, E_v) \leq 0$ for all $v\}$.*

(a) The rational function $Z(\mathbf{t})$ admits a periodic constant in the cone $\mathcal{S}$, which equals the normalized Seiberg–Witten invariant

$$\mathrm{pc}^{\mathcal{S}}_h(Z) = \mathfrak{sw}^{norm}_h(M). \tag{5.1}$$

(b) Consider the chamber decomposition associated with the denominator of $Z(\mathbf{t})$ as in Theorem 4.2, and let C be a chamber such that $int(C \cap \mathcal{S}) \neq \emptyset$. Then $Z(\mathbf{t})$ admits a periodic constant in C, which equals both $\mathrm{pc}^{\mathcal{S}}_h(Z)$ (satisfying (5.1)) and also

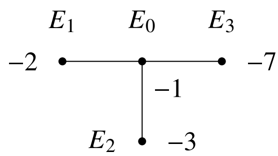
$$\mathrm{pc}^C_h(Z) = \sum_k \iota_k \cdot \mathcal{L}^C_{h-[b_k]}(\mathcal{T}, -b_k) = \sum_k \iota_k \cdot \mathcal{L}^C_{[b_k]-h}(\mathcal{F} \setminus \mathcal{T}, b_k). \tag{5.2}$$

In particular, $\mathrm{pc}^C_h(Z)$ does not depend on the choice of C (under the above assumption).

Proof Write $l' = \tilde{l} + r_h$ with $\tilde{l} \in L$ in (3.10). Since $\sum_{l\in L,\, l\not\geq 0} p_{l'+l} = \sum_{l''\not\geq \tilde{l},\, [l'']=r_h} p_{l''}$, (a) follows from Theorem 3.1. For (b) use Corollary 4.1 and Proposition 4.1. □

We note that the Lipman cone $\mathcal{S}$ can indeed be cut in several chambers (of the denominator of Z). This can happen even in the simple case of Brieskorn germs. Below we provide such an example together with several exemplifying details of the construction.

Example 5.2 Lipman cone cut in several chambers. Consider the 3-manifold $S^3_{-1}(T_{2,3})$ (where $T_{2,3}$ is the right-handed trefoil knot), or, equivalently, the link of the hypersurface singularity $z_1^2 + z_2^3 + z_3^7 = 0$. Its plumbing graph Γ and matrix $-I^{-1}$ are:



$$-I^{-1} = \begin{pmatrix} 42 & 21 & 14 & 6 \\ 21 & 11 & 7 & 3 \\ 14 & 7 & 5 & 2 \\ 6 & 3 & 2 & 1 \end{pmatrix}$$

where the row/column vectors of $-I^{-1}$ are E_0^*, E_1^*, E_2^* and E_3^* in the $\{E_v\}$ basis. The polytopes defined in (4.1), or in (4.8), with parameter $l = (l_0, l_1, l_2, l_3) \subset \mathbb{Z}^4$, sit in $\mathbb{R}^3$. Let u_1, u_2, u_3 be the basis of $\mathbb{R}^3$. Then the polytopes are the following convex closures:

$$\begin{aligned}
\mathcal{P}_0^{(l)} &= conv\,(0, (l_0/21)\, u_1, (l_0/14)\, u_2, (l_0/6)\, u_3) \\
\mathcal{P}_1^{(l)} &= conv\,(0, (l_1/11)\, u_1, (l_1/7)\, u_2, (l_1/3)\, u_3) \\
\mathcal{P}_2^{(l)} &= conv\,(0, (l_2/7)\, u_1, (l_2/5)\, u_2, (l_2/2)\, u_3) \\
\mathcal{P}_3^{(l)} &= conv\,(0, (l_3/3)\, u_1, (l_3/2)\, u_2, (l_3/1)\, u_3)\,.
\end{aligned}$$

Since $E_0^* + \varepsilon(-E_0)$ is in the interior of the (real) Lipman cone for $0 < \varepsilon \ll 1$, we get that the Lipman cone is cut in several chambers. The periodic constant can be computed with any of them. In fact, by the continuity of the quasipolynomials associated with the chambers, any quasipolynomial associated with any ray in the Lipman cone, even if it is situated at its boundary, provides the periodic constant. One such degenerated polytope provided by a ray on the boundary of $\mathcal{S}$ is of special interest. Namely, if we take $l = \lambda E_0^* \in \mathcal{S}$ for $\lambda > 0$, then $\mathcal{P}^{(l)} = \bigcup_{v=0}^3 \mathcal{P}_v^{(l)}$ is the same as $\mathcal{P}_0^{(l)} = conv(0, 2\lambda u_1, 3\lambda u_2, 7\lambda u_3)$. Moreover, if C is any chamber which contains the ray $\mathbb{R}_{\geq 0}E_0^*$ at its boundary, then for any $l = \lambda E_0^*$ one has $\mathcal{L}^C(\mathbf{A}, \mathcal{T}, l) = \mathcal{L}(\widetilde{\mathcal{P}}_0, \mathcal{T}, \lambda)$, where the last is the classical Ehrhart polynomial of the tetrahedron $\widetilde{\mathcal{P}}_0 := conv(0, 2u_1, 3u_2, 7u_3)$. Here we witness an additional coincidence of $\widetilde{\mathcal{P}}_0$ with the Newton polytope Γ_N^- of the equation $z_1^2 + z_2^3 + z_3^7$.

We compute $\mathcal{L}(\widetilde{\mathcal{P}}_0, \mathcal{T}, \lambda)$ as follows. From (3.6)–(3.7) and Corollary 4.1, we get that

$$\chi(\lambda E_0^*) + \text{geometric genus of } \{z_1^2 + z_2^3 + z_3^7 = 0\} = \mathcal{L}(\widetilde{\mathcal{P}}_0, \mathcal{T}, \lambda) - \mathcal{L}(\widetilde{\mathcal{P}}_0, \mathcal{T}, \lambda - 1). \quad (5.3)$$

Since this geometric genus is 1, and the free term of $\mathcal{L}(\widetilde{\mathcal{P}}_0, \mathcal{T}, \lambda)$ is zero (since for $\lambda = 0$ the zero polytope with boundary conditions contains no lattice point), and $-K = 2E_0 + E_1 + E_2 + E_3$, we get that $\mathcal{L}(\widetilde{\mathcal{P}}_0, \mathcal{T}, \lambda) = 7\lambda^3 + 10\lambda^2 + 4\lambda$. We emphasize that a formula as in (5.3), realizing a bridge between the Riemann–Roch expression χ (supplemented with the geometric genus) and the Ehrhart polynomial of the Newton diagram, was not known for Newton non–degenerate germs.

In the sequel we will provide several examples, when the Newton polytope is not even defined. □

5.3 Example: the case of lens spaces

As we will see in Theorem 5.1, the complexity of the problem depends basically on the number of nodes of Γ. In this subsection we treat the case when there are no nodes at all, that is M is a lens space. In this case the numerator of the rational function $f(\mathbf{t})$ is 1, hence everything is described by the 2–dimensional polytopes determined by the denominator. In the literature there are several results about lens spaces fitting in the present program, here we collect the relevant ones completing with the new interpretations. This subsection also serves as a preparatory part, or model, for the study of chains of arbitrary graphs.

Assume that the plumbing graph is

with all $k_v \geq 2$, and p/q is expressed via the (Hirzebruch, or negative) continued fraction

$$[k_1, \ldots, k_s] = k_1 - 1/(k_2 - 1/(\cdots - 1/k_s)\cdots). \tag{5.4}$$

Then M is the lens space $L(p,q)$. We also define q' by $q'q \equiv 1 \bmod p$, and $0 \leq q' < p$. Furthermore, we set $g_v = [E_v^*] \in H$. Then g_s generates $H = \mathbb{Z}_p$, and any element of H can be written as ag_s for some $0 \leq a < p$. Recall the definitions of r_h and s_h from 2.1.2.3 as well.

From the analytic point of view $(X,0)$ is a cyclic quotient singularity $(\mathbb{C}^2, o)/\mathbb{Z}_p$, where the action is $\xi * (x,y) = (\xi x, \xi^q y)$ (here ξ runs over p–roots of unity).

5.3.1 The Seiberg–Witten invariant

Since $(X,0)$ is rational, in this case $Z(\mathbf{t}) = P(\mathbf{t})$ (cf. subsection 3.1). Moreover, in (3.7) $H^1(\mathcal{O}_{\widetilde{Y}}) = 0$, hence

$$\mathfrak{sw}_{-h*\sigma_{can}}(M) = -\frac{(K+2r_h)^2 + |\mathcal{V}|}{8} = -\frac{K^2 + |\mathcal{V}|}{8} + \chi(r_h). \tag{5.5}$$

On the other hand, in [69, 72] a similar formula is proved for the Seiberg–Witten invariant: one only has to replace in (5.5) $\chi(r_h)$ by $\chi(s_h)$. In particular, for lens spaces, and for any $h \in H$ one has

$$\chi(r_h) = \chi(s_h). \tag{5.6}$$

(Note that, in general, for other links, $\chi(r_h) > \chi(s_h)$ might happen, see Example 6.2. Here, (5.6) follows from the vanishing of the geometric genus of the universal abelian cover of $(X, 0)$.)

In general, the coefficients of the representatives s_{ag_s} and r_{ag_s} $(0 \le a < p)$ are rather complicated arithmetical expressions; for s_{ag_s} see [69, 10.3] (where g_s is defined with opposite sign). The value $\chi(s_{ag_s})$ is computed in [69, 10.5.1] as

$$\chi(s_{ag_s}) = \frac{a(1-p)}{2p} + \sum_{j=1}^{a} \left\{ \frac{jq'}{p} \right\}. \tag{5.7}$$

For completeness of the discussion we also recall that $K = E_1^* + E_s^* - \sum_v E_v$ and

$$(K^2 + |\mathcal{V}|)/4 = (p-1)/(2p) - 3 \cdot \mathbf{s}(q,p), \tag{5.8}$$

cf. [69, 10.5], where $\mathbf{s}(q,p)$ denotes the Dedekind sum

$$\mathbf{s}(q,p) = \sum_{l=0}^{p-1} \left(\left(\frac{l}{p}\right)\right)\left(\left(\frac{ql}{p}\right)\right), \text{ where } ((x)) = \begin{cases} \{x\} - 1/2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

In particular, $\mathfrak{sw}_{-h*\sigma_{can}}(M)$ is determined via the formulae (5.5) – (5.8).

The non–equivariant picture looks as follows: $\sum_h \mathfrak{sw}_{-h*\sigma_{can}} = \lambda$, the Casson–Walker invariant of M, hence (5.5) gives

$$\lambda = -p(K^2 + |\mathcal{V}|)/8 + \textstyle\sum_h \chi(r_h).$$

This is compatible with (5.8) and formulae $\lambda(L(p,q)) = p \cdot \mathbf{s}(q,p)/2$ and $\sum_h \chi(r_h) = (p-1)/4 - p \cdot \mathbf{s}(q,p)$, cf. [69, 10.8].

5.3.2 The polytope and its quasipolynomial

We compare the above data with Ehrhart theory. In this case $Z(\mathbf{t}) = (1-\mathbf{t}^{E_1^*})^{-1}(1-\mathbf{t}^{E_s^*})^{-1}$. The vectors $a_1 = E_1^*$ and $a_s = E_s^*$ determine the polytopes $\mathcal{P}^{(l')}$ and a chamber decomposition.

For $1 \le v \le w \le s$ let n_{vw} denote the numerator of the continued fraction $[k_v, \ldots, k_w]$ (or, the determinant of the corresponding bamboo subgraph). For example, $n_{1s} = p$, $n_{2s} = q$ and $n_{1,s-1} = q'$. We also set $n_{v+1,v} := 1$. Then $pE_1^* = \sum_{v=1}^s n_{v+1,s} E_v$ and $pE_s^* = \sum_{v=1}^s n_{1,v-1} E_v$.

In particular, for any $l' = \sum_v l'_v E_v \in \mathcal{S}'$, the (non–convex) polytopes are

$$\mathcal{P}^{(l')} = \bigcup_{v=1}^{s} \left\{ (x_1, x_s) \in \mathbb{R}^2_{\ge 0} \,:\, x_1 n_{v+1,s} + x_s n_{1,v-1} \le p l'_v \right\} \subset \mathbb{R}^2_{\ge 0}. \tag{5.9}$$

The representation $\mathbb{Z}^2 \xrightarrow{\rho} \mathbb{Z}_p$ is $(x_1, x_s) \mapsto (qx_1 + x_s)g_s$.

Though $\mathcal{P}^{(l')}$ is a plane polytope, the direct computation of its equivariant Ehrhart multi-variable polynomial (associated with a chamber, or just with the Lipman cone) is still highly non–trivial. Here we will rely again on Theorem 3.1. On a subset of type $l'_0+\mathcal{S}'$ the identity (3.10) provides the counting function. The right hand side of (3.10) depends on all the coordinates of l', hence all the triangles $\mathcal{P}_v$ contribute in $\mathcal{P}^{(l')}$. Since this can happen only in a unique combinatorial way, we get that there is a chamber C which contains the Lipman cone. Let $\mathcal{L}^{e,C}$ be its quasipolynomial, and $\mathcal{L}^{e,\mathcal{S}}$ its restriction to $\mathcal{S}$. Since the numerator of $Z(\mathbf{t})$ is 1, $Q^C_h=\mathcal{L}^C_h$. Since this agrees with the right hand side of (3.10) on a cone of type $l'_0+\mathcal{S}'$, and the Lipman cone is in C, we get that

$$Q_h(l')=Q^C_h(l')=\mathcal{L}^{\mathcal{S}}_h(l')=-\mathfrak{sw}_{-h*\sigma_{can}}(M)-\frac{(K+2l')^2+|\mathcal{V}|}{8} \tag{5.10}$$

for any $l'\in(r_h+L)\cap\mathcal{S}'$ and $h\in H$. Using the identity (5.5), this reads as

$$\mathcal{L}^{\mathcal{S}}_h(\mathcal{T},l')=\chi(l')-\chi(r_h),\quad l'\in(r_h+L)\cap\mathcal{S}'. \tag{5.11}$$

Note that for any fixed h and any l' there exists a unique $q=q_{l',h}\in\square$ such that $l'+q:=l''\in r_h+L$. Indeed, take for q the representative of r_h-l' in $\square$. Then (4.18) and (5.11) imply

$$\mathcal{L}^{\mathcal{S}}_h(\mathcal{T},l')=\mathcal{L}^{\mathcal{S}}_h(\mathcal{T},l'')=\chi(l'+q_{l',h})-\chi(r_h). \tag{5.12}$$

This formula emphasizes the quasi–periodic behavior of $\mathcal{L}^{\mathcal{S}}_h(\mathcal{T},l')$ as well.

If l' is an element of L then $q_{l',h}=r_h$, hence (5.12) gives in this case

$$\mathcal{L}^{\mathcal{S}}_h(\mathcal{T},l)=\chi(l+r_h)-\chi(r_h)=\chi(l)-(l,r_h)\quad\text{for } l\in L\cap\mathcal{S}. \tag{5.13}$$

In particular, $\mathrm{pc}(\mathcal{L}^{\mathcal{S}}_h(\mathcal{T}))=\chi(r_h)-\chi(r_h)=0$ (a fact compatible with $H^1(\mathcal{O}_{\widetilde{Y}})=0$).

Even the non–equivariant case looks rather interesting. Let $\mathcal{L}^{\mathcal{S}}_{ne}(\mathcal{T})=\sum_{h\in H}\mathcal{L}^{\mathcal{S}}_h(\mathcal{T})$ be the Ehrhart polynomial of $\mathcal{P}^{(l')}$ (with boundary condition $\mathcal{T}$), where we count all the lattice points independently of their class in H. Then, (5.13) gives for $l\in L\cap\mathcal{S}$

$$\mathcal{L}^{\mathcal{S}}_{ne}(\mathcal{T},l)=p\cdot\chi(l)-(l,\textstyle\sum_h r_h)=-p\cdot(l,l)/2-p\cdot(l,K)/2-(l,\sum_h r_h). \tag{5.14}$$

In fact, $\sum_h r_h$ can explicitly be computed. Indeed, set $d_v=\gcd(p,n_{1,v-1})$ and $p_v=p/d_v$. Then one checks that $aE^*_s=\sum_v n_{1,v-1}\frac{a}{p}E_v$, $r_h=\sum_v\left\{n_{1,v-1}\frac{a}{p}\right\}E_v$ and $\sum_h r_h=\sum_v d_v\frac{p_v-1}{2}E_v$.

The coefficients of the polynomial $\mathcal{L}^{\mathcal{S}}_{ne}(\mathcal{T},l)$ can be compared with the coefficients given by general theory of Ehrhart polynomials applied for $\mathcal{P}^{(l)}$. E.g., the leading coefficient gives

$$-p\cdot(l,l)/2=\text{ Euclidian area of }\ \mathcal{P}^{(l)}.$$

Knowing that in $\mathcal{P}^{(l)}$ all the $\mathcal{P}_v$'s contribute, and it depends on s parameters, and the intersection of their boundary is messy, the simplicity and conceptual form of (5.14) is rather remarkable.

5.4 Reduction theorem for $Z(\mathbf{t})$

The number of terms in the denominator $\prod_i(1-\mathbf{t}^{a_i})$ of the series equals the number of variables of the corresponding partition function (associated with vectors a_i), and it is also the rank of the lattice where the corresponding polytope sit. In the case of the series $Z(\mathbf{t})$ associated with plumbing graph, this is the number of *end vertices* of Γ. On the other hand, the number of variables of $Z(\mathbf{t})$ is the number $|\mathcal{V}|$ of vertices of Γ. Furthermore, in the Ehrhart theoretical part, the associated (non–convex) polytope will be a union of $|\mathcal{V}|$ simplicial polytopes. Hence, the number of facets and the complexity of the polytope increases considerably with the number of vertices as well.

Nevertheless, the Theorem 5.1 eliminates a part of this abundance of parameters: it says that from the periodic constant point of view, the number of variables of the series, and also the number of simplicial polytopes in the union, can be reduced to the number of *nodes* of the graph. Hence, in fact, the complexity level can be measured by the number of nodes.

This approach is purely combinatorial, using the specialty of Γ. However, in [41, Thm 5.3] one can find another approach which uses the Reduction Theorem [41, Thm 3.7] for lattice cohomology, taking into the picture the hidden geometry which measures the rationality of the graph.

5.4.1 Reduction to the node variables

Let $\mathcal{N}$ denote the set of nodes as above. Let $\mathcal{S}_\mathcal{N}$ be the positive cone $\mathbb{R}_{\geq 0}\langle E_n^*\rangle_{n\in\mathcal{N}}$ generated by the dual base elements indexed by $\mathcal{N}$, and $V_\mathcal{N} := \mathbb{R}\langle E_n^*\rangle_{n\in\mathcal{N}}$ be its supporting linear subspace in $L\otimes\mathbb{R}$. Clearly $\mathcal{S}_\mathcal{N}\subset\mathcal{S}$. Furthermore, consider $L_\mathcal{N} := \mathbb{Z}\langle E_n\rangle_{n\in\mathcal{N}}$ generated by the node base elements, and the projection $\pi_\mathcal{N} : L\otimes\mathbb{R}\to L_\mathcal{N}\otimes\mathbb{R}$ on the node coordinates.

Lemma 5.1 *The restriction of $\pi_\mathcal{N}$ to $V_\mathcal{N}$, namely $\pi_\mathcal{N} : V_\mathcal{N}\to L_\mathcal{N}\otimes\mathbb{R}$, is an isomorphism.*

Proof Follows from the negative definiteness of the intersection form of the plumbing, which guarantees that any minor situated centrally on the diagonal is non–degenerate. □

Our goal is to prove that restricting the counting function to the subspace $V_\mathcal{N}$, the non–node variables of $Z(\mathbf{t})$ and $Q(l')$ became non–visible, hence they can be eliminated. This fact will provide a remarkable simplification in the periodic constant computation. But, *before* any elimination–substitution, we have first to decompose our series $Z(\mathbf{t})$ into $\sum_{h\in H} Z_h(\mathbf{t})[h]$ if we wish to preserve the information about all the H invariants, cf. the comment at the end of 4.2.2.2.

Theorem 5.1 *(a) The restriction of $\mathcal{L}_h^e(\mathbf{A},\mathcal{T},l')$ to $V_\mathcal{N}$ depends only on those coordinates which are indexed by the nodes (that is, it depends only on $\pi_\mathcal{N}(l')$ whenever $l'\in V_\mathcal{N}$).*

(b) The same is true for the counting function Q_h associated with $Z_h(\mathbf{t})$ as well. In other words, if we consider the restriction

$$Z_h(\mathbf{t}_{\mathcal{N}}) := Z_h(\mathbf{t})|_{t_v=1 \text{ for all } v\notin \mathcal{N}}$$

then for any $l'\in L_{\mathcal{N}}$, the counting functions $\sum_{l''\ngeq l'} p_{l''}[l'']$ of $Z_h(\mathbf{t})$ and $Z_h(\mathbf{t}_{\mathcal{N}})$ are the same.

(c) Consider the chamber decomposition of $\mathcal{S}_{\mathcal{N}}$ by intersections of type $C_{\mathcal{N}} := C\cap\mathcal{S}_{\mathcal{N}}$, where C denotes a chamber (of Z) such that $int(C\cap\mathcal{S})\neq\emptyset$, and the intersection of C with the relative interior of $\mathcal{S}_{\mathcal{N}}$ is also non–empty. Then

$$\mathrm{pc}^{C}(Z_h(\mathbf{t})) = \mathrm{pc}^{C_{\mathcal{N}}}(Z_h(\mathbf{t}_{\mathcal{N}})). \tag{5.15}$$

The theorem applies as follows. Assume that we are interested in the computation of $\mathrm{pc}^C_h(Z(\mathbf{t}))$ for some chamber C (e.g. when $C\subset\mathcal{S}$, cf. Corollary 5.1). Assume that C intersects the relative interior of $\mathcal{S}_{\mathcal{N}}$. Then, the restriction to $C\cap\mathcal{S}_{\mathcal{N}}$ of the quasipolynomial associated with C has two properties: it still preserves sufficient information to determine $\mathrm{pc}^C_h(Z(\mathbf{t}))$ (via the periodic constant of the restriction, see (5.15)), but it also has the advantage that for these dilation parameters l' the union $\cup_{v\in\mathcal{V}}\mathcal{P}_v^{(l'),\triangleleft}$ equals the union of significantly less polytopes, namely $\cup_{n\in\mathcal{N}}\mathcal{P}_v^{(l'),\triangleleft}$.

For example, when we have only one node, one has to handle one simplex instead of $|\mathcal{V}|$ many.

Proof (a) We show that for any $l'\in V_{\mathcal{N}}$ one has the inclusions

$$\mathcal{P}_v^{(l'),\triangleleft} \subset \bigcup_{n\in\mathcal{N}} \mathcal{P}_n^{(l'),\triangleleft} \text{ for any } v\notin\mathcal{N}. \tag{5.16}$$

We consider two cases. First we assume that v is on a leg (chain) connecting an end $e(v)\in\mathcal{E}$ with a node $n(v)$ (where $e(v)=v$ is also possible). Then, clearly, (5.16) follows from

$$\mathcal{P}_v^{(l'),\triangleleft} \subset \mathcal{P}_{n(v)}^{(l'),\triangleleft} \quad \text{for any } l'\in\mathcal{S}_{\mathcal{N}}. \tag{5.17}$$

Let $E^*_{uv} = (E^*_u)_v = -(E^*_u, E^*_v)$ be the v–cordinate of E^*_u. Note that $E^*_{uv}=E^*_{vu}$. Using the definition of the polytopes, (5.17) is equivalent with the implication (cf. 4.3.1)

$$\Big(\sum_{e\in\mathcal{E}} x_e E^*_{ve} < l'_v\Big) \Longrightarrow \Big(\sum_{e\in\mathcal{E}} x_e E^*_{n(v)e} < l'_{n(v)}\Big) \quad \text{for any } l'\in\mathcal{S}_{\mathcal{N}} \text{ and } x_e\geq 0. \tag{5.18}$$

Let $\mathcal{W}$ be the set of vertices on this leg (including $e(v)$ but not $n(v)$). Then, one verifies that there exist positive rational numbers α and $\{\alpha_w\}_{w\in\mathcal{W}}$, such that

$$E^*_v = \alpha\, E^*_{n(v)} + \sum_{w\in\mathcal{W}} \alpha_w E_w. \tag{5.19}$$

The numbers α and $\{\alpha_w\}_{w\in\mathcal{W}}$ can be determined from the linear system obtained by intersecting the identity (5.19) by $\{E_w\}_w$ and $E_{n(v)}$. Intersecting (5.19) by E^*_e ($e\in\mathcal{E}$), we get that $E^*_{ve}=\alpha E^*_{n(v)e}$ for any $e\neq e(v)$, and $E^*_{v,e(v)}=\alpha E^*_{n(v),e(v)}+\alpha_{e(v)}$. Hence

$$\sum_{e\in\mathcal{E}} x_e E^*_{ve} = \alpha \sum_{e\in\mathcal{E}} x_e E^*_{n(v)e} + x_{e(v)}\alpha_{e(v)}. \tag{5.20}$$

On the other hand, intersecting (5.19) with E^*_n, for $n \in \mathcal{N}$, we get $E^*_{vn} = \alpha E^*_{n(v)n}$. Since l' is a linear combination of E^*_n's, we get that

$$-l'_v = (l', E^*_v) = \alpha(l', E^*_{n(v)}) = -\alpha l'_{n(v)}. \tag{5.21}$$

Since $x_{e(v)}\alpha_{e(v)} \geq 0$, (5.20) and (5.21) imply (5.18). This ends the proof of this case.

Next, we assume that v is on a chain connecting two nodes $n(v)$ and $m(v)$. Let $\mathcal{W}$ be the set of vertices on this bamboo (not including $n(v)$ and $m(v)$). Then we will show that

$$\mathcal{P}_v^{(l'),\triangleleft} \subset \mathcal{P}_{n(v)}^{(l'),\triangleleft} \cup \mathcal{P}_{m(v)}^{(l'),\triangleleft} \text{ for any } l' \in \mathcal{S}_{\mathcal{N}}. \tag{5.22}$$

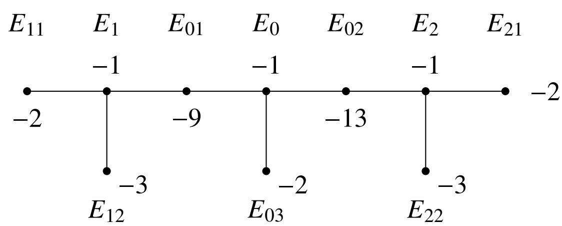
This follows as above from the existence of positive rational numbers α, β and $\{\alpha_w\}_{w\in\mathcal{W}}$ with

$$E^*_v = \alpha\, E^*_{n(v)} + \beta\, E^*_{m(v)} + \sum_{w\in\mathcal{W}} \alpha_w E_w. \tag{5.23}$$

(b) follows from (a) and from the fact that all b_k entries in the numerator of $Z(\mathbf{t})$ belong to $\mathcal{S}_{\mathcal{N}}$.

(c) If $\mathrm{pc}^C Z_h(\mathbf{t})$ is computed as $\widetilde{Q}_h(0)$ for some quasipolynomial $\widetilde{Q}_h$ defined on $\widetilde{L} \subset L$, then part (b) guarantees that $\mathrm{pc}^{C_{\mathcal{N}}} Z_h(\mathbf{t}_{\mathcal{N}})$ can be computed as $(\widetilde{Q}_h|_{\widetilde{L}\cap\mathcal{S}_{\mathcal{N}}})(0)$, which equals $\widetilde{Q}_h(0)$. □

Example 5.5 Consider the following graph Γ:



By Theorem 5.1 we are interested only in those polytopes $\mathcal{P}_v \subset \mathbb{R}^5$ which are associated with the nodes E_1, E_2 and E_0. Let $l \in \mathcal{S}_{\mathcal{N}}$, i.e. $l = \lambda_1 E^*_1 + \lambda_2 E^*_2 + \lambda_0 E^*_0$. Then one can verify that $\mathcal{S}_{\mathcal{N}}$ is divided by the plane $\lambda_1 = (13/9)\lambda_2$. Hence, in general $\mathcal{S}_{\mathcal{N}}$ can also be divided into several chambers. (On the other hand, for graphs with at most two nodes this does not happen.) □

5.6 Ehrhart theoretical interpretation of the Seiberg–Witten invariant

Let Γ be a connected negative definite plumbing tree. Let $\mathcal{N}$ and $\mathcal{E}$ be the set of nodes and end–vertices as above. We assume that $\mathcal{N} \neq \emptyset$. If δ_n denotes the valency of a node n, then $|\mathcal{E}| = 2 + \sum_{n\in\mathcal{N}}(\delta_n - 2)$.

We consider the matrix J with entries $J_{nm} := -(E^*_n, E^*_m)$ for $n, m \in \mathcal{N}$. By 2.1.2 it is a principal minor of $-I^{-1}$ (with rows and columns corresponding to the nodes).

Another incarnation of the matrix J already appeared in subsection 6.3.5, as the negative of the inverse of the orbifold intersection matrix. Indeed, let for any $n \in \mathcal{N}$ take that component of $\Gamma \setminus \cup_{m\in\mathcal{N}\setminus n}\{m\}$ which contains n. It is a star–shaped graph, let e_n be its orbifold Euler number. Furthermore, for any two nodes n and m which are connected by a chain, let α_{nm} be the determinant of that chain (not including the nodes). Then define the orbifold intersection matrix (of size $|\mathcal{N}|$) as $I^{orb}_{nn} = e_n$, $I^{orb}_{nm} = 1/\alpha_{nm}$ if the two nodes $n \neq m$ are connected by a chain, and $I^{orb}_{nm} = 0$ otherwise; cf. [14, 4.1.4] or 6.3.1. One can show (see [14, 4.1.4]) that I^{orb} is invertible, negative definite, and $\det(-I^{orb})$ is the product of $\det(-I)$ with the determinants of all (maximal) chains and legs of Γ. This fact and 2.2 imply that $J = (-I^{orb})^{-1}$.

5.6.1 The Ehrhart polynomial

In the sequel we assume that $L = L'$, that is $H = 0$.

By 5.1, $\mathcal{P}^{(l)}$ sits in $\mathbb{R}^{|\mathcal{E}|}$. Moreover, by Theorem 5.4.1, we can take l of the form $l = \sum_{n\in\mathcal{N}} \lambda_n E^*_n$, from the subcone of the Lipman cone generated by $\{E^*_n\}_{n\in\mathcal{N}}$.

Then 5.4.1 guarantees that the associated polytope is $\mathcal{P}^{(l)} = \bigcup_{n\in\mathcal{N}} \mathcal{P}^{(l_n)}_n$, $\mathcal{P}^{(l_n)}_n$ depending only on the component $l_n = -(l, E^*_n)$. Note that the coefficients $\{\lambda_n\}_n$ and the entries $\{l_n\}_n$ are connected exactly by the transformation law $(l_n)_n = J\,(\lambda_n)_n$.

Take any chamber C such that $int(C \cap \mathcal{S}) \neq \emptyset$, as in 5.1. Let $\widehat{\mathcal{L}}^C(\mathcal{P}, \mathcal{T}, (\lambda_n)_n)$ be the Ehrhart quasipolynomial $\mathcal{L}^C(\mathcal{P}, \mathcal{T}, (l_n)_n)$, associated with the denominator of Z, after changing the variables to $(\lambda_n)_n$ via $(l_n)_n = J\,(\lambda_n)_n$. It is convenient to normalize the coefficient of $\prod_n \lambda_n^{m_n}$ by a factor $\prod_n m_n!$, hence we write

$$\widehat{\mathcal{L}}^C(\mathcal{P}, \mathcal{T}, (\lambda_n)_n) = \sum_{\substack{\sum_n m_n \leq |\mathcal{E}| \\ m_n\geq 0;\ n\in\mathcal{N}}} \widehat{\mathfrak{a}}^C_{(m_n)_n} \prod_n \frac{\lambda_n^{m_n}}{m_n!},$$

for certain periodic functions $\widehat{\mathfrak{a}}^C_{(m_n)_n}$ in variables $(\lambda_n)_n$. By 3.10, 4.1 and 5.1

$$\chi\Big(\sum_{n\in\mathcal{N}} \lambda_n E^*_n\Big) + \mathrm{pc}^{\mathcal{S}}(Z) = \Delta((\lambda_n)_n), \tag{5.24}$$

where

$$\Delta((\lambda_n)_n) = \sum_{\substack{0\leq k_n\leq \delta_n-2 \\ \forall n\in\mathcal{N}}} (-1)^{\sum_n k_n} \prod_n \binom{\delta_n - 2}{k_n} \widehat{\mathcal{L}}^C(\mathcal{P}, \mathcal{T}, (\lambda_n - k_n)_n) =$$

$$\sum_{\substack{\sum_n m_n \leq |\mathcal{E}| \\ m_n\geq 0;\ n\in\mathcal{N}}} \Bigg(\sum_{\substack{0\leq p_n\leq m_n \\ n\in\mathcal{N}}} (-1)^{\sum_n p_n} \cdot \prod_n \binom{m_n}{p_n} \Bigg(\sum_{k_n=0}^{\delta_n-2} (-1)^{k_n} \binom{\delta_n - 2}{k_n} k_n^{p_n} \Bigg) \Bigg) \cdot \widehat{\mathfrak{a}}^C_{(m_n)_n} \prod_n \frac{\lambda_n^{m_n-p_n}}{m_n!}.$$

On the other hand, since $\chi(l) = -(K + l, l)/2$, the left hand side of (5.24) is the quadratic function

$$\sum_{n,m\in\mathcal{N}} (J_{nm}/2)\lambda_n\lambda_m + \sum_{n\in\mathcal{N}} (-(K, E^*_n)/2)\lambda_n + \mathrm{pc}^{\mathcal{S}}(Z).$$

Now we identify these coefficients with those of $\Delta((\lambda_n)_n)$ above. The additional ingredient is the combinatorial formula (6.31), which also shows that for the non–zero summands one necessarily has $p_n \geq \delta_n - 2$ for any n. One gets the following result.

Theorem 5.2

$$\begin{aligned} \widehat{\mathfrak{a}}^{C}_{(\delta_n,(\delta_m-2)_{m\neq n})} &= J_{nn} \\ \widehat{\mathfrak{a}}^{C}_{(\delta_n-1,\delta_m-1,(\delta_q-2)_{q\neq n,m})} &= J_{nm} \ \textit{for } n \neq m \\ \widehat{\mathfrak{a}}^{C}_{(\delta_n-1,(\delta_m-2)_{m\neq n})} &= -\tfrac{1}{2}(K,E_n^*) + \tfrac{1}{2}\textstyle\sum_{m\in\mathcal{N}}(\delta_m-2)J_{nm} \end{aligned}$$

$$\widehat{\mathfrak{a}}^{C}_{(\delta_n-2)_n} = \mathrm{pc}^{\mathcal{S}}(Z) - \sum_{n\in\mathcal{N}} \frac{(\delta_n-2)(K,E_n^*)}{4} + \sum_{n\in\mathcal{N}} \frac{(\delta_n-2)(3\delta_n-7)J_{nn}}{24} + \sum_{\substack{n,m\in\mathcal{N}\\ m\neq n}} \frac{(\delta_n-2)(\delta_m-2)J_{nm}}{8}.$$

Recall that $\mathrm{pc}^{\mathcal{S}}(Z) = -(K^2+|\mathcal{V}|)/8 - \lambda(M)$, where $\lambda(M)$ is the Casson invariant of M. Hence $\widehat{\mathfrak{a}}^{C}_{(\delta_n-2)_n}$ equals the normalized Casson invariant modulo some 'easy terms'.

We emphasize that these formulae also show that the above coefficients are constants (as periodic functions in $(\lambda_n)_n$) and independent of the chosen chamber C in the Lipman cone.

Chapter 6
Seiberg–Witten and Ehrhart theoretical computations and examples

Applying the general theory developed in the previous chapter, we make detailed computations for graphs with less than two nodes.

In the one–node case (star–shaped graphs) we provide a detailed presentation of all the involved (Seiberg–Witten and Ehrhart) invariants, and we establish closed formulae in terms of the Seifert invariants. Here we make connection with already known topological results regarding the Seiberg–Witten invariants of Seifert 3–manifolds, and also with analytic invariants of weighted homogeneous singularities.

In the two node case again we make complete presentations in terms of the analogs of the Seifert invariants of the chains and star–shaped subgraphs, including closed formulae for $\mathfrak{sw}(M)$. But, this case has a very interesting additional surprise in store.

It turns out that the corresponding combinatorial series $Z(\mathbf{t})$ associated with Γ, reduced to the two variables of the nodes, is the *Hilbert (characteristic) series of an affine monoid of rank two (and some of its modules).* In particular, the Seiberg–Witten invariant appears as the periodic constant of Hilbert series associated with affine monoids (and certain modules indexed by H), and, in some sense, measures the non–normality of these monoids.

At the end of the chapter, we provide some examples in which we demonstrate the calculation of the periodic constant (or equivalently, the Seiberg–Witten invariant) from the topological Poincaré series $Z(\mathbf{t})$.

6.1 The one–node case, star–shaped graphs

6.1.1 Seifert invariants and other notations

Assume that the graph is star–shaped with d legs. Each leg is a chain with normalized Seifert invariant (α_i, ω_i), where $0 < \omega_i < \alpha_i$, $\gcd(\alpha_i, \omega_i) = 1$. We also use ω_i' satisfying $\omega_i\omega_i' \equiv 1 \pmod{\alpha_i}$, $0 < \omega_i' < \alpha_i$.

If we consider the Hirzebruch/negative continued fraction expansion, cf. (5.4)

$$\alpha_i/\omega_i=[b_{i1},\dots,b_{i\nu_i}]=b_{i1}-1/(b_{i2}-1/(\cdots-1/b_{i\nu_i})\cdots)\quad (b_{ij}\geq 2),$$

then the i^{th} leg has ν_i vertices, say $v_{i1},\dots,v_{i\nu_i}$, with decorations $-b_{i1},\dots,-b_{i\nu_i}$, where v_{i1} is connected by the central vertex. The corresponding base elements in L are $\{E_{ij}\}_{j=1}^{\nu_i}$. Let b be the decoration of the central vertex; this vertex also defines $E_0\in L$. The plumbed 3–manifold M associated with such a star–shaped graph has a Seifert structure. We will assume that M is a rational homology sphere, or, equivalently, the central vertex has genus zero.

The classes in $H=L'/L$ of the dual base elements are denoted by $g_{ij}=[E^*_{ij}]$ and $g_0=[E^*_0]$. For simplicity we also write $E_i:=E_{i\nu_i}$ and $g_i:=g_{i\nu_i}$. A possible presentation of H is

$$H=\mathrm{ab}\langle\, g_0,g_1,\dots,g_d\mid -b\cdot g_0=\sum_{i=1}^{d}\omega_i\cdot g_i;\; g_0=\alpha_i\cdot g_i\ (1\leq i\leq d)\rangle, \tag{6.1}$$

cf. [86] (or use (6.3)). The orbifold Euler number of M is defined as $e=b+\sum_i\omega_i/\alpha_i$. The negative definiteness of the intersection form implies $e<0$. We write $\alpha:=\mathrm{lcm}(\alpha_1,\dots,\alpha_d)$, $\mathfrak{d}:=|H|$ and $\mathfrak{o}$ for the order of g_0 in H. One has (see e.g. [86])

$$\mathfrak{d}=\alpha_1\cdots\alpha_d|e|,\qquad \mathfrak{o}=\alpha|e|. \tag{6.2}$$

Each leg has similar invariants as the graph of a lens space, cf. Example 5.3, and we can introduce similar notation. For example, the determinant of the i^{th} leg is α_i. We write $n^i_{j_1j_2}$ for the determinant of the subchain of the i^{th} leg connecting the vertices v_{ij_1} and v_{ij_2} (including these vertices too). Then, using the correspondence between intersection pairing of the dual base elements and the determinants of the subgraphs, cf. (2.2) or [69, 11.1], one has

$$\begin{array}{llll}(a) & (E^*_0,E^*_{ij}-n^i_{j+1,\nu_i}E^*_{i\nu_i})=0 & (b) & g_{ij}=n^i_{j+1,\nu_i}g_{i\nu_i}\quad (1\leq i\leq d,\ 1\leq j\leq \nu_i)\\ (c) & (E^*_i,E^*_0)=\frac{1}{\alpha_ie} & (d) & (E^*_0,E^*_0)=\frac{1}{e}\,.\end{array} \tag{6.3}$$

Part (b) also explains why we do not need to insert the generators g_{ij} $(j<\nu_i)$ in (6.1).

For any $l'\in L'$ we set $\widetilde{c}(l'):=-(E^*_0,l')$, the E_0-coefficient of l'. Furthermore, if $l'=c_0E^*_0+\sum_{i,j}c_{ij}E^*_{ij}\in L'$, then we define its *reduced transform* by

$$l'_{red}:=c_0E^*_0+\sum_{i,j}c_{ij}\cdot n^i_{j+1,\nu_i}E^*_i\,.$$

By (6.3) we get $[l']=[l'_{red}]$ in H, $\widetilde{c}(l')=\widetilde{c}(l'_{red})$, and if $l'_{red}=\sum_{i=0}^d c_iE^*_i$, then $\widetilde{c}(l'_{red})$ is

$$\widetilde{c}:=\frac{1}{|e|}\cdot\Big(c_0+\sum_{i=1}^{d}\frac{c_i}{\alpha_i}\Big). \tag{6.4}$$

If $h\in H$, and $l'_h\in L'$ is any of its lifting (that is, $[l'_h]=h$), then $l'_{h,red}$ is also a lifting of the same h with $\widetilde{c}(l'_h)=\widetilde{c}(l'_{h,red})$. In general, $\widetilde{c}=\widetilde{c}(l'_h)$ depends on the lifting, nevertheless replacing l'_h by $l'_h\pm E_0$ we modify $\widetilde{c}$ by ±1, hence we can always achieve $\widetilde{c}\in[0,1)$, where it is determined uniquely by h. For example, since $r_h\in\square$, its E_0–coefficient $\widetilde{c}(r_h)$ is in $[0,1)$.

Finally, we consider

$$\gamma := \frac{1}{|e|} \cdot \Big(d - 2 - \sum_{i=1}^{d} \frac{1}{\alpha_i} \Big). \tag{6.5}$$

It has several 'names'. Since the canonical class is given by $K = -\sum_v E_v + \sum_v (\delta_v - 2) E_v^*$, by (6.3) we get that the E_0–coefficient of $-K$ is $(K, E_0^*) = \gamma + 1$. The number $-\gamma$ is sometimes called the 'log discrepancy' of E_0, γ the 'exponent' of the weighted homogeneous germ $(X, 0)$, and $\mathfrak{o}\gamma$ is the Goto–Watanabe a–invariant of the universal abelian cover of $(X, 0)$, see [35, (3.1.4)] and [20, (3.6.13)]; while in [86] $e\gamma$ appears as an orbifold Euler characteristic.

6.1.2 Interpretation of Z(t)

By Theorem 5.15, for the periodic constant computation, we can reduce ourself to the variable of the single node, it will be denoted by t.

First we analyze the equivariant rational function associated with the denominator of Z^e

$$Z_H(t) = \prod_{i=1}^{d} \big(1 - t^{-(E_i^*, E_0^*)}[g_i]\big)^{-1} = \sum_{x_1,\dots,x_d \geq 0} t^{\sum_i x_i/(\alpha_i |e|)} \Big[\sum_i x_i g_i \Big] \in \mathbb{Z}[[t^{1/\mathfrak{o}}]][H].$$

The right hand side of the above expression can be transformed as follows (cf. [78, §3]). If we fix a lift $\sum_{i=0}^{d} c_i E_i^*$ of h as above, then using the presentation (6.1) one gets that $\sum_{i=1}^{d} x_i g_i$ equals h if and only if there exist $\ell, \ell_1, \dots, \ell_d \in \mathbb{Z}$ such that

$$\begin{aligned} &(a) \quad\;\; -c_0 = \ell_1 + \cdots + \ell_d - \ell b \\ &(b)\; x_i - c_i = -\omega_i \ell - \alpha_i \ell_i \qquad (i = 1, \dots, d). \end{aligned}$$

Since $x_i \geq 0$, from (b) we get $\tilde{\ell}_i := \lfloor \frac{c_i - \omega_i \ell}{\alpha_i} \rfloor - \ell_i \geq 0$. Moreover, if we set for $\mathbf{c} = (c_0, c_1, \dots, c_d)$

$$N_{\mathbf{c}}(\ell) := 1 + c_0 - \ell b + \sum_{i=1}^{d} \Big\lfloor \frac{c_i - \omega_i \ell}{\alpha_i} \Big\rfloor, \tag{6.6}$$

then the number of realizations of $h = \sum_i c_i g_i$ in the form $\sum_i x_i g_i$ is given by the number of integers $(\tilde{\ell}_1, \dots, \tilde{\ell}_d)$ satisfying $\tilde{\ell}_i \geq 0$ and $\sum_i \tilde{\ell}_i = N_{\mathbf{c}}(\ell) - 1$. This is $\binom{N_{\mathbf{c}}(\ell)+d-2}{d-1}$. Moreover, the non–negative integer $\sum_i x_i/(\alpha_i |e|)$ equals $\ell + \widetilde{c}$. Therefore,

$$Z_h^{/H}(t) = \sum_{\ell \geq -\widetilde{c}} \binom{N_{\mathbf{c}}(\ell) + d - 2}{d - 1} t^{\ell + \widetilde{c}}. \tag{6.7}$$

This expression is independent of the choice of $\mathbf{c} = \{c_i\}_{i=0}^{d}$. Similarly, for any function ϕ, the expression $\sum_{\ell \geq -\widetilde{c}} \phi(N_{\mathbf{c}}(\ell)) t^{\ell + \widetilde{c}}$ is independent of the choice of $\mathbf{c}$, it depends only on $h = \sum_i c_i g_i$.

Furthermore, one checks that $N_{\mathbf{c}}(\ell) \leq 1 + (\ell + \widetilde{c})|e|$, hence if $\ell + \widetilde{c} < 0$ then $N_{\mathbf{c}}(\ell) \leq 0$, therefore $\binom{N_{\mathbf{c}}(\ell)+d-2}{d-1} = 0$ as well. Hence, in (6.7) the inequality $\ell + \widetilde{c} \geq 0$ below the sum, in fact, is not restrictive.

Next, we consider the numerator $(1-[g_0]t^{1/|e|})^{d-2}$ of $Z^e(t)$. A similar computation as above done for $Z^e(t)$ (see [86] and [78, §3]), or by multiplying (6.7) by the numerator and using $\sum_{k=0}^{d-2}(-1)^k\binom{d-2}{k}\binom{N-k+d-2}{d-1}=\binom{N}{1}$, gives

$$Z_h(t)=\sum_{\ell\geq -\widetilde{c}} \max\{0,N_{\mathbf{c}}(\ell)\}\, t^{\ell+\widetilde{c}}. \tag{6.8}$$

In order to compute the periodic constant of $Z_h(t)$ we decompose $Z_h(t)$ into its 'polynomial and negative degree parts', cf. 4.2.1. Namely, we write $Z_h(t)=Z_h^+(t)+Z_h^-(t)$, where

$$\begin{aligned} Z_h^+(t) &= \textstyle\sum_{\ell\geq -\widetilde{c}} \max\left\{0,-N_{\mathbf{c}}(\ell)\right\} t^{\ell+\widetilde{c}} \quad \text{(finite sum with positive exponents)} \\ Z_h^-(t) &= \textstyle\sum_{\ell\geq -\widetilde{c}} N_{\mathbf{c}}(\ell)\, t^{\ell+\widetilde{c}}. \end{aligned} \tag{6.9}$$

In Z_h^- it is convenient to fix a choice with $\widetilde{c}\in[0,1)$, hence the summation is over $\ell\geq 0$. Then a computation shows that it is a rational function of negative degree

$$Z_h^-(t)=\left(\frac{1-e\widetilde{c}}{1-t}-\frac{e\cdot t}{(1-t)^2}-\sum_{i=1}^{d}\sum_{r_i=0}^{\alpha_i-1}\left\{\frac{c_i-\omega_i r_i}{\alpha_i}\right\}t^{r_i}\cdot\frac{1}{1-t^{\alpha_i}}\right)\cdot t^{\widetilde{c}}. \tag{6.10}$$

(This expression can be compared with the Laurent expansion of Z_h at $t=1$ which was already considered in the literature. Dolgachev, Pinkham, Neumann and Wagreich [29, 100, 86, 108] determine the first two terms (the pole part), while [78, 69] the third terms as well. Nevertheless the above $Z_h^+ + Z_h^-$ decomposition does not coincide with the 'pole+regular part' decomposition of the Laurent expansion terms, and focuses on different aspects.)

Since the degree of Z_h^- is negative (or by a direct computation) $\mathrm{pc}(Z_h^-)=0$, cf. 4.2.1.

On the other hand, since $e<0$, in $Z_h^+(t)$ the sum is finite. (The degree of Z_0^+ is $\leq\gamma$, see e.g. [82]. Since $N_{\mathbf{c}(r_{h,red})}(\ell)\geq N_0(\ell)$, the degree of Z_h^+ is $\leq \gamma+\widetilde{c}(r_h)$ too). By 4.2.1,

$$\mathrm{pc}(Z_h)=Z_h^+(1)=\sum_{\ell\geq-\widetilde{c}}\max\left\{0,-N_{\mathbf{c}}(\ell)\right\} \tag{6.11}$$

for *any* lifting $\mathbf{c}$ of $h=\sum_i c_i g_i$. In this sum the bound $\ell\geq-\widetilde{c}$ is really restrictive.

We consider the non–equivariant version, the projection of $Z^e\in\mathbb{Z}[[t^{1/\mathfrak{o}}]][H]$ into $\mathbb{Z}[[t^{1/\mathfrak{o}}]]$ too

$$Z_{ne}(t)=\sum_h Z_h(t)=\frac{(1-t^{1/|e|})^{d-2}}{\prod_{i=1}^{d}(1-t^{1/(|e|\alpha_i)})}\in\mathbb{Z}[[t^{1/\mathfrak{o}}]].$$

We can get its $Z_{ne}^+ + Z_{ne}^-$ decomposition either by summation of Z_h^+ and Z_h^-, or as follows. Write

$$Z_{ne}(t)=\frac{1}{(1-t^{1/|e|})^2}\prod_{i=1}^{d}\frac{1-t^{1/|e|}}{1-t^{1/(|e|\alpha_i)}}=\frac{1}{(1-t^{1/|e|})^2}\sum_{\substack{0\leq x_i<\alpha_i\\ 0\leq i\leq d}} t^{\frac{1}{|e|}\cdot S(x)}, \tag{6.12}$$

where $S(x):=\sum_i\frac{x_i}{\alpha_i}$. Then its decomposition into $Z_{ne}^+(t)+Z_{ne}^-(t)$ is

$$Z^-_{ne}(t) = \sum_{\substack{0\le x_i<\alpha_i\\ 0\le i\le d}} t^{\frac{1}{|e|}\cdot\{S(x)\}}\cdot\Big(\frac{1}{(1-t^{1/|e|})^2}-\frac{\lfloor S(x)\rfloor}{1-t^{1/|e|}}\Big) \tag{6.13}$$

$$Z^+_{ne}(t) = \sum_{\substack{0\le x_i<\alpha_i\\ 0\le i\le d}} t^{\frac{1}{|e|}\cdot\{S(x)\}}\cdot\frac{t^{\frac{1}{|e|}\cdot\lfloor S(x)\rfloor}-\lfloor S(x)\rfloor t^{\frac{1}{|e|}}+\lfloor S(x)\rfloor-1}{(1-t^{1/|e|})^2}. \tag{6.14}$$

After dividing in $Z^+_{ne}(t)$ (or by L'Hospital rule), we get

$$\mathrm{pc}(Z_{ne}) = Z^+_{ne}(1) = \frac{1}{2}\cdot\sum_{\substack{0\le x_i<\alpha_i\\ 0\le i\le d}}\lfloor S(x)\rfloor\cdot\lfloor S(x)-1\rfloor. \tag{6.15}$$

6.1.3 Analytic interpretations

Rational homology sphere negative definite Seifert 3–manifolds can be realized analytically as links of weighted homogeneous singularities, or by equisingular deformations of weighted homogeneous singularities provided by splice–quotient equations [86, 88].

Consider the smooth germ at the origin of $\mathbb{C}^d$ with coordinate ring $\mathbb{C}\{z\}=\mathbb{C}\{z_1,\dots,z_d\}$, where z_i corresponds to the i^{th} end. Then H acts on it by the diagonal action $h*z_i=\theta(g_i)(h)z_i$. Similarly, we can introduce a multidegree $deg(z_i)=E^*_i\in L'$, hence the Poincaré series of $\mathbb{C}\{z\}$ associated with this multidegree is $\prod_i(1-\mathbf{t}^{E^*_i})^{-1}$. Moreover, considering the action of H on it, $\widetilde{Z}(\mathbf{t})=\prod_i(1-[g_i]\mathbf{t}^{E^*_i})^{-1}$ is the equivariant Poincaré series of $\mathbb{C}^d$, the invariant part $\widetilde{Z}_0(\mathbf{t})$ being the Poincaré series of the corresponding quotient space $\mathbb{C}^d/H$.

In $\mathbb{C}^d$ one can consider the *splice equations* as follows. Consider a matrix $\{\lambda_{ij}\}_{ij}$ of full rank and of size $d\times(d-2)$. Then the equations $\sum_{i=1}^d\lambda_{ij}z_i^{\alpha_i}=0$, for $j\in\{1,\dots,d-2\}$, determine in $\mathbb{C}^d$ an isolated complete intersection singularity $(Y,0)$ on which the group H acts as well. Then $(X,0)=(Y,0)/H$ is a normal surface singularity with the topological type of the Seifert manifold we started with. The equivariant Poincaré series of $(Y,0)$ is $Z(\mathbf{t})$ ([86]). For $(X,0)$, [15] proves the identity $P(\mathbf{t})=Z(\mathbf{t})$ mentioned in Subsection 3.1, hence $Z(\mathbf{t})$ is also the Poincaré series of the equivariant divisorial filtration associated with all the vertices.

Theorem 5.1 reduces the filtration to the $\mathbb{Z}$–filtration: the divisorial filtration associated with the central vertex. In the weighted homogeneous case this filtration is also induced by the weighted homogeneous equations. Then, $Z_H(t)$ is the Poincaré series of $\mathbb{C}^d/H$, $Z(t)$ is the equivariant Poincaré series of Y, hence $Z_0(t)$ is the Poincaré series of X, cf. [29, 86, 100].

By 3.1, $\{\mathrm{pc}(Z_h)\}_{h\in H}$ are the equivariant geometric genera of the universal abelian cover Y of X, hence $\mathrm{pc}(Z_0)$ and $\mathrm{pc}(Z_{ne})$ are the geometric genera of X and Y respectively, cf. [67].

6.1.4 Seiberg–Witten theoretical interpretations

Fix $h \in H$. Then, for any lifting $\sum_i c_i g_i$ of h, Corollary 5.1 and Equation 6.11 give

$$\mathrm{pc}(Z_h) = \sum_{\ell \geq -\tilde{c}} \max\left\{0, -N_{\mathbf{c}}(\ell)\right\} = -\mathfrak{sw}_{-h*\sigma_{can}}(M) - \frac{(K+2r_h)^2 + |\mathcal{V}|}{8}. \tag{6.16}$$

Recall that $\sum_h \mathfrak{sw}_{-h*\sigma_{can}}(M)$ is the *Casson–Walker invariant* $\lambda(M)$. Hence, for the non–equivariant version we get

$$\mathrm{pc}(Z_{ne}) = \frac{1}{2} \cdot \sum_{\substack{0 \leq x_i < \alpha_i \\ 0 \leq i \leq d}} \lfloor S(x) \rfloor \cdot \lfloor S(x) - 1 \rfloor = -\lambda(M) - \mathfrak{d} \cdot \frac{K^2 + |\mathcal{V}|}{8} + \sum_h \chi(r_h). \tag{6.17}$$

For explicit formulae of $\lambda(M)$ and $K^2 + |\mathcal{V}|$ in terms of Seifert invariants see e.g. [78, 2.6]).

Remark 6.1 (6.16) can be compared with a known formulae of the Seiberg–Witten invariants involving the representative s_h. This will also lead us to an expression for $\chi(r_h) - \chi(r_s)$ in terms of $N_{\mathbf{c}}(\ell)$. Let $\mathbf{c}(s_h) = (c_0, \ldots, c_d)$ be the coefficients of $s_{h,red}$, cf. 6.1.1. The set of all reduced coefficients $\mathbf{c}(s_h)$, when h runs in H, is characterized in [69, 11.5] by the inequalities

$$\begin{cases} c_0 \geq 0, \quad \alpha_i > c_i \geq 0 \ \ (1 \leq i \leq d) \\ N_{\mathbf{c}}(\ell) \leq 0 \ \text{ for any } \ell < 0. \end{cases} \tag{6.18}$$

Moreover, for this special lifting $\mathbf{c}(s_h)$ of h, in [69, §11] is proved

$$\sum_{\ell \geq 0} \max\left\{0, -N_{\mathbf{c}(s_h)}(\ell)\right\} = -\mathfrak{sw}_{-h*\sigma_{can}}(M) - \frac{(K+2s_h)^2 + |\mathcal{V}|}{8}. \tag{6.19}$$

Using the discussion from the end of 6.1.1, this can be rewritten for *any* lifting $\mathbf{c}$ of h as

$$\sum_{\ell \geq -\tilde{c} + \lfloor \tilde{c}(s_h) \rfloor} \max\left\{0, -N_{\mathbf{c}}(\ell)\right\} = -\mathfrak{sw}_{-h*\sigma_{can}}(M) - \frac{(K+2s_h)^2 + |\mathcal{V}|}{8}. \tag{6.20}$$

This compared with (6.16) gives for any lifting $\mathbf{c}$ of h

$$\sum_{-\tilde{c} + \lfloor \tilde{c}(s_h) \rfloor > \ell \geq -\tilde{c}} \max\left\{0, -N_{\mathbf{c}}(\ell)\right\} = \chi(r_h) - \chi(s_h). \tag{6.21}$$

Example 6.2 The sum in (6.21), in general, can be non–zero. This happens, for example, in the case of the link of a rational singularity whose universal abelian cover is not rational. Here is a concrete example, cf. [74, 4.5.4]: take the Seifert manifold with $b = -2$ and three legs, all of them with Seifert invariants $(\alpha_i, \omega_i) = (3, 1)$. For $h = \sum_{i=1}^3 g_i$ one has $s_h = \sum_{i=1}^3 E_i^*$, the E_0-coefficient of s_h is 1, $r_h = s_h - E_0$, and $\chi(s_h) = 0$, $\chi(r_h) = 1$. □

6.2.1 Ehrhart theoretical interpretations

We fix $h \in H$ as above and *we assume that* $\widetilde{c} \in [0,1)$. Note that $Z_h(t)$ has the form $t^{\widetilde{c}} \sum_{\ell \geq 0} p_\ell t^\ell$; here the exponents $\{\widetilde{c} + \ell\}_{\ell \geq 0}$ are the possible E_0–coordinates of the elements $(r_h + L) \cap \mathcal{S}'$.

Let us compute the counting function for Z_h. If $S(t) = \sum_r p_r t^r$ is a series, we write $Q(S)(r') = \sum_{r<r'} p_r$, for $r' \in \mathbb{Q}_{\geq 0}$.

Lemma 6.1 *For any* $n \in \mathbb{N}_{\geq 0}$ *one has the following facts.*

(a) $Q(Z_h)(n) = Q(Z_h)(n + \widetilde{c})$.

(b) $Q(Z_h^+)(n)$ *is a step function (hence piecewise polynomial) with*

$$Q(Z_h^+)(n) = \mathrm{pc}(Z_h) \quad \textit{for } n > \deg(Z_h^+).$$

(c) $Q(Z_h^-)(n)$ *is a quasipolynomial:*

$$Q(Z_h^-)(n) = (1 - e\widetilde{c})n - e \cdot \frac{n(n-1)}{2} - \sum_{i=1}^{d} \sum_{r_i=0}^{\alpha_i - 1} \left\{ \frac{c_i - \omega_i r_i}{\alpha_i} \right\} \left\lceil \frac{n - r_i}{\alpha_i} \right\rceil \tag{6.22}$$

$$= -\frac{en^2}{2} + \frac{en}{2}(\gamma + 1 - 2\widetilde{c}) - \sum_{i=1}^{d} \sum_{r_i=0}^{\alpha_i - 1} \left\{ \frac{c_i - \omega_i r_i}{\alpha_i} \right\} \left(\left\{ \frac{r_i - n}{\alpha_i} \right\} - \frac{r_i}{\alpha_i} \right).$$

In particular, if $n = m\alpha$ for $m \in \mathbb{Z}$, and $n > \deg(Z_h^+)$, then the double sum is zero, hence

$$Q(Z_h)(n) = -\frac{en^2}{2} + \frac{en}{2}(\gamma + 1 - 2\widetilde{c}) + \mathrm{pc}(Z_h). \tag{6.23}$$

This is compatible with the expression provided by Theorem 3.1 and Theorem 5.1. Indeed, let us fix any chamber C such that $int(C \cap \mathcal{S}') \neq \emptyset$, and C contains the ray $\mathcal{R} = \mathbb{R}_{\geq 0} \cdot E_0^*$. Since the numerator of $f(\mathbf{t})$ is $(1 - \mathbf{t}^{E_0^*})^{d-2}$, all the b_k–vectors belong to $\mathcal{R}$. In particular, $\cap_k (b_k + C)$ intersects $\mathcal{R}$ along a semi–line $\mathcal{R}^{\geq c} = \mathbb{R}_{\geq const} \cdot E_0^*$ of $\mathcal{R}$. Since $Q_h(l')$ is quasipolynomial on $\cap_k (b_k + C)$, cf. (4.17), and a restriction of it is determined by (3.10) whose right hand side is a quasipolynomial too, we obtain that the identity (3.10) is valid on $\mathcal{R}^{\geq c}$ as well.

Recall that for any $h \in H$ and $l' \in L'$ we have a unique $q_{l',h} \in \square$ with $l' + q_{l',h} \in r_h + L$. Hence we get

$$Q_h(l') = -\mathfrak{sw}_{-h*\sigma_{can}}(M) - \frac{(K + 2l' + 2q_{l',h})^2 + |\mathcal{V}|}{8} \quad (l' \in \mathcal{R}^{\geq c}). \tag{6.24}$$

The term $q_{l',h}$ is responsible for the non–polynomial behavior. Nevertheless, if we assume that $l' = m\mathfrak{o}E_0^* \in \mathcal{R}^{\geq c} \cap L$, $m \in \mathbb{Z}$, then $q_{l',h} = r_h$, hence by (6.16)

$$Q_h(l') = -\frac{(l', l' + K + 2r_h)}{2} + \mathrm{pc}(Z_h). \tag{6.25}$$

By Theorem 5.1 $Q_h(l')$ from (6.25) depends only on the E_0-coefficient of $l' = m\mathfrak{o}E_0^*$, which is exactly $m\alpha$. One sees that in fact (6.25) agrees with (6.23) if we set $n = m\alpha$.

The non–equivariant version can be obtained by summation of (6.23). For this we need $\sum_h \widetilde{c}(r_h)$. Consider the group homomorphism $\varphi : H \to \mathbb{Q}/\mathbb{Z}$ given by $h \mapsto [\widetilde{c}(r_h)]$, or $[E^*_v] \mapsto [-(E^*_0, E^*_v)]$. Its image is generated by the classes of $1/(\alpha_i|e|)$, hence its order is $\mathfrak{o}$. Hence, $\widetilde{c}(r_h)$ vanishes exactly $\mathfrak{d}/\mathfrak{o}$ times (whenever $h \in \ker(\varphi)$). In all other cases $\widetilde{c}(r_h) \neq 0$, and $\widetilde{c}(r_h) + \widetilde{c}(r_{-h}) = 1$. In particular, $2\sum_h \widetilde{c}(r_h) = \mathfrak{d} - \mathfrak{d}/\mathfrak{o}$. Therefore, the summation of (6.23) provides

$$Q(Z_{ne})(n) = -\frac{\mathfrak{d}en^2}{2} + \frac{\mathfrak{d}en}{2}\Big(\gamma + \frac{1}{\mathfrak{o}}\Big) + \mathrm{pc}(Z_{ne}) \quad (\text{for } n \in \alpha\mathbb{Z}). \tag{6.26}$$

Next, we will identify the coefficients of (6.23) and (6.26) with the first three coefficient of the Ehrhart quasipolynomial $\mathcal{L}^C_h(\mathcal{T})$ via the identity (4.17).

For simplicity we will assume that $\mathfrak{o} = 1$, in particular all the b_k–vectors belong to L.

If $l' \in \mathcal{R}$, then by Theorem 5.1 the counting function $\mathcal{L}^C_h(\mathcal{T}, l')$ of the polytope $\mathcal{P}^{(l')}$ depends only on the E_0–coefficient of l'; let us denote this coefficient by l'_0.

Hence, this $\mathcal{L}^C_h(\mathcal{T}, l'_0)$ is the Ehrhart quasipolynomial of the d–dimensional simplicial polytope, being its h–class counting function. Via (6.3) the definition (4.8) of this polytope becomes

$$\mathcal{P}_0 = \Big\{(x_1, \ldots, x_d) \in (\mathbb{R}_{\geq 0})^d \;:\; \sum_i \frac{x_i}{|e|\alpha_i} < l'_0\Big\}. \tag{6.27}$$

Let

$$\mathcal{L}^C_h(\mathcal{T}, l'_0) = \sum_{j=0}^{d} \mathfrak{a}_{h,j}(l'_0) \cdot \frac{(l'_0)^j}{j!} \tag{6.28}$$

be the coefficients of the Ehrhart quasipolynomial: each $\mathfrak{a}_{h,j}(l'_0)$ is a periodic function in l'_0 and is normalized by $1/j!$. Since the numerator of f is $(1 - t^{1/|e|})^{d-2}$, by (4.17) we obtain for $l' \in \mathcal{R}$

$$Q_h(l') = \sum_{j=0}^{d} \mathfrak{a}_{h,j}(l'_0) \cdot \frac{1}{j!} \sum_{k=0}^{d-2} (-1)^k \binom{d-2}{k} \Big(l'_0 - \frac{k}{|e|}\Big)^j. \tag{6.29}$$

This equals the expression (6.24) above. The non–polynomial behavior of these two expressions indicate that $\mathfrak{a}_j(l'_0)$ is indeed non–constant periodic, and can be determined explicitly.

Since we are interested primarily in the Seiberg–Witten invariant, namely in $\mathrm{pc}(Z_h)$, we perform this explicit identification only via the expressions (6.23) and (6.25). Hence, similarly as in these cases, we take $l' = m\mathfrak{o}E^*_0 \in \mathcal{R}^{\geq c} \cap L$, and we identify (6.23) with (6.29) evaluated for l', whose E_0–coefficient is $l'_0 = m\alpha = n$. In this case $\mathfrak{a}_{h,j}(n)$ is a *constant*, denoted by $\mathfrak{a}_{h,j}$, and

$$-\frac{en^2}{2} + \frac{ne}{2}(\gamma + 1 - 2\widetilde{c}) + \mathrm{pc}(Z_h) = \sum_{j=0}^{d} \mathfrak{a}_{h,j} \cdot \frac{1}{j!} \sum_{k=0}^{d-2} (-1)^k \binom{d-2}{k} \Big(n - \frac{k}{|e|}\Big)^j. \tag{6.30}$$

Here the following combinatorial expression is helpful (see e.g. [101, p. 7-8])

$$\frac{(-1)^d}{(d-2)!}\cdot\sum_{k=0}^{d-2}(-1)^k\binom{d-2}{k}k^j=\begin{cases}0 & \text{if } j<d-2,\\ 1 & \text{if } j=d-2,\\ (d-2)(d-1)/2 & \text{if } j=d-1,\\ (d-2)(d-1)d(3d-5)/24 & \text{if } j=d.\end{cases} \tag{6.31}$$

We obtain

$$\begin{aligned}\frac{\mathfrak{a}_{h,d}}{|e|^d}&=\frac{1}{|e|}\\ \frac{\mathfrak{a}_{h,d-1}}{|e|^{d-1}}&=\frac{d-2}{2|e|}-\frac{1}{2}(\gamma+1-2\widetilde{c})\\ \frac{\mathfrak{a}_{h,d-2}}{|e|^{d-2}}&=\mathrm{pc}(Z_h)+\frac{(d-2)(3d-7)}{24|e|}-\frac{d-2}{4}(\gamma+1-2\widetilde{c}).\end{aligned} \tag{6.32}$$

In particular, the $\mathfrak{a}_{h,d-2}$ can be identified (up to 'easy' extra terms) with $\mathrm{pc}(Z_h)$ (with analytical interpretation $\dim(H^1(\widetilde{Y},\mathcal{O}_{\widetilde{Y}})_{\theta(h)})$ and Seiberg–Witten theoretical interpretation (6.16)). The first coefficients can also be identified with the equivariant volume of $\mathcal{P}_0$, (a fact already known in the non–equivariant cases). Usually (in the non–equivariant case, and when we count the points of all the facets) the second coefficient can be related with the volumes of the facets. Here we eliminate from this count some of the facets, and we are in the equivariant situation as well.

In the non–equivariant case, if $\sum_{j=0}^{d}\mathfrak{a}_j\frac{n^j}{j!}$ is the classical Ehrhart polynomial of $\mathcal{P}_0$, then

$$\begin{aligned}\frac{\mathfrak{a}_d}{|e|^d}&=\prod_i\alpha_i\\ \frac{\mathfrak{a}_{d-1}}{|e|^{d-1}}&=\prod_i\alpha_i\cdot\Big(-\frac{1}{\alpha}+\sum_i\frac{1}{\alpha_i}\Big)/2\\ \frac{\mathfrak{a}_{d-2}}{|e|^{d-2}}&=\prod_i\alpha_i\Big(\frac{\mathrm{pc}(Z_{ne})}{\prod_i\alpha_i}-\frac{(d-2)(3d-5)}{24}+\frac{d-2}{4}(-\frac{1}{\alpha}+\sum_i\frac{1}{\alpha_i})\Big).\end{aligned} \tag{6.33}$$

In this non–equivariant case the identities (6.33) are valid even without the assumption $\mathfrak{o}=1$ by Theorem 5.2.

The formulae in (6.32) and (6.33) can be further simplified if we replace $\mathcal{P}_0$ by $|e|\mathcal{P}_0$, or if we substitute in the Ehrhart polynomial the new variable $\lambda:=|e|l'_0$ instead of l'_0; cf. Section 5.6.

6.3 The two–node case

6.3.1 Notations and the group *H*

We consider the graph Γ from Figure 6.1.

The nodes E_0 and $\widetilde{E}_0$ have decorations b_0 and $\widetilde{b}_0$ respectively. Similarly as in the one–node case, we encode the decorations of maximal chains by continued fraction expansions. In fact, it is convenient to consider the two maximal star–shaped graphs Γ_0 and $\widetilde{\Gamma}_0$, and the corresponding normalized Seifert invariants of their legs. Hence, let the normalized Seifert invariants of the legs with ends E_i ($1\leq i\leq d$) be (α_i,ω_i), while of the legs with ends $\widetilde{E}_j$ ($1\leq j\leq \widetilde{d}$) be $(\widetilde{\alpha}_j,\widetilde{\omega}_j)$.

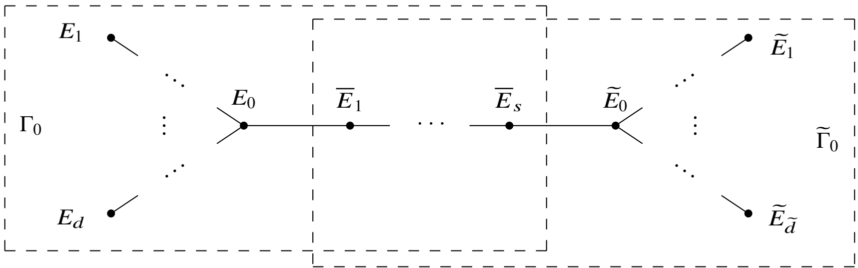


Fig. 6.1 Graph with two nodes

The chain connecting the nodes, viewed in Γ_0 has normalized Seifert invariants (α_0, ω_0), while viewed as a leg in $\widetilde{\Gamma}_0$, it has Seifert invariants $(\alpha_0, \widetilde{\omega}_0)$. One has $\omega_0\widetilde{\omega}_0 = \alpha_0\tau + 1$. Clearly, α_0 is the determinant of the chain, and

$$\omega_0 := \det(\overset{\overline{E}_2}{\bullet\!-} \;\cdots\; \overset{\overline{E}_s}{-\!\bullet}), \quad \widetilde{\omega}_0 := \det(\overset{\overline{E}_1}{\bullet\!-} \;\cdots\; \overset{\overline{E}_{s-1}}{-\!\bullet}), \quad \tau := \det(\overset{\overline{E}_2}{\bullet\!-} \;\cdots\; \overset{\overline{E}_{s-1}}{-\!\bullet}).$$

We denote the orbifold Euler numbers of the star–shaped subgraphs Γ_0 and $\widetilde{\Gamma}_0$ by

$$e = b_0 + \frac{\omega_0}{\alpha_0} + \sum_i \frac{\omega_i}{\alpha_i} \quad \text{and} \quad \widetilde{e} = \widetilde{b}_0 + \frac{\widetilde{\omega}_0}{\alpha_0} + \sum_j \frac{\widetilde{\omega}_j}{\widetilde{\alpha}_j}.$$

Consider the *orbifold intersection matrix* $I^{orb} = \begin{pmatrix} e & 1/\alpha_0 \\ 1/\alpha_0 & \widetilde{e} \end{pmatrix}$, cf. [14, 4.1.4]. Then, the negative definiteness of I (or Γ) implies that I^{orb} is negative definite too, hence

$$\varepsilon := \det I^{orb} = e\widetilde{e} - \frac{1}{\alpha_0^2} > 0.$$

Then the determinant of the graph is $\det(\Gamma) = \det(-I) = \varepsilon \cdot \alpha_0 \prod_i \alpha_i \prod_j \widetilde{\alpha}_j$, cf. [14].

Using (2.2) we have the following intersection number of the dual base elements:

$$\begin{array}{c} (E_0^*)^2 = \frac{\widetilde{e}}{\varepsilon}; \;\; (\widetilde{E}_0^*)^2 = \frac{e}{\varepsilon}; \;\; (E_0^*, \widetilde{E}_0^*) = -\frac{1}{\alpha_0\varepsilon}; \;\; (E_0^*, E_i^*) = \frac{\widetilde{e}}{\alpha_i\varepsilon}; \\ (E_0^*, \widetilde{E}_j^*) = -\frac{1}{\alpha_0\widetilde{\alpha}_j\varepsilon}; \;\; (\widetilde{E}_0^*, E_i^*) = -\frac{1}{\alpha_0\alpha_i\varepsilon}; \;\; (\widetilde{E}_0^*, \widetilde{E}_j^*) = \frac{e}{\widetilde{\alpha}_j\varepsilon}. \end{array} \tag{6.34}$$

Similarly as in 5.3 or 6.1.1, we can write $n^i_{k_1,k_2}$, $\widetilde{n}^j_{k_1,k_2}$ resp. $\overline{n}_{k_1,k_2}$ for the determinant of the sub–chains of the 'left' i^{th} leg, 'right' j^{th} leg and connecting chain connecting the vertices v_{k_1} and v_{k_2}. Let ν_i and $\tilde{\nu}_j$ be the number of vertices in the legs, cf. 6.1.1. Then (with the standard notations, where $E_{i\ell}$ and $\widetilde{E}_{j\ell}$ are the vertices of the legs) one has the following slightly technical Lemma, but whose proof is standard based on the arithmetical properties of continued fractions:

Lemma 6.2 *(a)* $E^*_{i\ell} = n^i_{\ell+1,\nu_i} E^*_i + \sum_{\ell<r\leq\nu_i} \frac{n^i_{1,\ell-1}n^i_{r+1,\nu_i} - n^i_{1,r-1}n^i_{\ell+1,\nu_i}}{\alpha_i} E_{ir}$ *for any* $1 \leq \ell < \nu_i$.
(There is a similar formula for $\widetilde{E}^*_{j\ell}$*.)*

(b) $\overline{E}^*_k = \overline{n}_{1,k-1}\overline{E}^*_1 - \overline{n}_{2,k-1}E^*_0 + \sum_{1\leq r<k} \frac{\overline{n}_{1,r-1}\overline{n}_{k+1,s}-\overline{n}_{1,k-1}\overline{n}_{r+1,s}}{\alpha_0}\overline{E}_r$ *, for* $1 < k \leq s$*.*
(This is true even for $k = s+1$ *with the identification* $\overline{E}^*_{k+1} = \widetilde{E}^*_0$*.)*

Next, we give a presentation of $H = L'/L$. Set $g_i := [E^*_i]$ $(1 \leq i \leq d)$, $\widetilde{g}_j := [\widetilde{E}^*_j]$ $(1 \leq j \leq \widetilde{d})$, $g_0 := [E^*_0]$ and $\widetilde{g}_0 := [\widetilde{E}^*_0]$. Moreover we need to choose an additional generator corresponding to a vertex sitting on the connecting chain: we choose $\overline{g} := [\overline{E}^*_1]$ (this motivates the choice in Lemma 6.2)(b) too). The above lemma implies

$$[E^*_{i\ell}] = n^i_{\ell+1,\nu_i}g_i, \quad [\widetilde{E}^*_{j\ell}] = \widetilde{n}^j_{\ell+1,\widetilde{\nu}_j}\widetilde{g}_j \quad \text{and} \quad [\overline{E}^*_k] = \overline{n}_{1,k-1}\overline{g} - \overline{n}_{2,k-1}g_0; \tag{6.35}$$

and similar arguments as in the star–shaped case provides the following presentation for H

$$\begin{aligned} H = \mathrm{ab}\langle\, g_0, \widetilde{g}_0, g_i, \widetilde{g}_j, \overline{g} \,|\, g_0 = \alpha_i\cdot g_i;\ \widetilde{g}_0 = \widetilde{\alpha}_j\cdot\widetilde{g}_j;\ \alpha_0\cdot\overline{g} = \omega_0\cdot g_0 + \widetilde{g}_0; \\ -\overline{g} - b_0\cdot g_0 = \textstyle\sum_i \omega_i\cdot g_i;\ -\widetilde{\omega}_0\cdot\overline{g} + \tau\cdot g_0 - \widetilde{b}_0\cdot\widetilde{g}_0 = \sum_j \widetilde{\omega}_j\cdot\widetilde{g}_j\rangle. \end{aligned} \tag{6.36}$$

Moreover, for any $l' \in L'$,

$$l' = c_0E^*_0 + \widetilde{c}_0\widetilde{E}^*_0 + \textstyle\sum_k \overline{c}_k\overline{E}^*_k + \sum_{i,\ell} c_{i\ell}E^*_{i\ell} + \sum_{j\ell}\widetilde{c}_{j\ell}\widetilde{E}^*_{j\ell},$$

if we define its *reduced transform* l'_{red} by

$$(c_0 - \sum_{k>1}\overline{n}_{2,k-1}\overline{c}_k)E^*_0 + \widetilde{c}_0\widetilde{E}^*_0 + (\overline{c}_1 + \sum_{k>1}\overline{n}_{1,k-1}\overline{c}_k)\overline{E}^*_1 + \sum_{i,\ell} c_{i\ell}n^i_{\ell+1,\nu_i}E^*_i + \sum_{j,\ell}\widetilde{c}_{j\ell}\widetilde{n}^j_{\ell+1,\widetilde{\nu}_j}\widetilde{E}^*_j,$$

then, by Lemma 6.2, $[l'] = [l'_{red}]$ in H. Moreover, if for any $l' \in L'$ we distinguish the E_0 and $\widetilde{E}_0$ coefficients, that is, we set $c(l') := -(E^*_0, l')$ and $\widetilde{c}(l') := -(\widetilde{E}^*_0, l')$, then $c(l') = c(l'_{red})$ and $\widetilde{c}(l') = \widetilde{c}(l'_{red})$ as well. Lemma 6.2(b) (applied for $k = s+1$) provide these coefficients for $\overline{E}_1$:

$$(\overline{E}^*_1, E^*_0) = \frac{1}{\varepsilon\alpha_0}\big(\omega_0\widetilde{e} - \frac{1}{\alpha_0}\big), \quad (\overline{E}^*_1, \widetilde{E}^*_0) = \frac{1}{\varepsilon\alpha_0}\big(e - \frac{\omega_0}{\alpha_0}\big). \tag{6.37}$$

We will use the coefficients $\mathbf{c} = (c_0, \widetilde{c}_0, \overline{c}, c_i, \widetilde{c}_j)$ to write an element $l'_{red} = c_0E^*_0 + \widetilde{c}_0\widetilde{E}^*_0 + \overline{c}\overline{E}^*_1 + \sum_i c_iE^*_i + \sum_j \widetilde{c}_j\widetilde{E}^*_j$. Then (6.34) and (6.37) imply that

$$\begin{pmatrix} c \\ \widetilde{c} \end{pmatrix} = \begin{pmatrix} c(l'_{red}) \\ \widetilde{c}(l'_{red}) \end{pmatrix} = (-I^{orb})^{-1}\cdot\begin{pmatrix} A \\ \widetilde{A} \end{pmatrix} = \frac{1}{\varepsilon}\begin{pmatrix} -\widetilde{e} & 1/\alpha_0 \\ 1/\alpha_0 & -e \end{pmatrix}\cdot\begin{pmatrix} A \\ \widetilde{A} \end{pmatrix}, \tag{6.38}$$

where

$$A := c_0 + \sum_i \frac{c_i}{\alpha_i} + \frac{\omega_0}{\alpha_0}\overline{c}, \qquad \widetilde{A} := \widetilde{c}_0 + \sum_j \frac{\widetilde{c}_j}{\widetilde{\alpha}_j} + \frac{1}{\alpha_0}\overline{c}.$$

Therefore, any $h \in H$ has a lift of type $l'_{h,red}$. Although the corresponding coefficients c and $\widetilde{c}$ depend on the lift, by adding $\pm E_0$ and $\pm\widetilde{E}_0$ to $l'_{h,red}$ we can achieve $c, \widetilde{c} \in [0,1)$, and these values are uniquely determined by h. For example, the reduced transform $(r_h)_{red}$ of r_h satisfies $c((r_h)_{red}) = c(r_h) \in [0,1)$ and $\widetilde{c}((r_h)_{red}) = \widetilde{c}(r_h) \in [0,1)$ since $r_h \in \square$.

As we will see, for different elements of $h \in H$, we have to shift the rank two lattices by vectors of type $(c, \widetilde{c})$, hence the vectors $(c, \widetilde{c})$ will play a crucial role later.

6.3.2 Interpretation of $Z(\mathbf{t})$

If we wish to compute the periodic constant of $Z^e(\mathbf{t})$, by Theorem 5.1 we can eliminate all the variables of $Z^e(\mathbf{t})$ except the variables of the nodes; these remaining two variables are denoted by $(t,\widetilde{t})$. Therefore the equivariant form of reciprocal of the denominator is

$$\begin{aligned} Z_H(t,\widetilde{t}) &= \prod_i \left(1-t^{-(E_i^*,E_0^*)}\widetilde{t}^{-(E_i^*,\widetilde{E}_0^*)}[g_i]\right)^{-1}\cdot\prod_j\left(1-t^{-(\widetilde{E}_j^*,E_0^*)}\widetilde{t}^{-(\widetilde{E}_j^*,\widetilde{E}_0^*)}[\widetilde{g}_j]\right)^{-1}\\ &= \sum_{x_i,\widetilde{x}_j\geq 0} t^{\frac{-\widetilde{e}}{\varepsilon}\sum_i\frac{x_i}{\alpha_i}+\frac{1}{\alpha_0\varepsilon}\sum_j\frac{\widetilde{x}_j}{\widetilde{\alpha}_j}}\ \widetilde{t}^{\frac{1}{\alpha_0\varepsilon}\sum_i\frac{x_i}{\alpha_i}+\frac{-e}{\varepsilon}\sum_j\frac{\widetilde{x}_j}{\widetilde{\alpha}_j}}\left[\textstyle\sum_i x_ig_i+\sum_j\widetilde{x}_j\widetilde{g}_j\right].\end{aligned}$$

We fix a lift $c_0E_0^*+\widetilde{c}_0\widetilde{E}_0^*+\overline{c}\,\overline{E}_1^*+\sum_i c_iE_i^*+\sum_j\widetilde{c}_j\widetilde{E}_j^*$ of h. Then the class of $\sum_i x_iE_i^*+\sum_j\widetilde{x}_j\widetilde{E}_j^*$ equals h if and only if its difference with the lift is a linear combination of the relation in 6.36. In other words, if there exist $\ell_0,\widetilde{\ell}_0,\overline{\ell},\ell_i,\widetilde{\ell}_j\in\mathbb{Z}$ such that

$$\begin{array}{lll} (a)\ -c_0=\sum_i\ell_i-b_0\ell_0+\tau\widetilde{\ell}_0+\omega_0\overline{\ell} & (c)\ x_i-c_i\ =-\omega_i\ell_0-\alpha_i\ell_i & (i=1,\dots,d)\\ (b)\ -\widetilde{c}_0=\sum_j\widetilde{\ell}_j-\widetilde{b}_0\widetilde{\ell}_0+\overline{\ell} & (d)\ \widetilde{x}_j-\widetilde{c}_j=-\widetilde{\omega}_j\widetilde{\ell}_0-\widetilde{\alpha}_j\widetilde{\ell}_j & (j=1,\dots,\widetilde{d})\\ (e)\ \ -\overline{c}=-\ell_0-\widetilde{\omega}_0\widetilde{\ell}_0-\alpha_0\overline{\ell}. & & \end{array}$$

From (e) we deduce that

$$\ell_0+\widetilde{\omega}_0\widetilde{\ell}_0\equiv\overline{c}\ (\mathrm{mod}\,\alpha_0). \tag{6.39}$$

Since $x_i,\widetilde{x}_j\geq 0$, (c) and (d) implies $\frac{c_i-\omega_i\ell_0}{\alpha_i}\geq\ell_i$, $\frac{\widetilde{c}_j-\widetilde{\omega}_j\widetilde{\ell}_0}{\widetilde{\alpha}_j}\geq\widetilde{\ell}_j$. Recall also that $\omega_0\widetilde{\omega}_0=\alpha_0\tau+1$. Therefore if we set $m_i:=\lfloor\frac{c_i-\omega_i\ell_0}{\alpha_i}\rfloor-\ell_i$ and $\widetilde{m}_j:=\lfloor\frac{\widetilde{c}_j-\widetilde{\omega}_j\widetilde{\ell}_0}{\widetilde{\alpha}_j}\rfloor-\widetilde{\ell}_j$ non–negative integers then the number of the realization of h in the form $\sum_i x_ig_i+\sum_j\widetilde{x}_j\widetilde{g}_j$ is determined by the number of non–negative integral $(d+\widetilde{d})$–tuples $(m_i,\widetilde{m}_j)$ satisfying

$$\begin{aligned} N_{\mathbf{c}}(\ell_0,\widetilde{\ell}_0) &:= c_0+\tfrac{\omega_0}{\alpha_0}\overline{c}-(b_0+\tfrac{\omega_0}{\alpha_0})\ell_0-\tfrac{1}{\alpha_0}\widetilde{\ell}_0+\textstyle\sum_i\lfloor\frac{c_i-\omega_i\ell_0}{\alpha_i}\rfloor\ =\sum_i m_i\,,\\ \widetilde{N}_{\mathbf{c}}(\ell_0,\widetilde{\ell}_0) &:= \widetilde{c}_0+\tfrac{1}{\alpha_0}\overline{c}-(\widetilde{b}_0+\tfrac{\widetilde{\omega}_0}{\alpha_0})\widetilde{\ell}_0-\tfrac{1}{\alpha_0}\ell_0+\textstyle\sum_j\lfloor\frac{\widetilde{c}_j-\widetilde{\omega}_j\widetilde{\ell}_0}{\widetilde{\alpha}_i}\rfloor=\sum_j\widetilde{m}_j\,.\end{aligned}$$

This number is $\binom{N_{\mathbf{c}}(\ell_0,\widetilde{\ell}_0)+d-1}{d-1}\binom{\widetilde{N}_{\mathbf{c}}(\ell_0,\widetilde{\ell}_0)+\widetilde{d}-1}{\widetilde{d}-1}$ if $N_{\mathbf{c}}$ and $\widetilde{N}_{\mathbf{c}}$ are non–negative, otherwise it is 0. Note that (6.39) guarantees that both $N_{\mathbf{c}}$ and $\widetilde{N}_{\mathbf{c}}$ are integers. Furthermore, (c) and (d) and (6.38) show that the exponent of t and $\widetilde{t}$ in the formula of $Z_h^{/H}(t,\widetilde{t})$ are equal to ℓ_0+c and $\widetilde{\ell}_0+\widetilde{c}$ respectively. Hence

$$Z_h^{/H}(t,\widetilde{t})=\sum\binom{N_{\mathbf{c}}(\ell,\widetilde{\ell})+d-1}{d-1}\binom{\widetilde{N}_{\mathbf{c}}(\ell,\widetilde{\ell})+\widetilde{d}-1}{\widetilde{d}-1}t^{\ell+c}\,\widetilde{t}^{\widetilde{\ell}+\widetilde{c}},$$

where the sum runs over $(\ell,\widetilde{\ell})\in\mathbb{Z}^2$ with $\ell+\widetilde{\omega}_0\widetilde{\ell}\equiv\overline{c}$ (mod α_0).

The numerator of $Z(t,\widetilde{t})$ is $\left(1-t^{-(E_0^*,E_0^*)}\widetilde{t}^{-(E_0^*,\widetilde{E}_0^*)}[g_0]\right)^{d-1}\cdot\left(1-t^{-(\widetilde{E}_0^*,E_0^*)}\widetilde{t}^{-(\widetilde{E}_0^*,\widetilde{E}_0^*)}[\widetilde{g}_0]\right)^{\widetilde{d}-1}$. Hence we get Z^e by multiplying this expression by $\sum_h Z_h^{/H}[h]$. Recall that $h=c_0g_0+\widetilde{c}_0\widetilde{g}_0+\overline{c}\,\overline{g}+\sum_i c_ig_i+\sum_j\widetilde{c}_j\widetilde{g}_j$ is paired with $\mathbf{c}$. Set $h':=h+kg_0+\widetilde{k}\widetilde{g}_0$ which corresponds to $\mathbf{c}'=\mathbf{c}+(k,\widetilde{k},0,0,0)$. Hence $Z_{h'}[h']$ is the next sum according to the decompositions

$h' = h + kg_0 + \widetilde{k}\widetilde{g}_0$:

$$\sum_{k=0}^{d-1}(-1)^k\binom{d-1}{k}\sum_{\widetilde{k}=0}^{\widetilde{d}-1}(-1)^{\widetilde{k}}\binom{\widetilde{d}-1}{\widetilde{k}}\cdot$$

$$\sum_h\left(\sum_{\ell+\widetilde{\omega}_0\widetilde{\ell}\equiv\widetilde{c}\ (\alpha_0)}\binom{N_{\mathbf{c}}(\ell,\widetilde{\ell})+d-1}{d-1}\binom{\widetilde{N}_{\mathbf{c}}(\ell,\widetilde{\ell})+\widetilde{d}-1}{\widetilde{d}-1}t^{\ell+c+\frac{-\widetilde{e}k+\widetilde{k}/\alpha_0}{\varepsilon}}\,\widetilde{t}^{\widetilde{\ell}+\widetilde{c}+\frac{-e\widetilde{k}+k/\alpha_0}{\varepsilon}}\right)[h']$$

$$=\sum_{k=0}^{d-1}(-1)^k\binom{d-1}{k}\sum_{\widetilde{k}=0}^{\widetilde{d}-1}(-1)^{\widetilde{k}}\binom{\widetilde{d}-1}{\widetilde{k}}\cdot$$

$$\sum_h\left(\sum_{\ell+\widetilde{\omega}_0\widetilde{\ell}\equiv\widetilde{c}\ (\alpha_0)}\binom{N_{\mathbf{c}'}(\ell,\widetilde{\ell})-k+d-1}{d-1}\binom{\widetilde{N}_{\mathbf{c}'}(\ell,\widetilde{\ell})-\widetilde{k}+\widetilde{d}-1}{\widetilde{d}-1}t^{\ell+c'}\,\widetilde{t}^{\widetilde{\ell}+\widetilde{c}'}\right)[h'].$$

Rearranging and using the combinatorial formula $\sum_{k=0}^{d-1}(-1)^k\binom{N-k+d-1}{d-1}\binom{d-1}{k}=1$ for $N\geq 0$ and $=0$ otherwise, we get the following.

Theorem 6.1 *For any $h\in H$ one has*

$$Z_h(t,\widetilde{t})=\sum_{(\ell,\widetilde{\ell})\in\mathcal{S}_{\mathbf{c}}}t^{\ell+c}\,\widetilde{t}^{\widetilde{\ell}+\widetilde{c}},\quad \textit{where} \tag{6.40}$$

$$\mathcal{S}_{\mathbf{c}}:=\left\{(\ell,\widetilde{\ell})\in\mathbb{Z}^2\ :\ N_{\mathbf{c}}(\ell,\widetilde{\ell})\geq 0,\ \widetilde{N}_{\mathbf{c}}(\ell,\widetilde{\ell})\geq 0\ \textit{and}\ \ell+\widetilde{\omega}_0\widetilde{\ell}\equiv\widetilde{c}\ (\textit{mod}\ \alpha_0)\right\}. \tag{6.41}$$

It is straightforward to verify that the right hand side of (6.40) does not depend on the choice of $\mathbf{c}$, it depends only on h. The identity (6.40) is remarkable: it realizes the bridge between the series Z^e and the equivariant Hilbert series of *affine monoids and their modules*.

6.3.3 The structure of $\mathcal{S}_{\mathbf{c}}$

Recall that for any $h\in H$ we consider a lift of h identified by a certain $\mathbf{c}$ which detemines the pair $(c,\widetilde{c})$ (cf. (6.38)), and the integers $N_{\mathbf{c}}(\mathfrak{l})$ and $\widetilde{N}_{\mathbf{c}}(\mathfrak{l})$, where $\mathfrak{l}=(\ell,\widetilde{\ell})\in\mathbb{Z}^2$. We define

$$\mathbb{Z}^2(\mathbf{c}):=\{(\ell,\widetilde{\ell})\in\mathbb{Z}^2\ :\ \ell+\widetilde{\omega}_0\widetilde{\ell}\equiv\widetilde{c}\ (\text{mod}\ \alpha_0)\}.$$

If $h=0$ then we always choose the zero lift with $\mathbf{c}=\mathbf{0}$.

If, in the definition of $N_{\mathbf{c}}(\mathfrak{l})$ and $\widetilde{N}_{\mathbf{c}}(\mathfrak{l})$, we replace each $[y]$ by y, we get the entries of

$$\begin{pmatrix}A-e\ell_0-\widetilde{\ell}/\alpha_0\\ \widetilde{A}-\ell_0/\alpha_0-\widetilde{e}\widetilde{\ell}_0\end{pmatrix}=-I^{orb}\begin{pmatrix}\ell+c\\ \widetilde{\ell}+\widetilde{c}\end{pmatrix}.$$

This motivates to define

$$\overline{\mathcal{S}}_{\mathbf{c}}:=\left\{\mathfrak{l}\in\mathbb{Z}^2(\mathbf{c})\ :\ -I^{orb}\begin{pmatrix}\ell+c\\ \widetilde{\ell}+\widetilde{c}\end{pmatrix}\geq 0\right\}. \tag{6.42}$$

Clearly $\mathcal{S}_{\mathbf{c}} \subset \overline{\mathcal{S}}_{\mathbf{c}}$. We also consider C^{orb}, the real cone $\{\mathfrak{l} \in \mathbb{R}^2 \ : \ -I^{orb} \cdot \mathfrak{l} \geq 0\}$. Then $\overline{\mathcal{S}}_{\mathbf{c}} = \big(C^{orb} - (c, \widetilde{c})\big) \cap \mathbb{Z}^2(\mathbf{c})$.

Lemma 6.3 *(1) $\mathcal{S}_{\mathbf{0}}$ and $\overline{\mathcal{S}}_{\mathbf{0}}$ are affine monoids. $\overline{\mathcal{S}}_{\mathbf{0}}$ is the normalization of $\mathcal{S}_{\mathbf{0}}$.*
(2) $\mathcal{S}_{\mathbf{c}}$ and $\overline{\mathcal{S}}_{\mathbf{c}}$ are finitely generated $\mathcal{S}_{\mathbf{0}}$-modules, $\mathcal{S}_{\mathbf{c}}$ is a submodule of $\overline{\mathcal{S}}_{\mathbf{c}}$.

Proof (1) is elementary. By Corollary [21, 2.12] $\overline{\mathcal{S}}_{\mathbf{c}}$ is finitely generated over $\overline{\mathcal{S}}_0$, but $\overline{\mathcal{S}}_0$ itself is finitely generated as an $\mathcal{S}_0$ module. □

Lemma 6.4 *There exists $\mathfrak{v}_1$ and $\mathfrak{v}_2$ elements of $\mathbb{Z}^2$ with the following properties:*
(a) $\mathfrak{v}_1$ and $\mathfrak{v}_2$ belong to $\mathcal{S}_{\mathbf{0}}$ and $\mathbb{R}_{\geq 0}\mathfrak{v}_1 + \mathbb{R}_{\geq 0}\mathfrak{v}_2 = C^{orb}$.
(b) For any $\mathfrak{l} \in \overline{\mathcal{S}}_{\mathbf{c}}$ one has:

$$(i)\ \ N_{\mathbf{c}}(\mathfrak{l} + \mathfrak{v}_1) = N_{\mathbf{c}}(\mathfrak{l}); \quad (\widetilde{i})\ \ \widetilde{N}_{\mathbf{c}}(\mathfrak{l} + \mathfrak{v}_2) = \widetilde{N}_{\mathbf{c}}(\mathfrak{l});$$
$$(ii)\ \ N_{\mathbf{c}}(\mathfrak{l} + \mathfrak{v}_2) \geq 0; \quad (\widetilde{ii})\ \ \widetilde{N}_{\mathbf{c}}(\mathfrak{l} + \mathfrak{v}_1) \geq 0.$$

Proof We choose $\mathfrak{v}_1$ and $\mathfrak{v}_2$ such that $\widetilde{N}_{\mathbf{0}}(\mathfrak{v}_1) \geq \widetilde{d} - 1$ and $N_{\mathbf{0}}(\mathfrak{v}_2) \geq d - 1$, and with
(A) $\mathfrak{v}_1 = (\ell_1, \widetilde{\ell}_1) \in \mathbb{Z}^2(\mathbf{c})$ such that $\{-\omega_i \ell_1/\alpha_i\} = 0$ for all i, and $N_{\mathbf{0}}(\mathfrak{v}_1) = 0$;
(B) $\mathfrak{v}_2 = (\ell_2, \widetilde{\ell}_2) \in \mathbb{Z}^2(\mathbf{c})$ such that $\{-\widetilde{\omega}_j \widetilde{\ell}_2/\widetilde{\alpha}_j\} = 0$ for all j, and $\widetilde{N}_{\mathbf{0}}(\mathfrak{v}_2) = 0$.
Then $\mathfrak{v}_1$ and $\mathfrak{v}_2$ satisfy (a), and (b)(i), and (b)($\widetilde{i}$). Furthermore, note that $N_{\mathbf{c}}(\mathfrak{l} + \mathfrak{v}_2) \geq N_{\mathbf{c}}(\mathfrak{l}) + N_{\mathbf{0}}(\mathfrak{v}_2)$ and for any $\mathfrak{l} \in \overline{\mathcal{S}}_{\mathbf{c}}$ one has $N_{\mathbf{c}}(\mathfrak{l}) \geq -(d-1)$, hence all the conditions will be satisfied. □

Remark 6.2 Usually, the ‘universal restrictions’ $\widetilde{N}_{\mathbf{0}}(\mathfrak{v}_1) \geq \widetilde{d} - 1$ and $N_{\mathbf{0}}(\mathfrak{v}_2) \geq d - 1$ in the proof of Lemma 6.4 provide rather ‘large’ vectors $\mathfrak{v}_1$ and $\mathfrak{v}_2$. Nevertheless, usually much smaller vectors also satisfy (a) and (b). Here is another choice. Besides (A) and (B) we impose the following:

(C) Let $\square = \square(\mathfrak{v}_1, \mathfrak{v}_2) = \{\mathfrak{l} = q_1\mathfrak{v}_1 + q_2\mathfrak{v}_2 \ : \ 0 \leq q_1, q_2 < 1\}$ be the semi–open cube in C^{orb}. Then we require $N_{\mathbf{0}}(\mathfrak{v}_2) \geq 0$ and $N_{\mathbf{c}}(\mathfrak{l}_\square + \mathfrak{v}_2) \geq 0$ for any $\mathfrak{l}_\square \in (\square - (c, \widetilde{c})) \cap \mathbb{Z}^2(\mathbf{c})$; and symmetrically: $\widetilde{N}_{\mathbf{0}}(\mathfrak{v}_1) \geq 0$ and $\widetilde{N}_{\mathbf{c}}(\mathfrak{l}_\square + \mathfrak{v}_1) \geq 0$ for any $\mathfrak{l}_\square \in (\square - (c, \widetilde{c})) \cap \mathbb{Z}^2(\mathbf{c})$.

The wished inequality for any $\mathfrak{l} \in \overline{\mathcal{S}}_{\mathbf{c}}$ then follows from $N_{\mathbf{c}}(\mathfrak{l}_\square + k_1\mathfrak{v}_1 + k_2\mathfrak{v}_2 + \mathfrak{v}_2) = N_{\mathbf{c}}(\mathfrak{l}_\square + k_2\mathfrak{v}_2 + \mathfrak{v}_2) \geq N_{\mathbf{c}}(\mathfrak{l}_\square + \mathfrak{v}_2) + k_2 N_{\mathbf{0}}(\mathfrak{v}_2)$ (and its symmetric version).

In the sequel the next two subsets of $\overline{\mathcal{S}}_{\mathbf{c}}$ will be crucial.

$$\mathcal{S}^-_{\mathbf{c},1} := \big\{\mathfrak{l} \in (\square - (c, \widetilde{c})) \cap \mathbb{Z}^2(\mathbf{c}) \ : \ N_{\mathbf{c}}(\mathfrak{l}) < 0\big\},$$
$$\mathcal{S}^-_{\mathbf{c},2} := \big\{\mathfrak{l} \in (\square - (c, \widetilde{c})) \cap \mathbb{Z}^2(\mathbf{c}) \ : \ \widetilde{N}_{\mathbf{c}}(\mathfrak{l}) < 0\big\}.$$

Again, both sets $\mathcal{S}^-_{\mathbf{c},1}$ and $\mathcal{S}^-_{\mathbf{c},2}$ are independent of the choice of $\mathbf{c}$, they depend only on h.

Proposition 6.1 *Let $\mathfrak{v}_1$ and $\mathfrak{v}_2$ be as in Lemma 6.4. Then*

$$(1)\ \overline{\mathcal{S}}_{\mathbf{c}} = \bigsqcup_{\mathfrak{l} \in (\square - (c, \widetilde{c})) \cap \mathbb{Z}^2(\mathbf{c})} \mathfrak{l} + \mathbb{Z}_{\geq 0}\mathfrak{v}_1 + \mathbb{Z}_{\geq 0}\mathfrak{v}_2$$

$$(2)\ \overline{\mathcal{S}}_{\mathbf{c}} \setminus \mathcal{S}_{\mathbf{c}} = \Big(\bigsqcup_{\mathfrak{l} \in \mathcal{S}^-_{\mathbf{c},1}} \mathfrak{l} + \mathbb{Z}_{\geq 0}\mathfrak{v}_1\Big) \cup \Big(\bigsqcup_{\mathfrak{l} \in \mathcal{S}^-_{\mathbf{c},2}} \mathfrak{l} + \mathbb{Z}_{\geq 0}\mathfrak{v}_2\Big),$$

$$\textit{but}\ \Big(\bigsqcup_{\mathfrak{l} \in \mathcal{S}^-_{\mathbf{c},1}} \mathfrak{l} + \mathbb{Z}_{\geq 0}\mathfrak{v}_1\Big) \cap \Big(\bigsqcup_{\mathfrak{l} \in \mathcal{S}^-_{\mathbf{c},2}} \mathfrak{l} + \mathbb{Z}_{\geq 0}\mathfrak{v}_2\Big) = \bigsqcup_{\mathfrak{l} \in \mathcal{S}^-_{\mathbf{c},1} \cap \mathcal{S}^-_{\mathbf{c},2}} \mathfrak{l}\,.$$

Proof The statements follow from the choice of $\mathfrak{v}_1$ and $\mathfrak{v}_2$ and properties (a) and (b) of Lemma 6.4. Compare also with the structure theorem [21, 4.36] of $\mathcal{S}_{\mathbf{0}}$ modules. □

6.3.4 The periodic constant and $\mathfrak{sw}$ in the equivariant case.

Set $\mathbf{t} = (t, \widetilde{t})$. Using (6.40) and Proposition 6.1 one can write $Z_h(\mathbf{t})/\mathbf{t}^{(c,\widetilde{c})}$ in the next form:

$$\sum_{\mathfrak{l}\in(\square-(c,\widetilde{c}))\cap\mathbb{Z}^2\cap(\equiv_{\mathbf{c}})} \frac{\mathbf{t}^{\mathfrak{l}}}{(1-\mathbf{t}^{\mathfrak{v}_1})(1-\mathbf{t}^{\mathfrak{v}_2})} - \sum_{\mathfrak{l}\in\mathcal{S}^-_{\mathbf{c},1}} \frac{\mathbf{t}^{\mathfrak{l}}}{1-\mathbf{t}^{\mathfrak{v}_1}} - \sum_{\mathfrak{l}\in\mathcal{S}^-_{\mathbf{c},2}} \frac{\mathbf{t}^{\mathfrak{l}}}{1-\mathbf{t}^{\mathfrak{v}_2}} + \sum_{\mathfrak{l}\in\mathcal{S}^-_{\mathbf{c},1}\cap\mathcal{S}^-_{\mathbf{c},2}} \mathbf{t}^{\mathfrak{l}}.$$

Next, we apply the decomposition established in subsection 4.10. Here it is important to *choose* $\mathbf{c}$ *in such a way that* $c\in[0,1)$ *and* $\widetilde{c}\in[0,1)$.

Note that $\mathfrak{v}_1\in\mathbb{R}_{>0}(1/\alpha_0,-e)$ and $\mathfrak{v}_2\in\mathbb{R}_{>0}(-\widetilde{e},1/\alpha_0)$, hence $\mathfrak{v}_2$ sits in the cone determined by $\mathfrak{v}_1$ and $(1,0)$. Then, as in 4.10, we set $\Xi_1 := \{(\ell,\widetilde{\ell}) \ : \ 0\leq\ell<\text{first coordinate of }\mathfrak{v}_1\}$ and $\Xi_2 := \{(\ell,\widetilde{\ell}) \ : \ 0\leq\widetilde{\ell}<\text{second coordinate of }\mathfrak{v}_2\}$, and for any $\mathfrak{l}\in\mathcal{S}^-_{\mathbf{c},i}$ the unique $n_{\mathfrak{l},i}$ such that $\mathfrak{l}-n_{\mathfrak{l},i}\mathfrak{v}_i\in\Xi_i$, $i=1,2$. Then subsection 4.10 provides the following decomposition

$$\begin{aligned} Z^+_h(\mathbf{t}) &= \mathbf{t}^{(c,\widetilde{c})}\Big(\textstyle\sum_{\mathfrak{l}\in\mathcal{S}^-_{\mathbf{c},1}}\sum_{j=1}^{n_{\mathfrak{l},1}}\mathbf{t}^{\mathfrak{l}-j\mathfrak{v}_1} + \sum_{\mathfrak{l}\in\mathcal{S}^-_{\mathbf{c},2}}\sum_{j=1}^{n_{\mathfrak{l},2}}\mathbf{t}^{\mathfrak{l}-j\mathfrak{v}_2} + \sum_{\mathfrak{l}\in\mathcal{S}^-_{\mathbf{c},1}\cap\mathcal{S}^-_{\mathbf{c},2}}\mathbf{t}^{\mathfrak{l}}\Big)\\ Z^-_h(\mathbf{t}) &= \mathbf{t}^{(c,\widetilde{c})}\Big(\textstyle\sum_{\mathfrak{l}\in(\square-(c,\widetilde{c}))\cap\mathbb{Z}^2\cap(\equiv_{\mathbf{c}})}\frac{\mathbf{t}^{\mathfrak{l}}}{(1-\mathbf{t}^{\mathfrak{v}_1})(1-\mathbf{t}^{\mathfrak{v}_2})} - \sum_{\mathfrak{l}\in\mathcal{S}^-_{\mathbf{c},1}}\frac{\mathbf{t}^{\mathfrak{l}-n_{\mathfrak{l},1}\mathfrak{v}_1}}{1-\mathbf{t}^{\mathfrak{v}_1}} - \sum_{\mathfrak{l}\in\mathcal{S}^-_{\mathbf{c},2}}\frac{\mathbf{t}^{\mathfrak{l}-n_{\mathfrak{l},2}\mathfrak{v}_2}}{1-\mathbf{t}^{\mathfrak{v}_2}}\Big)\ . \end{aligned}$$

Therefore, by 4.5 and Theorem 4.3 we get

$$\mathrm{pc}^{C^{orb}}_h(Z) = \mathrm{pc}^{C^{orb}}(Z_h(\mathbf{t})/\mathbf{t}^{(c,\widetilde{c})}) = Z^+_h(1,1) = \sum_{\mathfrak{l}\in\mathcal{S}^-_{\mathbf{c},1}} n_{\mathfrak{l},1} + \sum_{\mathfrak{l}\in\mathcal{S}^-_{\mathbf{c},2}} n_{\mathfrak{l},2} + |\mathcal{S}^-_{\mathbf{c},1}\cap\mathcal{S}^-_{\mathbf{c},2}|\ .$$

Corollary 6.1 *Choose* $\mathbf{c}$ *in such a way that* $c\in[0,1)$ *and* $\widetilde{c}\in[0,1)$. *Then one has the following combinatorial formula for the normalized Seiberg–Witten invariant of* M

$$\mathfrak{sw}^{norm}_h(M) = \sum_{\mathfrak{l}\in\mathcal{S}^-_{\mathbf{c},1}} n_{\mathfrak{l},1} + \sum_{\mathfrak{l}\in\mathcal{S}^-_{\mathbf{c},2}} n_{\mathfrak{l},2} + |\mathcal{S}^-_{\mathbf{c},1}\cap\mathcal{S}^-_{\mathbf{c},2}|.$$

Proof Use Corollary 5.1, the Theorem 5.1 and the above computation. □

6.3.5 The periodic constant and $\lambda(M)$ in the non–equivariant case

Though the non–equivariant Z_{ne} can be obtained by the sum $\sum_h Z_h$ treated in the previous subsection, here we provide a more direct procedure, which leads to a new formula. Write $J := (-I^{orb})^{-1}$ and $\mathbf{t}^{\binom{a}{b}}$ for $t^a\widetilde{t}^b$. Applying the reduction 5.1 for the definition (3.8) of Z, we get

$$Z_{ne}(\mathbf{t}) = \frac{(1-\mathbf{t}^{J\binom{1}{0}})^{d-1}(1-\mathbf{t}^{J\binom{0}{1}})^{\widetilde{d}-1}}{\prod_i(1-\mathbf{t}^{J\binom{1/\alpha_i}{0}})\prod_j(1-\mathbf{t}^{J\binom{0}{1/\widetilde{\alpha}_j}})}.$$

Set $S(x) := \sum_i x_i/\alpha_i$ and $\widetilde{S}(\widetilde{x}) := \sum_j \widetilde{x}_j/\widetilde{\alpha}_j$. Similarly as in 6.12, $Z_{ne}(\mathbf{t})$ can be written as

$$\sum_{\substack{0\le x_i<\alpha_i, 0\le i\le d\\ 0\le \widetilde{x}_j<\widetilde{\alpha}_j, 0\le j\le \widetilde{d}}} f(x,\widetilde{x}), \quad \text{where } f(x,\widetilde{x}) = \frac{\mathbf{t}^{J\binom{S(x)}{\widetilde{S}(\widetilde{x})}}}{(1-\mathbf{t}^{J\binom{1}{0}})(1-\mathbf{t}^{J\binom{0}{1}})}.$$

By the substitution $u_1 = \mathbf{t}^{J\binom{1}{0}}$ and $u_2 = \mathbf{t}^{J\binom{0}{1}}$, $f(x,\widetilde{x})$ transforms into $u_1^{S(x)}u_2^{\widetilde{S}(\widetilde{x})}/(1-u_1)(1-u_2)$. The division of this fraction (with remainder) is elementary, hence $f(x,\widetilde{x})$ equals

$$\mathbf{t}^{J\binom{S_{rat}}{\widetilde{S}_{rat}}}\left(\sum_{n=0}^{S_{int}-1}\sum_{k=0}^{\widetilde{S}_{int}-1}\mathbf{t}^{J\binom{n}{k}} - \sum_{k=0}^{S_{int}-1}\frac{\mathbf{t}^{J\binom{k}{0}}}{1-\mathbf{t}^{J\binom{0}{1}}} - \sum_{\widetilde{k}=0}^{\widetilde{S}_{int}-1}\frac{\mathbf{t}^{J\binom{0}{k}}}{1-\mathbf{t}^{J\binom{1}{0}}} + \frac{1}{(1-\mathbf{t}^{J\binom{1}{0}})(1-\mathbf{t}^{J\binom{0}{1}})}\right),$$

where $S_{int} := \lfloor S(x)\rfloor$, $\widetilde{S}_{int} := \lfloor \widetilde{S}(\widetilde{x})\rfloor$, $S_{rat} := \{S(x)\}$ and $\widetilde{S}_{rat} := \{\widetilde{S}(\widetilde{x})\}$.

Then, by 4.1 $\mathrm{pc}^{C^{orb}}(\mathbf{t}^{J\binom{S_{rat}}{\widetilde{S}_{rat}}}/(1-\mathbf{t}^{J\binom{1}{0}})(1-\mathbf{t}^{J\binom{0}{1}})) = 0$. Moreover, 4.10 gives a unique integer $s(k)\ge 0$ for $k\in\{0,\dots,S_{int}-1\}$ such that $\mathbf{t}^{J\binom{k+S_{rat}}{-s(k)+\widetilde{S}_{rat}}}/1-\mathbf{t}^{J\binom{0}{1}}$ has vanishing periodic constant with respect to C^{orb}. It turns out that $s(k) = \lfloor -\widetilde{e}\alpha_0(k+S_{rat})+\widetilde{S}_{rat}\rfloor$. Similarly $s(\widetilde{k}) = \lfloor -e\alpha_0(\widetilde{k}+\widetilde{S}_{rat})+S_{rat}\rfloor$ in the case of $\mathbf{t}^{J\binom{-s(\widetilde{k})+S_{rat}}{\widetilde{k}+\widetilde{S}_{rat}}}/1-\mathbf{t}^{J\binom{1}{0}}$. Therefore, by 4.3, for

$$\mathrm{pc}(Z_{ne}) = -\lambda(M) - \mathfrak{d}\cdot\frac{K^2+|\mathcal{V}|}{8} + \sum_h \chi(r_h)$$

we get

$$\sum_{\substack{0\le x_i<\alpha_i, 0\le i\le d\\ 0\le \widetilde{x}_j<\widetilde{\alpha}_j, 0\le j\le \widetilde{d}}}\left(S_{int}\widetilde{S}_{int} + \sum_{k=0}^{S_{int}-1}\lfloor -\widetilde{e}\alpha_0(k+S_{rat})+\widetilde{S}_{rat}\rfloor + \sum_{\widetilde{k}=0}^{\widetilde{S}_{int}-1}\lfloor -e\alpha_0(\widetilde{k}+\widetilde{S}_{rat})+S_{rat}\rfloor\right).$$

6.3.6 Ehrhart theoretical interpretation

In general, in contrast with the one–node case 6.2.1, the direct determination of the counting function of $Z_h(\mathbf{t})$, or equivalently, of the complete equivariant Ehrhart quasipolynomial associated with the corresponding polytope, is rather hard. Nevertheless, those coefficients which are relevant to us (e.g. those ones which contain the information about the Seiberg–Witten invariants of the 3–manifold) can be identified using the right hand side of (3.10). The computation is more transparent when $L' = L$. In that case, the two–variable Ehrhart polynomial has degree $d+\widetilde{d}$, and a specific $d+\widetilde{d}-2$ degree coefficient is exactly the normalized Seiberg–Witten invariant of the 3–manifold. We will not provide here the formulae, since this identification was already established for any negative definite plumbing graph with arbitrary number of nodes, see Section 5.6, where several other coefficients were computed as well.

6.4 Examples

Example 6.5 Consider the following plumbing graph.

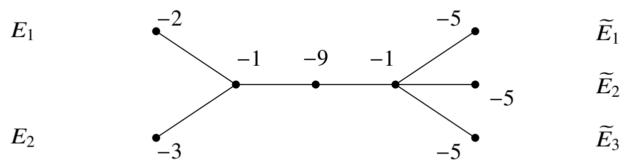


Fig. 6.2 Graph for Example 6.5.

The corresponding Seifert invariants are $\alpha_1 = 2$, $\alpha_2 = 3$, $\widetilde{\alpha}_j = 5$, $\alpha_0 = 9$ and $\omega_i = \widetilde{\omega}_j = \omega_0 = \widetilde{\omega}_0 = 1$ for all i and j. Hence $e = -1/18$, $\widetilde{e} = -13/45$ and $\varepsilon = 1/(3^3 \cdot 10)$. For $h = 0$ we choose $\mathbf{c} = 0$. Then

$$\mathcal{S}_0 = \left\{ \begin{array}{l} (\ell, \widetilde{\ell}) \in \mathbb{Z}^2 \\ 8\ell - \widetilde{\ell} + 9 \cdot ([\frac{-\ell}{2}] + [\frac{-\ell}{3}]) \geq 0 \\ 8\widetilde{\ell} - \ell + 27 \cdot [\frac{-\widetilde{\ell}}{5}] \geq 0 \\ \ell + \widetilde{\ell} \equiv 0 \ (\mathrm{mod}\ 9) \end{array} \right\} \quad \text{and} \quad \overline{\mathcal{S}}_0 = \left\{ \begin{array}{l} (\ell, \widetilde{\ell}) \in \mathbb{Z}^2 \\ \ell - 2\widetilde{\ell} \geq 0 \\ -5\ell + 13\widetilde{\ell} \geq 0 \\ \ell + \widetilde{\ell} \equiv 0 \ (\mathrm{mod}\ 9) \end{array} \right\}.$$

If we take the generators $\mathfrak{v}_1 = (60, 30)$ and $\mathfrak{v}_2 = (26, 10)$ (via conditions (A)-(B)-(C) following Lemma 6.4), one can calculate explicitly the sets

$$\mathcal{S}^-_{\mathbf{0},1} = \left\{ \begin{array}{l} (13,5), (19,8), (25,11), \\ (31,14), (37,17), (43,20), \\ (49,23), (55,26), (61,29), \\ (67,32) \end{array} \right\} \quad \text{and} \quad \mathcal{S}^-_{\mathbf{0},2} = \left\{ \begin{array}{l} (6,3), (19,8), (12,6), \\ (25,11), (24,12), (37,17), \\ (42,21), (55,26) \end{array} \right\}.$$

This generates the next counting function of $\overline{\mathcal{S}}_{\mathbf{0}} \setminus \mathcal{S}_{\mathbf{0}}$, namely $\sum_{(\ell,\widetilde{\ell}) \in \overline{\mathcal{S}}_{\mathbf{0}} \setminus \mathcal{S}_{\mathbf{0}}} t^{\ell} \widetilde{t}^{\widetilde{\ell}} =$

$$\sum_{(\ell,\widetilde{\ell}) \in \overline{\mathcal{S}}_{\mathbf{0}} \setminus \mathcal{S}_{\mathbf{0}}} t^{\ell} \widetilde{t}^{\widetilde{\ell}} = \frac{t^{13}\widetilde{t}^5 + t^{19}\widetilde{t}^8 + t^{25}\widetilde{t}^{11} + t^{31}\widetilde{t}^{14} + t^{37}\widetilde{t}^{17} + t^{43}\widetilde{t}^{20} + t^{49}\widetilde{t}^{23} + t^{55}\widetilde{t}^{26} + t^{61}\widetilde{t}^{29} + t^{67}\widetilde{t}^{32}}{1 - t^{60}\widetilde{t}^{30}} +$$
$$+ \frac{t^6\widetilde{t}^3 + t^{12}\widetilde{t}^6 + t^{19}\widetilde{t}^8 + t^{24}\widetilde{t}^{12} + t^{25}\widetilde{t}^{11} + t^{37}\widetilde{t}^{17} + t^{42}\widetilde{t}^{21} + t^{55}\widetilde{t}^{26}}{1 - t^{26}\widetilde{t}^{10}} - t^{19}\widetilde{t}^8 - t^{25}\widetilde{t}^{11} - t^{37}\widetilde{t}^{17} - t^{55}\widetilde{t}^{26},$$

which by 6.3.4 provides $Z_0^+(t, \widetilde{t}) = t\widetilde{t}^{-1} + t^3\widetilde{t}^2 + t^{-2}\widetilde{t}^2 + t^{-1}\widetilde{t} + t^{11}\widetilde{t}^7 + t^{16}\widetilde{t}^{11} + t^{-10}\widetilde{t} + t^{29}\widetilde{t}^{16} + t^3\widetilde{t}^6 + t^{19}\widetilde{t}^8 + t^{25}\widetilde{t}^{11} + t^{37}\widetilde{t}^{17} + t^{55}\widetilde{t}^{26}$. Hence $\mathrm{pc}_0^{C^{orb}}(Z) = Z_0^+(1,1) = 13$.

It can be verified that there exists a splice–quotient type normal surface singularity whose link is given by the above graph. It is a complete intersection in $(\mathbb{C}^4, 0)$ with equations $z^3 + (y_2 + 2y_3)^2 - y_1 y_2 (2y_2 + 3y_3) = y_1^5 + (2y_2 + 3y_3) y_2 y_3 = 0$. Its geometric genus is 13 according to the above computation and [80]. □

Example 6.6 Let Γ be the graph in Figure 6.3.

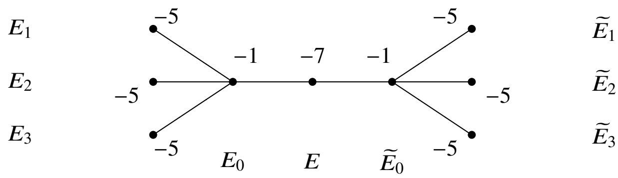


Fig. 6.3 Graph for Example 6.6.

The corresponding generalized Seifert invariants are $\alpha_i = \widetilde{\alpha}_j = 5, \omega_0 = \widetilde{\omega}_0 = \omega_i = \widetilde{\omega}_j = 1$, $e = \widetilde{e} = -9/35$ and $\varepsilon = 8/(7 \cdot 35)$ for all $i, j \in \{1, \ldots, 3\}$. Let $h \in H$ determined by the following coefficients: $c_0 = -2$, $\widetilde{c}_0 = 1$, $\overline{c} = 2$, $c_i = 3$ and $\widetilde{c}_j = -2$ for any i, j. Then $(c, \widetilde{c}) = (3/4, 3/4)$ which is uniquely determined by the h. It is immediate that

$$\mathcal{S}_{\mathbf{c}} = \left\{ \begin{array}{l} (\ell, \widetilde{\ell}) \in \mathbb{Z}^2 \\ 6\ell - \widetilde{\ell} + 21 \cdot [\frac{3-\ell}{5}] \geq 12 \\ 6\widetilde{\ell} - \ell + 21 \cdot [\frac{-2-\widetilde{\ell}}{5}] \geq 9 \\ \ell + \widetilde{\ell} \equiv 2 \pmod 7 \end{array} \right\} \text{ and } \overline{\mathcal{S}}_{\mathbf{c}} = \left\{ \begin{array}{l} (\ell, \widetilde{\ell}) \in \mathbb{Z}^2 \\ 9\ell - 5\widetilde{\ell} \geq -3 \\ -5\ell + 9\widetilde{\ell} \geq -3 \\ \ell + \widetilde{\ell} \equiv 2 \pmod 7 \end{array} \right\} .$$

If we choose $\mathfrak{v}_1 := (5, 9)$ and $\mathfrak{v}_2 := (9, 5)$ as generators for C^{orb}, one can calculate $\mathcal{S}^-_{\mathbf{c},1}$ and $\mathcal{S}^-_{\mathbf{c},2}$ explicitly, i.e.

$$\mathcal{S}^-_{\mathbf{c},1} = \{(1,1), (4,5), (5,4), (9,7)\} \text{ and } \mathcal{S}^-_{\mathbf{c},2} = \{(1,1), (4,5), (5,4), (7,9)\} .$$

Therefore, the counting function of $\overline{\mathcal{S}}_{\mathbf{c}} \setminus \mathcal{S}_{\mathbf{c}}$ is

$$-t^{3/4}\widetilde{t}^{3/4}\big((t\widetilde{t} + t^4\widetilde{t}^5 + t^5\widetilde{t}^4 + t^9\widetilde{t}^7)/(1 - t^5\widetilde{t}^9) + (t\widetilde{t} + t^4\widetilde{t}^5 + t^5\widetilde{t}^4 + t^7\widetilde{t}^9)/(1 - t^9\widetilde{t}^5) - t\widetilde{t} - t^4\widetilde{t}^5 - t^5\widetilde{t}^4\big).$$

Finally, using 6.3.4 we get $Z^+_h(t, \widetilde{t}) = -t^{3/4}\widetilde{t}^{3/4}(-\widetilde{t}^5 - t^4\widetilde{t}^{-2} - t^{-5} - t^{-2}\widetilde{t}^4 - t\widetilde{t} - t^4\widetilde{t}^5 - t^5\widetilde{t}^4)$, hence $\mathrm{pc}^{C^{orb}}_h(Z) = Z^+_h(1, 1) = 7$. □

Chapter 7

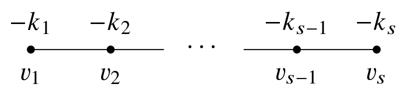
Decomposition of topological Poincaré series, non-normal affine monoids and their modules

In this chapter we generalize the computations from section 6.3 as follows. We construct a non-normal affine monoid together with its modules associated with a negative definite plumbed 3-manifold M. In terms of their structure, we describe the H-equivariant parts of the reduced topological Poincaré series. In particular, we give combinatorial formulas for the Seiberg–Witten invariants of M. The results of this chapter is based on the work of [44].

7.1 Generalized Seifert invariants and the group H

Consider a connected negative definite plumbing tree Γ. Some determinants associated with subgraphs of Γ will play a special role in the sequel. Therefore we set the following notations: since Γ is a tree, for any two vertices $v, w \in \mathcal{V}$ there is a unique minimal connected subgraph $[v, w]$ with vertices $\{v_i\}_{i=0}^{k}$ such that $v = v_0$ and $w = v_k$. Similarly, we also introduce notations $[v, w)$, $(v, w]$ and (v, w) for the complete subgraphs with vertices $\{v_i\}_{i=0}^{k-1}$, $\{v_i\}_{i=1}^{k}$ and $\{v_i\}_{i=1}^{k-1}$ respectively, and denote the corresponding determinants of the subgraphs by $\det_{[v,w]}$, $\det_{[v,w)}$ and $\det_{(v,w)}$.

For example, recall from section 6.3.1 (see also example 5.3 and 6.1.1) that for a subgraph

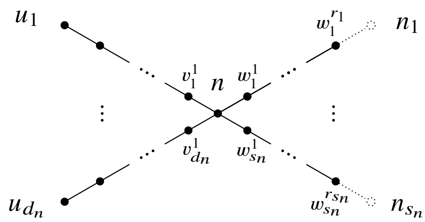


with vertices $\{v_i\}_{i=1}^{s}$ and $k_i \geq 2$ for all i, the arithmetical properties of the graph can be encoded by the normalized Seifert invariant (α, ω), where $0 < \omega < \alpha$ and $\gcd(\alpha, \omega) = 1$, using Hirzebruch/negative continued fraction expansion

$$\alpha/\omega = [k_1, \dots, k_s] = k_1 - 1/(k_2 - 1/(\cdots - 1/k_s)\cdots).$$

In particular, the plumbed 3-manifold associated with the above graph itself is the lens space $L(\alpha, \omega)$. We also consider $\widetilde{\omega}$ satisfying $\omega\widetilde{\omega} \equiv 1 \pmod{\alpha}$, $0 < \widetilde{\omega} < \alpha$. Clearly, these invariants are graph determinants, namely $\alpha = \det_{[v_1,v_s]}$, $\omega = \det_{[v_2,v_s]}$, $\widetilde{\omega} = \det_{[v_1,v_{s-1}]}$. Moreover, $\omega\widetilde{\omega} = \alpha\tau + 1$ for $\tau = \det_{[v_2,v_{s-1}]}$.

Similarly to the case of star-shaped plumbing graphs (cf. section 6.1.1), we encode the information of Γ by the normalized Seifert invariants of the chains and legs and the orbifold Euler numbers attached to the nodes $n \in \mathcal{N}$. In fact, for any $n \in \mathcal{N}$ it is convenient to consider the maximal star-shaped subgraphs Γ_n of Γ which contains only one node $n \in \mathcal{N}$:



where $n_i \in \mathcal{N}_n$ $(1 \le i \le s_n)$ and $u_j \in \mathcal{E}_n$ $(1 \le j \le d_n)$. We denote the normalized Seifert invariants of the legs $(n, u_j]$ by $(\alpha_{u_j}, \omega_{u_j})$, ie. $\alpha_{u_j} = \det_{(n,u_j]}$ and $\omega_{u_j} = \det_{(v_j^1,u_j]}$. Moreover, the chain (n, n_i) connecting the nodes n and n_i considered as leg of Γ_n has normalized Seifert invariant $(\alpha_{n,n_i}, \omega_{n,n_i}) := (\det_{(n,n_i)}, \det_{(w_i^1,n_i)})$, while considered as leg of Γ_{n_i} has invariant $(\alpha_{n_i,n}, \omega_{n_i,n}) := (\det_{(n_i,n)}, \det_{(w_i^{r_i},n)})$, with relation $\omega_{n,n_i}\omega_{n_i,n} = \alpha_{n,n_i}\tau_{n,n_i} + 1$. Notice that $\alpha_{n,n_i} = \alpha_{n_i,n}$ and $\tau_{n,n_i} = \tau_{n_i,n}$.

7.1.1 Orbifold intersection matrix

We define the orbifold Euler number of the star-shaped subgraph Γ_n by

$$e_n = b_n + \sum_{v \in \mathcal{E}_n} \frac{\omega_v}{\alpha_v} + \sum_{n' \in \mathcal{N}_n} \frac{\omega_{n,n'}}{\alpha_{n,n'}}.$$

Notice that $e_n < 0$ for any $n \in \mathcal{N}$, since Γ_n itself is negative definite being a subgraph of a negative definite graph Γ. One can collect these informations by defining the *orbifold intersection matrix* $I^{orb} = (I^{orb}_{n,n'})_{n,n' \in \mathcal{N}}$ associated with Γ as

$$I^{orb}_{n,n'} := \begin{cases} e_n & \text{if } n = n', \\ \dfrac{1}{\alpha_{n,n'}} & \text{if } n' \in \mathcal{N}_n, \\ 0 & \text{otherwise.} \end{cases}$$

Then I^{orb} is again negative definite and $\det_\Gamma = \det(-I^{orb}) \cdot \det_{\Gamma\setminus\mathcal{N}}$ by [14, Lemma 4.1.4]. Furthermore, one has

$$(E_n^*, E_{n'}^*) = (I^{orb})^{-1}_{n,n'}. \tag{7.1}$$

7.1.2 The group H: generators and relations

We introduce a partial order on $\mathcal{N}$ such that for any two nodes $n, n' \in \mathcal{N}$ connected by a chain we pick $n < n'$ or $n' < n$. Using this partial order for any $n < n'$ we denote by $n_{n'}$ the vertex on the chain (n, n') such that $(E_n, E_{n_{n'}}) = 1$. When $(n, n') = \emptyset$ then $n_{n'} = n'$. In the sequel, we will need the following identities expressing relations regarding E_v^*'s using determinants of respective subgraphs of Γ. They were considered already in Lemma 6.2, although we will repeat them here as well in a reasonable form for the sake of readability.

Lemma 7.1 *Assume we have a connected negative definite plumbing tree* Γ.

(a) Let $n \in \mathcal{N}$ *and* $u \in \mathcal{E}_n$. *Then for any* $v \in [n, u)$ *we have*

$$E_v^* = \det{}_{(v,u]} E_u^* - \sum_{w \in (v,u]} \det{}_{(v,w)} E_w.$$

In particular, we have $E_n^* = \alpha_u E_u^* - \sum_{w \in (n,u]} \det_{(n,w)} E_w$ *and* $E_v^* = \omega_u E_u^* - \sum_{w \in (v,u]} \det_{(v,w)} E_w$ *for the first vertex* v *on* (n, u).

(b) Consider two nodes $n < n'$ *connected by a non-empty chain. Then for all* $v \in [n', n_{n'})$ *we have*

$$E_v^* = \det{}_{(v,n_{n'}]} E_{n_{n'}}^* - \det{}_{(v,n_{n'})} E_n^* - \sum_{w \in (v,n_{n'}]} \det{}_{(v,w)} E_w.$$

In particular, for the vertex m *on the chain* (n, n') *such that* $(E_m, E_{n'}) = 1$ *we get* $E_m^* = \omega_{n',n} E_{n_{n'}}^* - \tau_{n',n} E_n^* - \sum_{w \in (m,n_{n'}]} \det_{(m,w)} E_w$.

7.1.2.1 Lifts We introduce short notation $g_v := [E_v^*]$ for classes of dual basis elements in H. With these notation the class of every $l' = \sum_{v \in \mathcal{V}} l'_v E_v^*$ can be written as

$$[l'] = \sum_{n \in \mathcal{N}} \Big(a_n g_n + \sum_{u \in \mathcal{E}_n} a_u g_u + \sum_{n' > n} a_{n_{n'}} g_{n_{n'}} \Big)$$

with

$$a_n := l'_n - \sum_{\substack{n' > n \\ v \in (n', n_{n'})}} l'_v \det{}_{(v,n_{n'})}, \quad a_u := l'_u + \sum_{v \in (n,u)} l'_v \det{}_{(v,u]}, \quad a_{n_{n'}} := l'_{n_{n'}} + \sum_{\substack{n' > n \\ v \in (n', n_{n'})}} l'_v \det{}_{(v,n_{n'}]}$$

by Lemma 7.1. We call $a = \sum_{n \in \mathcal{N}} a_n E_n^* + \sum_{u \in \mathcal{E}} a_u E_u^* + \sum_{n < n'} a_{n_{n'}} E_{n_{n'}}^*$ the *reduced transform* of l' and a *reduced lift* of $[l'] \in H$. Thus, using the idea of Neumann [86] and section 6.3.1, the group $H = L'/L$ can be presented with generators

$$g_n = [E_n^*],\ n \in \mathcal{N}, \qquad g_u = [E_u^*],\ u \in \mathcal{E}, \qquad g_{n_{n'}} = [E_{n_{n'}}^*],\ n < n'$$

and relations

$$R_u := g_n - \alpha_u g_u = 0, \qquad (u \in \mathcal{E}_n)$$
$$R_{n_{n'}} := -\alpha_{n,n'} g_{n_{n'}} + \omega_{n,n'} g_n + g_{n'} = 0, \qquad (n < n')$$
$$R_n := -b_n g_n - \sum_{u \in \mathcal{E}_n} \omega_u g_u - \sum_{n<n'} g_{n_{n'}} - \sum_{n'<n} \left(\omega_{n,n'} g_{n'_n} - \tau_{n,n'} g_{n'}\right) = 0, \qquad (n \in \mathcal{N}).$$

In particular, $a = \sum_v a_v E_v^*$ and $x = \sum_v x_v E_v^*$ are reduced lifts of the same group element $h \in H$ if and only if there are $\ell_n, \ell_u, \ell_{n_{n'}} \in \mathbb{Z}$ such that

$$x_u - a_u = -\ell_u \alpha_u - \omega_u \ell_n, \qquad (u \in \mathcal{E}_n) \qquad (\mathrm{R}'_u)$$
$$x_n - a_n = \sum_{u \in \mathcal{E}_n} \ell_u - b_n \ell_n + \sum_{n>n'} \ell_{n_{n'}} + \sum_{n<n'} \omega_{n,n'} \ell_{n_{n'}} + \sum_{n<n'} \tau_{n,n'} \ell_{n'}, \qquad (n \in \mathcal{N}) \qquad (\mathrm{R}'_n)$$
$$x_{n_{n'}} - a_{n_{n'}} = -\ell_n - \omega_{n',n} \ell_{n'} - \alpha_{n,n'} \ell_{n_{n'}}, \qquad (n < n'). \qquad (\mathrm{R}'_{n_{n'}})$$

Remark 7.1 For any class $h \in H$ we will consider the unique representative $r_h \in L'$ characterized by $r_h \in \sum_v [0,1) E_v$ and $[r_h] = h$ (cf. [69, 5.4]). In general, r_h is not a reduced lift of h.

Remark 7.2 The group H can be generated by elements g_u, $u \in \mathcal{E}$. Indeed, we choose a special partial order on nodes of the graph Γ in the following way. We fix a 'root' node and we orient all of its outgoing paths away from the root. Then the partial order is defined such that the tail of an oriented chain is greater then its head. We proceed with induction on the number of the nodes of the graph. In every step we choose a minimal node n. If $\mathcal{E}_n \neq \emptyset$ then we express g_n from a relation R_u, $u \in \mathcal{E}_n$, otherwise there must be $\widetilde{n} < n$ in the original graph Γ, hence we express g_n from the relation $R_{\widetilde{n}_n}$. Moreover, either we have a unique node $n' > n$ and in this case $g_{n_{n'}}$ is expressed from $R_{n_{n'}}$, or n is the root node which finishes the induction. At the end of each step we remove the chain $[n, n')$ together with all legs attached to n in order to get a new graph with one node less.

7.1.2.2 Projections

We consider the reduction of the Poincaré series to only node variables, which involves the use of the projection $\pi_{\mathcal{N}} : \mathbb{R}\langle E_v \rangle_{v \in \mathcal{V}} \to \mathbb{R}\langle E_n \rangle_{n \in \mathcal{N}}$ along the subspace spanned by E_v for $v \notin \mathcal{N}$. Therefore, we introduce the projection $\mathbf{c}_a := \pi_{\mathcal{N}}(a)$ of a reduced lift a. One can express $\mathbf{c}_a$ with the basis $\{\pi_{\mathcal{N}}(E_n^*)\}_{n \in \mathcal{N}}$ as $\mathbf{c}_a = \sum_{n \in \mathcal{N}} A_n \pi_{\mathcal{N}}(E_n^*)$ so that

$$A_n = a_n + \sum_{u \in \mathcal{E}_n} \frac{a_u}{\alpha_u} + \sum_{n'>n} \frac{\omega_{n,n'}}{\alpha_{n,n'}} a_{n_{n'}} + \sum_{n'<n} \frac{1}{\alpha_{n',n}} a_{n'_n}. \tag{7.2}$$

Furthermore, in terms of basis $\{E_n\}_{n \in \mathcal{N}}$ we can write $\mathbf{c}_a = \sum_{n \in \mathcal{N}} c_n E_n$ with

$$(c_n)_{n \in \mathcal{N}} = (-I^{orb})^{-1} \cdot (A_n)_{n \in \mathcal{N}}. \tag{7.3}$$

Note that Lemma 7.1 implies that the difference of an element $l' \in L'$ and its reduced transform a lies on the sublattice $\mathbb{Z}\langle E_v \rangle_{v \notin \mathcal{N}}$. Thus, the projections of l' and a by $\pi_{\mathcal{N}}$ coincide, hence

$\mathbf{c}_a = \pi_{\mathcal{N}}(l')$ and we use notation $c_n(l') = c_n$ for the corresponding coefficients in the basis $\{E_n\}_{n\in\mathcal{N}}$. In particular, for the unique representative r_h of h we have $c_n(r_h) \in [0,1)$ for all $n \in \mathcal{N}$, and these values are uniquely determined by h.

7.2 The decomposition of the reduced Poincaré series

We will use multiplicative notation for the group H when we consider $\mathbb{Z}[H]$-coefficients of the zeta function f_H and the Poincaré series Z_H. Note that by Lemma 7.1(a) for $u \in \mathcal{E}_n$ we have $(E_u^*, E_{n'}^*) = \alpha_u(E_n^*, E_{n'}^*)$ for all $n' \in \mathcal{N}$, hence $\big(g_u \mathbf{t}_{\mathcal{N}}^{E_u^*}\big)^{\alpha_u} = g_n \mathbf{t}_{\mathcal{N}}^{E_n^*}$. Therefore, we can write

$$f_H(\mathbf{t}_{\mathcal{N}}) = \frac{\prod_{n\in\mathcal{N}}\big(1-g_n\mathbf{t}_{\mathcal{N}}^{E_n^*}\big)^{\delta_n-2}}{\prod_{u\in\mathcal{E}}(1-g_u\mathbf{t}_{\mathcal{N}}^{E_u^*})} = \prod_{u\in\mathcal{E}}\Big(\sum_{x_u=0}^{\alpha_u-1} g_u^{x_u}\mathbf{t}_{\mathcal{N}}^{x_uE_u^*}\Big)\prod_{n\in\mathcal{N}}\Big(1-g_n\mathbf{t}_{\mathcal{N}}^{E_n^*}\Big)^{\delta_{n,\mathcal{N}}-2}.$$

Taking its Taylor expansion at the origin we get

$$\begin{aligned} Z_H(\mathbf{t}_{\mathcal{N}}) &= \prod_{u\in\mathcal{E}}\Big(\sum_{x_u=0}^{\alpha_u-1} g_u^{x_u}\mathbf{t}_{\mathcal{N}}^{x_uE_u^*}\Big)\prod_{\delta_{n',\mathcal{N}}>2}\Big(1-g_{n'}\mathbf{t}_{\mathcal{N}}^{E_{n'}^*}\Big)^{\delta_{n',\mathcal{N}}-2}\prod_{\delta_{n,\mathcal{N}}=1}\Big(\sum_{x_n\geq 0} g_n^{x_n}\mathbf{t}_{\mathcal{N}}^{x_nE_n^*}\Big)\\ &= \sum_{x\in\mathcal{X}}\prod_{\delta_{n',\mathcal{N}}>2}(-1)^{x_{n'}}\binom{\delta_{n',\mathcal{N}}-2}{x_{n'}}\prod_{v\in\mathcal{E}\cup\mathcal{N}} g_v^{x_v}\mathbf{t}_{\mathcal{N}}^{x_vE_v^*}, \end{aligned}$$

where the sum is over the set

$$\mathcal{X} = \left\{ x = \sum_{v\in\mathcal{N}\cup\mathcal{E}} x_vE_v^* \in L' \;\middle|\; \begin{array}{ll} 0\leq x_u < \alpha_u, & u\in\mathcal{E}\\ 0\leq x_n, & n\notin\widehat{\mathcal{N}}\\ 0\leq x_{n'}\leq \delta_{n',\mathcal{N}}-2, & n'\in\widehat{\mathcal{N}} \end{array}\right\}.$$

In particular, the h-equivariant part of $Z_H(\mathbf{t}_{\mathcal{N}})$ equals

$$Z_h(\mathbf{t}_{\mathcal{N}}) = \sum_{x\in\mathcal{X}_h}\prod_{\delta_{n',\mathcal{N}}>2}(-1)^{x_{n'}}\binom{\delta_{n',\mathcal{N}}-2}{x_{n'}}\cdot\mathbf{t}_{\mathcal{N}}^{x}, \tag{7.4}$$

where $\mathcal{X}_h = \{x\in\mathcal{X}\mid [x]=h\}$.

To describe $\mathcal{X}_h$ more explicitely, we fix a reduced lift a of h (cf. section 7.1.2.1) and we define an affine lattice

$$\mathbb{Z}^{\mathcal{N}}(a) = \left\{\ell = \sum_{n\in\mathcal{N}}\ell_nE_n \in \mathbb{Z}\langle E_n\rangle_{n\in\mathcal{N}} \;\middle|\; \ell_n + \omega_{n',n}\ell_{n'} \equiv a_{n_{n'}} \ (\bmod\ \alpha_{n,n'}),\ \forall\, n<n'\right\}.$$

Moreover, for any $n\in\mathcal{N}$ consider the quasilinear function

$$\begin{aligned}N_a(\ell,n) := a_n + \sum_{n'>n} \frac{\omega_{n,n'}}{\alpha_{n,n'}} a_{n_{n'}} + \sum_{n'<n} \frac{1}{\alpha_{n,n'}} a_{n_{n'}} \\ - \Big(b_n + \sum_{n'\in\mathcal{N}_n} \frac{\omega_{n,n'}}{\alpha_{n,n'}}\Big)\ell_n - \sum_{n'\in\mathcal{N}_n} \Big(\frac{1}{\alpha_{n,n'}}\ell_{n'}\Big) + \sum_{u\in\mathcal{E}_n} \Big\lfloor \frac{a_u - \omega_u \ell_n}{\alpha_u} \Big\rfloor.\end{aligned} \tag{7.5}$$

Then we have the following parametrization of $\mathcal{X}_h$.

Proposition 7.1 *(a) There is a bijection*

$$\mathcal{S}_a = \left\{ \ell \in \mathbb{Z}^{\mathcal{N}}(a) \;\middle|\; \begin{array}{ll} 0 \le N_a(\ell,n), & n \notin \widehat{\mathcal{N}} \\ 0 \le N_a(\ell,n) \le \delta_{n,\mathcal{N}} - 2, & n \in \widehat{\mathcal{N}} \end{array} \right\} \longrightarrow \mathcal{X}_h$$

given by $x_n = N_a(\ell,n)$ for any $n \in \mathcal{N}$ and $x_u = \alpha_u \left\{ \frac{a_u - \omega_u \ell_n}{\alpha_u} \right\}$ for any $u \in \mathcal{E}$.

(b) We have

$$\mathbf{t}_{\mathcal{N}}^{x} = \mathbf{t}^{\mathbf{c}_a+\ell} = \prod_{n\in\mathcal{N}} t_n^{c_n+\ell_n}.$$

(c) Finally, the h-equivariant part of the reduced Poincaré series equals

$$Z_h(\mathbf{t}_{\mathcal{N}}) = \sum_{\ell\in\mathcal{S}_a} \prod_{\delta_{n',\mathcal{N}}>2} (-1)^{N_a(\ell,n')} \binom{\delta_{n',\mathcal{N}}-2}{N_a(\ell,n')} \cdot \mathbf{t}^{\mathbf{c}_a+\ell}. \tag{7.6}$$

Remark 7.3

(a) In fact, one can write (7.6) in the form

$$Z_h(\mathbf{t}_{\mathcal{N}}) = \sum_{\ell\in\mathcal{S}_a} \prod_{n\in\mathcal{N}} (-1)^{N_a(\ell,n)} \binom{\delta_{n,\mathcal{N}}-2}{N_a(\ell,n)} \cdot \mathbf{t}^{\mathbf{c}_a+\ell}, \tag{7.7}$$

if we regard $\binom{\delta_{n,\mathcal{N}}-2}{N_a(\ell,n)}$ to be the generalized binomial coefficient.

(b) In Section 7.3 we give a more explicit description of the set $\mathcal{S}_a$, which will lead to the rational form of $Z_h(\mathbf{t}_{\mathcal{N}})$ in Subsection 7.5.

Proof (a) Since both x and a are reduced lifts of h, there are $\ell_u, \ell_n, \ell_{n_{n'}} \in \mathbb{Z}$ such that (R'_u), (R'_n) and ($R'_{n_{n'}}$) hold with $x_{n_{n'}} = 0$ (cf. Section 7.1.2.1). We can eliminate ℓ_u's and $\ell_{n_{n'}}$'s as follows. The relation ($R'_{n_{n'}}$) is equivalent with

$$a_{n_{n'}} \equiv \ell_n + \omega_{n',n}\ell_{n'} \pmod{\alpha_{n,n'}} \qquad \text{and} \qquad \ell_{n_{n'}} = \frac{a_{n_{n'}} - \ell_n - \omega_{n',n}\ell_{n'}}{\alpha_{n,n'}}, \qquad (n < n'). \tag{7.8}$$

The conditions $0 \le x_u < \alpha_u$ and (R'_u) are equivalent with

$$x_u = \alpha_u \cdot \left\{ \frac{a_u - \omega_u \ell_n}{\alpha_u} \right\} \qquad \text{and} \qquad \ell_u = \left\lfloor \frac{a_u - \omega_u \ell_n}{\alpha_u} \right\rfloor \in \mathbb{Z}. \tag{7.9}$$

Moreover, by substituting $\ell_{n_{n'}}$ from (7.8) and ℓ_u from (7.9) into (R'_n) we get $x_n = N_a(\ell,n)$. Thus, we have defined a map from $\mathcal{S}_a \to \mathcal{X}_h$. To show that this map is invertible, note that

$$x_n + \sum_{u\in\mathcal{E}_n} \frac{x_u}{\alpha_u} = N_a(\ell, n) + \left\{\frac{a_u - \omega_u \ell_n}{\alpha_u}\right\} = A_n - e_n\ell_n - \sum_{n'\in\mathcal{N}_n} \frac{1}{\alpha_{n,n'}}\ell_{n'}, \tag{7.10}$$

where A_n was considered in (7.2) as the coefficient of the projection $\mathbf{c}_a = \pi_{\mathcal{N}}(a)$ in the $\{E_n^*\}_{n\in\mathcal{N}}$-basis. We can write (7.10) in matrix form $\left(x_n + \sum_{u\in\mathcal{E}_n} \frac{x_u}{\alpha_u}\right)_n = (A_n)_n - I^{orb} \cdot (\ell_n)_n$, which can be reformulated simply as

$$\mathbf{c}_x = \mathbf{c}_a + \ell. \tag{7.11}$$

This shows that $\mathcal{S}_a \longrightarrow \mathcal{X}_h$ is invertible.

(b) Follows from (7.11).

(c) Follows from (7.4) and the previous two parts of the proposition. □

Remark 7.4 (a) In the case when M is a Seifert rational homology sphere, or equivalently, the graph Γ is star-shaped, the reduced Poincaré series has only one variable associated with the central vertex n of Γ. Since $\mathcal{S}_a = \{\ell = \ell_n E_n \in \mathbb{Z}E_n \mid 0 \le N_a(\ell, n)\} \subset (-c_n + \mathbb{R}_{\ge 0}) \cap \mathbb{Z}$, we can rephrase (7.7) with

$$Z_h(t_n) = \sum_{\ell_n \ge -c_n} \max\{0, N_a(\ell, n) + 1\} \cdot t^{c_n + \ell_n},$$

hence we get back the formula given in section 6.1.2.

(b) Assume Γ has two nodes n_0 and $\widetilde{n}_0$ with $\delta_{n,\mathcal{N}} = 1$ and the subgraph $[n_0, \widetilde{n}_0]$ contains all the other nodes with $\delta_{n,\mathcal{N}} = 2$. Then $Z_h(\mathbf{t}_{\mathcal{N}}) = \sum_{\ell\in\mathcal{S}_a} \mathbf{t}^{\mathbf{c}_a+\ell}$ is the generating series of $\mathcal{S}_a$. In particular, this formula for the two-node case ($\widehat{\mathcal{N}} = \emptyset$) can be found in section 6.3.1.

(c) If Γ is the plumbing graph of the link of a 2-dimensional Newton nondegenerate hypersurface singularity, then [14] implies $\delta_{n,\mathcal{N}} \le 3$ for any $n \in \mathcal{N}$. Hence, by (7.6) the coefficients of $Z_h(\mathbf{t}_{\mathcal{N}})$ can be either $N_a(\ell, n) + 1$ if Γ is star-shaped, or ± 1 otherwise. Similar consequences regarding this example can be found in [104, Lemma 7.1.12].

7.3 Non-normal affine monoids and modules

One can consider the 'normalization' of the quasilinear function $N_a(\ell, n)$ by introducing the linear function

$$\overline{N}_a(\ell, n) := N_a(\ell, n) + \sum_{u\in\mathcal{E}_n} \left\{\frac{a_u - \omega_u \ell_n}{\alpha_u}\right\} \overset{(7.2)}{=} A_n - e_n\ell_n - \sum_{n'\in\mathcal{N}_n} \frac{1}{\alpha_{n,n'}}\ell_{n'}.$$

In particular, notice that for any $n \in \mathcal{N}$ with $\delta_{n,\mathcal{E}} = 0$ we have equality $\overline{N}_a(\ell, n) = N_a(\ell, n)$.

Associated with the pair (Γ, a), we define the sets

$$\mathcal{M}_a := \{\ell \in \mathbb{Z}^{\mathcal{N}}(a) \mid N_a(\ell, n) \ge 0,\ \forall n \in \mathcal{N}\} \text{ and } \overline{\mathcal{M}}_a := \{\ell \in \mathbb{Z}^{\mathcal{N}}(a) \mid \overline{N}_a(\ell, n) \ge 0,\ \forall n \in \mathcal{N}\}.$$

From $\overline{N}_a(\ell, n) \geq N_a(\ell, n)$ follows that $\mathcal{M}_a \subset \overline{\mathcal{M}}_a$. Moreover, if we consider the real cone $C^{orb} := \pi_{\mathcal{N}}(\mathcal{S}'_{\mathbb{R}}) = \{\ell = \sum_{n\in\mathcal{N}} \ell_n E_n \mid -I^{orb} \cdot (\ell_n)_n \geq 0\}$ then

$$\overline{\mathcal{M}}_a = (C^{orb} - \mathbf{c}_a) \cap \mathbb{Z}^{\mathcal{N}}(a).$$

Remark 7.5 Notice that by Remark 7.2 we can choose a such that every $a_{n_{n'}} = 0$, hence for such an a the lattice $\mathbb{Z}^{\mathcal{N}}(a)$ is independent of a.

Lemma 7.2 *(a) $\mathcal{M}_0$ and $\overline{\mathcal{M}}_0$ are affine monoids. $\overline{\mathcal{M}}_0$ is the normalization of $\mathcal{M}_0$.*
(b) $\mathcal{M}_a$ and $\overline{\mathcal{M}}_a$ are finitely generated $\mathcal{M}_0$-modules, $\mathcal{M}_a$ is a submodule of $\overline{\mathcal{M}}_a$.

Proof (a) is elementary. Part (b) follows from [21, Theorem 2.12], that is $\overline{\mathcal{M}}_a$ is finitely generated over $\overline{\mathcal{M}}_0$, but $\overline{\mathcal{M}}_0$ itself is finitely generated as an $\mathcal{M}_0$-module. □

7.3.1 The set of holes

The *holes* of the $\mathcal{M}_0$-module $\mathcal{M}_a$ is defined to be the set $\overline{\mathcal{M}}_a \setminus \mathcal{M}_a$. By [21, 4.36], this is small, in the sense that it is contained in finitely many hyperplanes parallel to the facets of $\mathcal{M}_0$, that is of C^{orb}. More details on the decompositions for the set of holes in general can be found in [21] and [37]. In the following, we describe $\overline{\mathcal{M}}_a \setminus \mathcal{M}_a$ for the present special case.

Lemma 7.3 *Fix $\kappa \geq 0$ and a reduced lift a. Then there exists $\mathfrak{v}_n \in \mathcal{M}_0$, $n \in \mathcal{N}$ such that $\mathbb{R}_{\geq 0}\langle \mathfrak{v}_n \rangle_{n\in\mathcal{N}} = C^{orb}$ and satisfying the following properties: for any $\ell \in \overline{\mathcal{M}}_a$ one has*

(a) $N_a(\ell + \mathfrak{v}_n, n') = N_a(\ell, n')$ for all $n' \neq n$,
(b) $N_a(\ell + \mathfrak{v}_n, n) \geq \kappa$.

Proof We can choose $\mathfrak{v}_n = \lambda_n \pi_{\mathcal{N}}(E_n^*)$ for λ_n sufficiently large. Indeed, for $\mathfrak{v}_n = \sum_{n'\in\mathcal{N}} \mathfrak{v}_{n,n'} E_{n'} \in \mathbb{Z}^{\mathcal{N}}(0)$ such that $\{\omega_u \mathfrak{v}_{n,n'}/\alpha_u\} = 0$ for any $n' \in \mathcal{N}$ and $u \in \mathcal{E}_{n'}$ we have $N_a(\ell + \mathfrak{v}_n, n') = N_a(\ell, n') + \overline{N}_0(\mathfrak{v}_n, n')$. By (7.1) note that $\overline{N}_0(\mathfrak{v}_n, n') = 0$ for any $n' \neq n$, which implies (a). Finally, we have $N_a(\ell, n) \geq \overline{N}_a(\ell, n) - |\mathcal{E}_n|$. Hence, if we choose λ_n such that $\overline{N}_0(\mathfrak{v}_n, n) = \lambda_n \geq |\mathcal{E}_n| + \kappa$ then we have $N_a(\ell + \mathfrak{v}_n, n) \geq \kappa$ for any $\ell \in \overline{\mathcal{M}}_a$. □

Remark 7.6 (a) The vectors $\mathfrak{v}_n$ given in the above proof does not depend on a, hence they can be chosen for all a.
(b) Alternatively, we can construct rather smaller vectors $\{\mathfrak{v}_n\}_{n\in\mathcal{N}}$ also satisfy (a) and (b). Require the vanishing $\{\omega_u \mathfrak{v}_{n,n'}/\alpha_u\} = 0$ only for $u \in \mathcal{E}_{n'}$ $n' \neq n$, and assume $N_0(\mathfrak{v}_n, n) \geq 0$. Moreover, we require also that $N_a(\ell + \mathfrak{v}_n, n) \geq \kappa$ for any $\ell \in (\square - \mathbf{c}_a) \cap \mathbb{Z}^{\mathcal{N}}(a)$, where $\square = \sum_{n\in\mathcal{N}} [0, 1)\mathfrak{v}_n$ is the semiopen cube generated by $\{\mathfrak{v}_n\}_{n\in\mathcal{N}}$. Then one can check that these conditions imply (a) and (b).

We define the sets

$$\mathcal{M}^-_{a,n} := \{\ell \in (\square - \mathbf{c}_a) \cap \mathbb{Z}^{\mathcal{N}}(a) \mid N_a(\ell, n) < 0\}, \quad \mathcal{M}^-_{a,\mathcal{I}} := \bigcap_{n\in\mathcal{I}} \mathcal{M}^-_{a,n} \text{ for every } \mathcal{I} \subseteq \mathcal{N},$$

and let $F_{\mathcal{I}} = \mathbb{Z}_{\geq 0}\langle \mathfrak{v}_{n'}\rangle_{n'\in\mathcal{N}\setminus\mathcal{I}}$ be the 'face' associated with $\mathcal{I}$. In particular, $\mathcal{M}^-_{a,\emptyset} = (\square - \mathbf{c}_a) \cap \mathbb{Z}^{\mathcal{N}}(a)$. One can see immediately that $\{\ell \in \mathbb{Z}^{\mathcal{N}}(a) \mid N_a(\ell, n) < 0\} = \bigsqcup_{\ell\in\mathcal{M}^-_{a,n}}(\ell + F_n)$. Moreover, we conclude the following generalization of Proposition 6.1.

Proposition 7.2 *Let $\{\mathfrak{v}_n\}_{n\in\mathcal{N}}$ as in Lemma 7.3. Then*

(a) The normalization $\overline{\mathcal{M}}_a$ is given by

$$\overline{\mathcal{M}}_a = \bigsqcup_{\ell\in\mathcal{M}^-_{a,\emptyset}} \ell + \mathbb{Z}_{\geq 0}\langle \mathfrak{v}_n\rangle_{n\in\mathcal{N}}.$$

(b) The set of holes $\overline{\mathcal{M}}_a \setminus \mathcal{M}_a$ is described by

$$\overline{\mathcal{M}}_a \setminus \mathcal{M}_a = \bigcup_{n\in\mathcal{N}} \Big(\bigsqcup_{\ell\in\mathcal{M}^-_{a,n}} \ell + F_n \Big),$$

where $\bigcap_{n\in\mathcal{I}} \Big(\bigsqcup_{\ell\in\mathcal{M}^-_{a,n}} \ell + F_n \Big) = \bigsqcup_{\ell\in\mathcal{M}^-_{a,\mathcal{I}}} \ell + F_{\mathcal{I}}.$

Proof (a) is implied by [21, Proposition 2.43]. Part (b) follows from the choice of $\{\mathfrak{v}_n\}_{n\in\mathcal{N}}$ and their properties from Lemma 7.3. □

Corollary 7.1 *The structure of any set $\mathcal{D}$ can be encoded by defining the generating series (fine Hilbert series) $\mathcal{H}_{\mathcal{D}}(\mathbf{t}) := \sum_{\ell\in\mathcal{D}} \mathbf{t}^\ell$. In the case of $\mathcal{M}_a$, Proposition 7.2 implies the following form:*

$$\mathcal{H}_{\mathcal{M}_a}(\mathbf{t}_{\mathcal{N}}) = \sum_{\emptyset\subseteq\mathcal{I}\subseteq\mathcal{N}} (-1)^{|\mathcal{I}|} \frac{\sum_{\ell\in\mathcal{M}^-_{a,\mathcal{I}}} \mathbf{t}^\ell}{\prod_{n\notin\mathcal{I}}(1 - \mathbf{t}^{\mathfrak{v}_n})}.$$

7.3.2 Multi-index filtration and generating series

We consider a filtration $\{\mathcal{M}_a(k) \mid k \in \mathbb{Z}\langle E^*_j\rangle_{j\in\mathcal{J}}\}$ on the $\mathcal{M}_0$-module $\mathcal{M}_a$ associated with a fixed index set $\mathcal{J} \subset \mathcal{N}$ by defining the submodules

$$\mathcal{M}_a(k) := \{\ell \in \mathcal{M}_a \mid N_a(\ell, n) \geq k_n, \ \forall n \in \mathcal{J}\},$$

where $k = \sum_{j\in\mathcal{J}} k_j E^*_j$.

Remark 7.7 We observe that $\mathbb{Z}^{\mathcal{N}}(a) = \mathbb{Z}^{\mathcal{N}}(a - k)$ and $N_a(\ell, n) \geq k_n$ is equivalent with $N_{a-k}(\ell, n) \geq 0$ for any $n \in \mathcal{J}$, therefore $\mathcal{M}_a(k) = \mathcal{M}_{a-k}$.

Clearly, $\mathcal{M}_a(k) \supset \mathcal{M}_a(k + E^*_j)$, moreover for any $\mathcal{I} \subset \mathcal{J}$ we have

$$\bigcap_{i\in\mathcal{I}} \mathcal{M}_a(k+E_i^*) = \mathcal{M}_a(k+E_{\mathcal{I}}^*), \tag{7.12}$$

where we use notation $E_{\mathcal{I}}^* := \sum_{i\in\mathcal{I}} E_i^*$. One can also consider the associated graded object

$$\mathrm{gr}_k\mathcal{M}_a := \mathcal{M}_a(k) \setminus \bigcup_{j\in\mathcal{J}} \mathcal{M}_a(k+E_j^*) \tag{7.13}$$

at level k. Notice that $\mathrm{gr}_k\mathcal{M}_a = \{\ell \in \mathcal{M}_a \mid N_a(\ell, j) = k_j,\ \forall j \in \mathcal{J}\}$.

We also define the generating set of holes of the graded pieces as follows. We fix k and we choose vectors $\mathfrak{v}_n \in \mathbb{Z}\langle E_{n'}\rangle_{n'\in\mathcal{N}}$ for any $n \in \mathcal{N}$ satisfying the properties of Lemma 7.3 for the lift $a-k$ and parameter $\kappa = 1$, in particular $N_a(\ell+\mathfrak{v}_n, n) \geq k_n + 1$. Then, for every subset $\mathcal{I} \supseteq \mathcal{J}$ we set

$$\mathrm{gr}_k\mathcal{M}^-_{a,\mathcal{I}} := \{\ell \in (\square - \mathbf{c}_{a-k}) \cap \mathbb{Z}^{\mathcal{N}}(a) \mid N_a(\ell, n) < 0\ \forall n \in \mathcal{I}\setminus\mathcal{J} \text{ and } N_a(\ell, n) = k_n\ \forall n \in \mathcal{J}\}.$$

The next lemma gives a rational form of the series $\mathcal{H}_{\mathcal{M}_a(k)}$ and $\mathcal{H}_{\mathrm{gr}_k\mathcal{M}_a}$ in terms of holes.

Lemma 7.4

(a) $\mathcal{H}_{\mathcal{M}_a(k)}(\mathbf{t}_{\mathcal{N}}) = \displaystyle\sum_{\emptyset\subseteq\mathcal{I}\subseteq\mathcal{N}} (-1)^{|\mathcal{I}|} \frac{\sum_{\ell\in\mathcal{M}^-_{a-k,\mathcal{I}}} \mathbf{t}^\ell}{\prod_{n\notin\mathcal{I}}(1-\mathbf{t}^{\mathfrak{v}_n})}$,

(b) $\mathcal{H}_{\mathrm{gr}_k\mathcal{M}_a}(\mathbf{t}_{\mathcal{N}}) = \displaystyle\sum_{\mathcal{J}\subseteq\mathcal{I}\subseteq\mathcal{N}} (-1)^{|\mathcal{I}\setminus\mathcal{J}|} \frac{\sum_{\ell\in\mathrm{gr}_k\mathcal{M}^-_{a,\mathcal{I}}} \mathbf{t}^\ell}{\prod_{n\notin\mathcal{I}}(1-\mathbf{t}^{\mathfrak{v}_n})}$.

Proof Remark 7.7 and Corollary 7.1 applied for $\mathcal{M}_{a-k}$ deduce the formula of (a). For part (b) we give an inclusion-exclusion formula for $\mathrm{gr}_k\mathcal{M}_a$, which will imply the formula for $\mathcal{H}_{\mathrm{gr}_k\mathcal{M}_a}(\mathbf{t}_{\mathcal{N}})$. Thus, denote $\mathcal{L}_{a-k} := \{\ell \in \mathbb{Z}^{\mathcal{N}}(a) \mid N_{a-k}(\ell, j) \leq 0\ \forall j \in \mathcal{J}\}$ and note that $\mathrm{gr}_k\mathcal{M}_a = \mathcal{M}_{a-k} \cap \mathcal{L}_{a-k}$. The inclusion-exclusion formula for $\mathrm{gr}_k\mathcal{M}_a$ will be deduced from the following, given by the Proposition 7.2(b),

$$\mathcal{M}_{a-k} = \sum_{\emptyset\subseteq\mathcal{I}'\subseteq\mathcal{N}} (-1)^{|\mathcal{I}'|} \left(\mathcal{M}^-_{a-k,\mathcal{I}'} + \mathbb{Z}_{\geq 0}\langle\mathfrak{v}_i\rangle_{i\notin\mathcal{I}'}\right)$$

by intersecting it with $\mathcal{L}_{a-k}$. Therefore, we deduce the identity

$$\left(\mathcal{M}^-_{a-k,\mathcal{I}'} + \mathbb{Z}_{\geq 0}\langle\mathfrak{v}_i\rangle_{i\notin\mathcal{I}'}\right) \cap \mathcal{L}_{a-k} = \left(\mathcal{M}^-_{a-k,\mathcal{I}'} \cap \mathcal{L}_{a-k}\right) + \mathbb{Z}_{\geq 0}\langle\mathfrak{v}_i\rangle_{i\notin\mathcal{I}'\cup\mathcal{J}}. \tag{7.14}$$

Indeed, for $\ell = \ell_0 + \sum_{i\notin\mathcal{I}'} \lambda_i\mathfrak{v}_i$ in the left hand side of (7.14) we have $0 \geq N_{a-k}(\ell, j) = N_{a-k}(\ell_0 + \lambda_j\mathfrak{v}_j, j)$ for any $j \in \mathcal{J}\setminus\mathcal{I}'$, which implies $\lambda_j = 0$ by the choice of vectors $\mathfrak{v}_n$, $n \in \mathcal{N}$ (cf. Lemma 7.3(b)). Furthermore, we have inclusion-exclusion formula for the graded holes

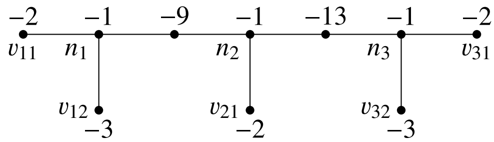
$$\mathrm{gr}_k\mathcal{M}^-_{a,\mathcal{I}} = \sum_{\substack{\mathcal{I}'\subseteq\mathcal{I}\\ \mathcal{I}'\cup\mathcal{J}=\mathcal{I}}} (-1)^{|\mathcal{I}'\cap\mathcal{J}|} \left(\mathcal{M}^-_{a-k,\mathcal{I}'} \cap \mathcal{L}_{a-k}\right).$$

Finally, combining the above results we get

$$\mathrm{gr}_k\mathcal{M}_a = \mathcal{M}_{a-k} \cap \mathcal{L}_{a-k} = \sum_{\mathcal{J}\subseteq\mathcal{I}\subseteq\mathcal{N}} (-1)^{|\mathcal{I}\setminus\mathcal{J}|} \left(\mathrm{gr}_k\mathcal{M}^-_{a,\mathcal{I}} + \mathbb{Z}_{\geq 0}\langle\mathfrak{v}_i\rangle_{i\notin\mathcal{I}}\right),$$

which implies the formula for $\mathcal{H}_{\mathrm{gr}_k \mathcal{M}_a}(\mathbf{t}_{\mathcal{N}})$. □

Example 7.4 Consider the following plumbing graph Γ:



Notice that H is trivial, hence we describe the non-normal affine monoid $\mathcal{M}_0$ associated with $a = 0$. Let n_1, n_2, n_3 be the nodes of Γ and denote by $v_{11}, v_{12}, v_{21}, v_{31}, v_{32}$ the corresponding ends as shown on the above figure. The generalized Seifert invariants are the followings: $(\alpha_{v_{11}}, \omega_{v_{11}}) = (2,1)$, $(\alpha_{v_{12}}, \omega_{v_{12}}) = (3,1)$, $(\alpha_{n_1,n_2}, \omega_{n_1,n_2}) = (9,1)$, $(\alpha_{v_{21}}, \omega_{v_{21}}) = (2,1)$, $(\alpha_{n_2,n_3}, \omega_{n_2,n_3}) = (13,1)$, $(\alpha_{v_{31}}, \omega_{v_{31}}) = (2,1)$ and $(\alpha_{v_{32}}, \omega_{v_{32}}) = (3,1)$. Then,

$$\mathcal{M}_0 = \left\{ \ell = \sum_{i=1}^{3} \ell_i E_{n_i} \in \mathbb{Z}\langle E_i\rangle_{i=1}^3 \ : \ \begin{array}{l} N_0(\ell, n_1) = \frac{8}{9}\ell_1 - \frac{1}{9}\ell_2 + \left\lfloor \frac{-\ell_1}{2} \right\rfloor + \left\lfloor \frac{-\ell_1}{3} \right\rfloor \geq 0 \\ N_0(\ell, n_2) = \frac{95}{117}\ell_2 - \frac{1}{9}\ell_1 - \frac{1}{13}\ell_3 + \left\lfloor \frac{-\ell_2}{2} \right\rfloor \geq 0 \\ N_0(\ell, n_3) = \frac{12}{13}\ell_3 - \frac{1}{13}\ell_2 + \left\lfloor \frac{-\ell_3}{2} \right\rfloor + \left\lfloor \frac{-\ell_3}{3} \right\rfloor \geq 0 \\ \ell_1 + \ell_2 \equiv 0 \ (\mathrm{mod}\ 9) \\ \ell_2 + \ell_3 \equiv 0 \ (\mathrm{mod}\ 13) \end{array} \right\}.$$

Through the example we will use short notation (ℓ_1, ℓ_2, ℓ_3) for $\ell = \ell_1 E_{n_1} + \ell_2 E_{n_2} + \ell_3 E_{n_3}$. One can take the generators $\mathfrak{v}_1 = 1/3 \cdot \pi_{\mathcal{N}}(E_1^*) = (62, 28, 24)$, $\mathfrak{v}_2 = \pi_{\mathcal{N}}(E_2^*) = (84, 42, 36)$ and $\mathfrak{v}_3 = 1/3 \cdot \pi_{\mathcal{N}}(E_3^*) = (24, 12, 14)$ satisfying properties of Lemma 7.3. They provide the following sets

$$\begin{aligned} \square = \{&(0,0,0), (12,6,7), (31,14,12), (42,21,18), (43,20,19), \\ &(54,27,25), (73,35,30), (85,41,37)\}, \\ \mathcal{M}^-_{0,n_1} &= \{(31,14,12), (43,20,19), (73,35,30), (85,41,37)\}, \\ \mathcal{M}^-_{0,n_2} = \emptyset \ \text{ and } \ \mathcal{M}^-_{0,n_3} &= \{(12,6,7), (43,20,19), (54,27,25), (85,41,37)\}. \end{aligned}$$

Hence, the holes of $\mathcal{M}_0$ are elements in the following forms:

$$\begin{array}{ll} (31,14,12) + \lambda_1 \cdot (84,42,36) + \lambda_2 \cdot (24,12,14), & (43,20,19) + \lambda_3 \cdot (84,42,36) + \lambda_4 \cdot (24,12,14), \\ (73,35,30) + \lambda_5 \cdot (84,42,36) + \lambda_6 \cdot (24,12,14), & (85,41,37) + \lambda_7 \cdot (84,42,36) + \lambda_8 \cdot (24,12,14), \\ (12,6,7) + \lambda_9 \cdot (62,28,24) + \lambda_{10} \cdot (84,42,36), & (43,20,19) + \lambda_{11} \cdot (62,28,24) + \lambda_{12} \cdot (84,42,36), \\ (54,27,25) + \lambda_{13} \cdot (62,28,24) + \lambda_{14} \cdot (84,42,36), & (85,41,37) + \lambda_{15} \cdot (62,28,24) + \lambda_{16} \cdot (84,42,36). \end{array}$$

for any $\lambda_i \in \mathbb{Z}_{\geq 0}$. The computations were performed using Maple. □

Remark 7.8 By Remark 7.6(b) we could choose 'smaller' generators, eg. one can take $\mathfrak{v}_2' := 1/2 \cdot \pi_{\mathcal{N}}(E_2^*) = (42, 21, 18)$ instead of $\mathfrak{v}_2$. However, the generators $\mathfrak{v}_1, \mathfrak{v}_2', \mathfrak{v}_3$ cannot be used for Lemma 7.4, since $N_0(\mathfrak{v}_2', n_2) = 0$. Notice that for our choice we have $N_0(\mathfrak{v}_2, n_2) = 1$.

7.5 Rational representation of $Z_h(\mathbf{t}_{\mathcal{N}})$

In this section we express $Z_h(\mathbf{t}_{\mathcal{N}})$ with rational functions given by the structure of the filtered $\mathcal{M}_0$-module $\mathcal{M}_a$. We assume that $\delta_{n,\mathcal{N}} \geq 1$ for every $n \in \mathcal{N}$ and consider the filtration in Section 7.3.2 associated with the subset $\widehat{\mathcal{N}} \subset \mathcal{N}$.

In the next theorem we give two formulas for $Z_h(\mathbf{t}_{\mathcal{N}})$. The first uses generating series of the submodules $\mathcal{M}_a(k)$, while the second formula is a rational representation in terms of generating functions associated with the graded pieces $\mathrm{gr}_k \mathcal{M}^-_{a,\mathcal{I}}$ of the holes.

Recall the notations $(-1)^k \binom{\delta-2}{k} := \prod_{n\in\widehat{\mathcal{N}}} (-1)^{k_n} \binom{\delta_{n,\mathcal{N}}-2}{k_n}$ and $0 \leq k \leq \delta - 2$ for $0 \leq k_n \leq \delta_{n,\mathcal{N}} - 2$ for all $n \in \widehat{\mathcal{N}}$.

Theorem 7.1 *Let a be a reduced lift of $h \in H$.*

(a) Let $\{\mathfrak{v}_n\}_{n\in\mathcal{N}}$ be a set of vectors satisfying properties of Lemma 7.3 for every lift $a-k$ with $0 \leq k \leq \delta - 1$ and parameter $\kappa = 0$, ie. $N_a(\ell + \mathfrak{v}_n, n) \geq k_n$ for all $n \in \mathcal{N}$ and $\ell \in \overline{\mathcal{M}}_{a-k}$. Then

$$Z_h(\mathbf{t}_{\mathcal{N}}) = \sum_{0\leq k\leq \delta-1} (-1)^k \binom{\delta-1}{k} \sum_{\emptyset\subseteq \mathcal{I}\subseteq\mathcal{N}} (-1)^{|\mathcal{I}|} \frac{\sum_{\ell\in\mathcal{M}^-_{a-k,\mathcal{I}}} \mathbf{t}^{\mathbf{c}_a+\ell}}{\prod_{n\notin\mathcal{I}}(1-\mathbf{t}^{\mathfrak{v}_n})}.$$

(b) Let $\{\mathfrak{v}_n\}_{n\in\mathcal{N}}$ be a set of vectors satisfying properties of Lemma 7.3 for every lift $a-k$ with $0 \leq k \leq \delta - 2$ and parameter $\kappa = 1$, ie. $N_a(\ell + \mathfrak{v}_n, n) \geq k_n + 1$ for all $n \in \mathcal{N}$ and $\ell \in \overline{\mathcal{M}}_{a-k}$. Then

$$Z_h(\mathbf{t}_{\mathcal{N}}) = \sum_{0\leq k\leq \delta-2} (-1)^k \binom{\delta-2}{k} \sum_{\widehat{\mathcal{N}}\subseteq \mathcal{I}\subseteq\mathcal{N}} (-1)^{|\mathcal{I}\setminus\widehat{\mathcal{N}}|} \frac{\sum_{\ell\in\mathrm{gr}_k\mathcal{M}^-_{a,\mathcal{I}}} \mathbf{t}^{\mathbf{c}_a+\ell}}{\prod_{n\notin\mathcal{I}}(1-\mathbf{t}^{\mathfrak{v}_n})}.$$

Proof (a) By the decomposition (7.7) we have

$$Z_h(\mathbf{t}_{\mathcal{N}}) = \mathbf{t}^{\mathbf{c}_a} \cdot \sum_{0\leq k\leq\delta-1} (-1)^k \binom{\delta-1}{k} \mathcal{H}_{\mathcal{M}_a(k)}(\mathbf{t}_{\mathcal{N}}).$$

Applying Lemma 7.4(a), we get the desired rational form.

(b) Note that $\mathcal{S}_a = \bigsqcup_{0\leq k\leq\delta-2} \mathrm{gr}_k \mathcal{M}_a$, thus $Z_h(\mathbf{t}_{\mathcal{N}}) = \sum_{0\leq k\leq\delta-2} (-1)^k \binom{\delta-2}{k} \mathcal{H}_{\mathrm{gr}_k\mathcal{M}_a}(\mathbf{t}_{\mathcal{N}})$ by (7.6). Finally, Lemma 7.4(b) implies the formula. □

Remark 7.9 Sometimes it is simpler to choose 'universal' generators $\mathfrak{v}_n$ of $\mathcal{M}_0$ satisfying the properties of Lemma 7.3(a) and $\overline{N}_0(\mathfrak{v}_n, n) \geq \delta_n - 1$ for every $n \in \mathcal{N}$. In general, they are larger, but their properties are easier to check.

Example 7.6 We continue the example 7.4. Recall that the chosen generators of $\mathcal{M}_0$ are $\mathfrak{v}_1 = (62, 28, 24)$, $\mathfrak{v}_2 = (84, 42, 36)$ and $\mathfrak{v}_3 = (24, 12, 14)$. Consider the filtration defined by Section 7.3.2 associated with the subset $\widehat{\mathcal{N}} = \{n_2\}$. Then, Theorem 7.1 (b) implies the following rational form

$$Z(\mathbf{t}_\mathcal{N}) = \frac{\sum_{\mathrm{gr}_0\mathcal{M}^-_{0,n_2}} \mathbf{t}^\ell}{(1-\mathbf{t}^{(62,28,24)})(1-\mathbf{t}^{(24,12,14)})} - \frac{\sum_{\mathrm{gr}_0\mathcal{M}^-_{0,\{n_1,n_2\}}} \mathbf{t}^\ell}{1-\mathbf{t}^{(24,12,14)}} - \frac{\sum_{\mathrm{gr}_0\mathcal{M}^-_{0,\{n_2,n_3\}}} \mathbf{t}^\ell}{1-\mathbf{t}^{(62,28,24)}} + \sum_{\mathrm{gr}_0\mathcal{M}^-_{0,\{n_1,n_2,n_3\}}} \mathbf{t}^\ell,$$

where $\mathrm{gr}_0\mathcal{M}^-_{0,\mathcal{I}} = \{\ell \in \square \cap \mathbb{Z}^3(0) \mid N_0(\ell, n_i) < 0 \ \ \forall n_i \in \mathcal{I} \setminus n_2 \ \text{ and } \ N_0(\ell, n_2) = 0\}$ for every $\{n_2\} \subseteq \mathcal{I} \subseteq \{n_1, n_2, n_3\}$. Thus, calculating the above associated graded sets we get

$$\begin{aligned}
\mathrm{gr}_0\mathcal{M}^-_{0,n_2} &= \{(0,0,0), (12,6,7), (31,14,12), (42,21,18), (43,20,19),\\
&\qquad (54,27,25), (73,35,30), (85,41,37)\},\\
\mathrm{gr}_0\mathcal{M}^-_{0,\{n_1,n_2\}} &= \{(31,14,12), (43,20,19), (73,35,30), (85,41,37)\},\\
\mathrm{gr}_0\mathcal{M}^-_{0,\{n_2,n_3\}} &= \{(12,6,7), (43,20,19), (54,27,25), (85,41,37)\} \text{ and}\\
\mathrm{gr}_0\mathcal{M}^-_{0,\{n_1,n_2,n_3\}} &= \{(43,20,19), (85,41,37)\}.
\end{aligned}$$

Chapter 8

Duality and the polynomial part of the topological Poincaré series

8.1 Ehrhart–Macdonald–Stanley duality for rational functions

8.1.1 Taylor expansion at infinity

Similarly as in the previous chapters we fix a free $\mathbb{Z}$–module L, another one $L' \supset L$ of the same rank (but in this context L' is not necessarily the dual of L, in fact L carries no intersection form at all). We write H for L'/L, d for the order of H, and we fix a basis $\{E_v\}_{v\in\mathcal{V}}$ in L.

We consider multivariable rational functions (in variables $\mathbf{t}^{L'}$) of type

$$z(\mathbf{t}) = \frac{\sum_{k=1}^{r} \iota_k \mathbf{t}^{b_k}}{\prod_{i=1}^{n}(1-\mathbf{t}^{a_i})}, \tag{8.1}$$

where $\{\iota_k\}_{k=1}^{r} \in \mathbb{Z}$, $\{b_k\}_{k=1}^{r}$, $\{a_i\}_{i=1}^{n} \in L'$ and for any $l' = \sum_v l'_v E_v \in L'$ we write $\mathbf{t}^{l'} = t_1^{l'_1} \dots t_s^{l'_s}$. We also assume that all the coordinates $a_{i,v}$ of a_i are strict positive.

Alongside the Taylor expansion $Tz(\mathbf{t})$ of $z(\mathbf{t})$ at the origin

$$Tz(\mathbf{t}) = \sum_{l'} z(l')\mathbf{t}^{l'} \in \mathbb{Z}[[\mathbf{t}^{1/d}]][\mathbf{t}^{-1/d}] := \mathbb{Z}[[t_1^{1/d}, \dots, t_s^{1/d}]][t_1^{-1/d}, \dots, t_s^{-1/d}],$$

we also define the Taylor expansion of $z(\mathbf{t})$ at infinity

$$T^{\infty}z(\mathbf{t}) = \sum_{\tilde{l}} r(\tilde{l})\mathbf{t}^{\tilde{l}} \in \mathbb{Z}[[\mathbf{t}^{-1/d}]][\mathbf{t}^{1/d}].$$

$T^{\infty}z$ is obtained by the substitution $\mathbf{s} = 1/\mathbf{t}$ into the Taylor expansion at $\mathbf{s} = 0$ of the function $z(1/\mathbf{s})$. E.g., if $z(t) = t^2/(1-t)$, then $T^{\infty}z(t) = -t(1 + t^{-1} + t^{-2} + \cdots)$. Note that $\sum_{\tilde{l}\geq 0} r(\tilde{l})$ is a finite sum.

8.1.2 Restrictions

The function z has equivariant decomposition $\sum_h z_h$ ($h \in H$) with respect to $H = L'/L$, where $z_h(\mathbf{t})$ is rational of form $\sum_{b' \in h+L} \iota_{b'} \mathbf{t}^{b'} / \prod_{i=1}^n (1 - \mathbf{t}^{|H|a_i})$ ($\iota_{b'} \neq 0$ for finite b'). While the decompositions $\sum_h Tz_h$ and $\sum_h (T^\infty z)_h$ of the series Tz and $T^\infty z$ are defined similarly as the decomposition of Z in 3.2. (Note that the operation $Tz \mapsto T^\infty z$, defined via the original z, preserves all the h–components.)

We also might eliminate some of the variables: for any subset $\mathcal{I} \subset \mathcal{V}$ we substitute $t_i = 1$ in $z(\mathbf{t})$ for all $i \notin \mathcal{I}$; in this way we obtain $z(\mathbf{t}_\mathcal{I})$. We call this procedure 'reduction'. This procedure at the level of Tz is a summation of some of the coefficients. Since the series associated with the denominator of z is supported in the cone $\mathbb{Z}_{\geq 0}\langle a_i \rangle$, the summation is finite.

Note that the H–decomposition of the restricted functions are not well–defined. That is, from the restricted function $z(\mathbf{t}_\mathcal{I}) := z(\mathbf{t})|_{t_i=1, i \notin \mathcal{I}}$ in general one cannot recover anymore the restriction of the H–component $z_h(\mathbf{t})$. (That is, from the exponent of $\mathbf{t}_\mathcal{I}^{l'}$ one cannot recover $[l'] \in H$. Here, and in the sequel, for any $l' \in L'$ we write $\mathbf{t}_\mathcal{I}^{l'}$ for $\prod_{v \in \mathcal{I}} t_v^{l'_v}$.) Hence, in the presence of decomposition and reduction the only well–defined object is $(z_h)|_{t_i=1, i \notin \mathcal{I}}$, the reduction of z_h, which will be denoted by $z_h(\mathbf{t}_\mathcal{I})$. Furthermore, if $\pi_\mathcal{I}$ is the natural projection associated with the reduction (elimination of the $\mathcal{V} \setminus \mathcal{I}$–components), then $\pi_\mathcal{I}(L')/\pi_\mathcal{I}(L)$ usually is not isomorphic to H. Hence, we keep H as an index set and we never consider $\pi_\mathcal{I}(L')/\pi_\mathcal{I}(L)$.

8.1.3 The modified counting function

We fix $h \in H$ and $\mathcal{I} \subset \mathcal{V}$. We define two functions associated with the coefficients of $Tz_h(\mathbf{t}_\mathcal{I})$: the first is called the *counting function*, already defined in 4.2.2.3:

$$Q_{h,\mathcal{I}} : L'_h := \{x \in L' \,:\, [x] = h\} \to \mathbb{Z}, \quad Q_{h,\mathcal{I}}(x) := \sum_{l'|_\mathcal{I} \ngeq x|_\mathcal{I},\, [l']=[x]} z(l'). \tag{8.2}$$

Recallthat the above sum is finite by strict positivity assumption on the coordinates $a_{i,v}$ of a_i. The definition selects only the coefficients of $Tz_h(\mathbf{t}_\mathcal{I})$, hence if we write $Tz_h(\mathbf{t}_\mathcal{I})$ as $\sum_{k' \in \pi_\mathcal{I} L'} z^{(h)}(k')\mathbf{t}_\mathcal{I}^{k'}$, then $Q_{h,\mathcal{I}}(x) = \sum_{k' \ngeq x|_\mathcal{I}} z^{(h)}(k')$. It depends only on $x|_\mathcal{I}$. This truncation and counting function does not (naturally) appear in Ehrhart theory, but this is what is imposed by Theorem 3.1.

Our second function is called the *modified counting function*; it is defined by

$$q_{h,\mathcal{I}} : L'_h \to \mathbb{Z}, \quad q_{h,\mathcal{I}}(x) := \sum_{l'|_\mathcal{I} < x|_\mathcal{I},\ [l']=[x]} z(l'), \tag{8.3}$$

where for any $a, b \in \mathbb{R}^{|\mathcal{I}|}$ we say that $a < b$ if $a_v < b_v$ for *all* $v \in \mathcal{I}$. Similarly as above, one also has $q_{h,\mathcal{I}}(x) = \sum_{k' < x|_\mathcal{I}} z^{(h)}(k')$. For any h and $\mathcal{I}$ by inclusion–exclusion principle

$$Q_{h,\mathcal{I}}(x) = \sum_{\emptyset \neq \mathcal{J} \subset \mathcal{I}} (-1)^{|\mathcal{J}|+1} q_{h,\mathcal{J}}(x). \tag{8.4}$$

This modified counting function will be related to the usual counting functions from the Ehrhart theory. It satisfies several useful properties (e.g. convexity, reciprocity, see below).

8.1.4 Ehrhart theory associated with z revisited: the 'modified' case

In this subsection we follow the theory explained in section 4.3 and 5, taking into account some modification given by the work of [46].

Recall that associated with the vectors $\{a_i\}_{i=1}^n$ (the exponents in the denominator of $z(\mathbf{t})$) we define two objects. Firstly, let $\mathfrak{l} : \mathbb{R}^n \to L' \otimes \mathbb{R}$ be the map given by $\mathfrak{l}(\mathbf{y}) = \sum_{i=1}^n y_i a_i$ and consider the representation $\rho : \mathbb{Z}^n \to H$ defined by the composition $\mathbb{Z}^n \xrightarrow{\mathfrak{l}|_{\mathbb{Z}^n}} L' \to L'/L$.

Then, the vectors $\{a_i\}_{i=1}^n$ and any $\mathcal{I} \subset \mathcal{V}, \mathcal{I} \neq \emptyset$ (which might vary, cf. (8.4)) determine the family of closed dilated convex polytopes with dilatation parameter $l' = \sum_v l'_v E_v \in L'$

$$\mathcal{P}_{\mathcal{I}}^{(l')} := \{\mathbf{y} \in (\mathbb{R}_{\geq 0})^n \ : \ \sum_i y_i a_{i,v} \leq l'_v \ \text{ for all } v \in \mathcal{I}\}. \tag{8.5}$$

$\mathcal{P}_{\mathcal{I}}^{(l')}$ depends only on $l'_{\mathcal{I}} := l'|_{\mathcal{I}}$. We denote by $\mathcal{F}_{\mathcal{I}}^{(l')}$ the set of facets of $\mathcal{P}_{\mathcal{I}}^{(l')}$.

By varying $l'_{\mathcal{I}}$ in some region we say that the polytopes $\mathcal{P}_{\mathcal{I}}^{(l')}$ remain *combinatorially stable (or preserve their combinatorial type)* if they are equivalent up to homeomorphisms, which preserve the stratification of the faces.

By the results of [107, 46], we consider the variation of the combinatorial type of $\mathcal{P}_{\mathcal{I}}^{(l')}$ will be determined by the following chamber decomposition of $\pi_{\mathcal{I}}(L' \otimes \mathbb{R}) = \mathbb{R}^{|\mathcal{I}|}$: let $\mathcal{B}_{\mathcal{I}}$ be the set of all bases $\sigma \subset \{a_i|_{\mathcal{I}}, E_v|_{\mathcal{I}} \ : \ i \in \{1, \ldots, n\}, v \in \mathcal{I}\}$ of $\mathbb{R}^{|\mathcal{I}|}$. Then a (big, open) chamber $\mathfrak{c}$ is a connected component of $\mathbb{R}^{|\mathcal{I}|} \setminus \cup_{\sigma \in \mathcal{B}_{\mathcal{I}}} \partial \mathbb{R}_{\geq 0}\sigma$, where $\partial \mathbb{R}_{\geq 0}\sigma$ is the boundary of the closed cone $\mathbb{R}_{\geq 0}\sigma$.

Then $\mathcal{P}_{\mathcal{I}}^{(l')}$ stays combinatorially stable if $l'_{\mathcal{I}}$ moves in such a chamber $\mathfrak{c}$. In such a case we can associate to the stable combinatorial type the set (dilated family) of facets $\mathcal{F}_{\mathcal{I}}^{(l')}$. Moreover, any choice of a subset of facets in a fixed stable topological type provides a 'stable' (dilated) subset of facets $\mathcal{G}_{\mathcal{I}}^{(l')}$ in each $\mathcal{F}_{\mathcal{I}}^{(l')}$; we denote this choice by $\mathcal{G}_{\mathcal{I}} \subset \mathcal{F}_{\mathcal{I}}$. Furthermore, for any $h \in H$ and choice $\mathcal{G}_{\mathcal{I}} \subset \mathcal{F}_{\mathcal{I}}$ we consider the counting function of specially chosen lattice points identified by

$$Q_{h,\mathcal{G}_{\mathcal{I}}}(l') := \text{cardinality of } \big((\mathcal{P}_{\mathcal{I}}^{(l')} \setminus \mathcal{G}_{\mathcal{I}}^{(l')}) \cap \rho^{-1}(h) \big). \tag{8.6}$$

According to the equivariant Ehrhart theory (cf. corollary 4.1), applied to the dilated polytopes $\mathcal{P}_{\mathcal{I}}^{(l')}$ (parametrized by chamber $\mathfrak{c}$), and for any $h \in H$ and $\mathcal{G}_{\mathcal{I}}$, the counting function $Q_{h,\mathcal{G}_{\mathcal{I}}}(l'_{\mathcal{I}})$ is a quasipolynomial, denoted by $Q^{\mathfrak{c}}_{h,\mathcal{G}_{\mathcal{I}}}$.

The above construction shows that the chamber decomposition is independent of the choices of $h \in H$ and $\mathcal{G}_{\mathcal{I}} \subset \mathcal{F}_{\mathcal{I}}$.

One can extend $Q^{\mathfrak{c}}_{h,\mathcal{G}_\mathcal{I}}$ continuously to the closure $\bar{\mathfrak{c}}$ of $\mathfrak{c}$ as a quasipolynomial; all these extensions glue together to a continuous piecewise quasipolynomial of $\mathbb{R}^{|\mathcal{I}|}$. This piecewise quasipolynomial is the expression $Q_{h,\mathcal{G}_\mathcal{I}}$ from (8.6). (This means that $Q_{h,\mathcal{G}_\mathcal{I}}(l'_\mathcal{I})$ in terms of $\{Q^{\mathfrak{c}}_{h,\mathcal{G}_\mathcal{I}}\}_\mathfrak{c}$ can be redefined as follows: for any $l'_\mathcal{I}\in\mathbb{R}^{|\mathcal{I}|}$ first find a chamber $\mathfrak{c}$ such that $l'_\mathcal{I}\in\bar{\mathfrak{c}}$ and then set $Q_{h,\mathcal{G}_\mathcal{I}}(l'_\mathcal{I}):=Q^{\mathfrak{c}}_{h,\mathcal{G}_\mathcal{I}}(l'_\mathcal{I})$.)

By the *equivariant Ehrhart–Macdonald–Stanley reciprocity law* (Theorem 4.2(c)), for any fixed h, $\mathcal{G}_\mathcal{I}\subset\mathcal{F}_\mathcal{I}$ and chamber $\mathfrak{c}$ with $l'_\mathcal{I}\in\mathfrak{c}$ one has

$$Q^{\mathfrak{c}}_{h,\mathcal{G}_\mathcal{I}}(l'_\mathcal{I})=(-1)^n\cdot Q^{\mathfrak{c}}_{-h,\mathcal{F}_\mathcal{I}\setminus\mathcal{G}_\mathcal{I}}(-l'_\mathcal{I}). \tag{8.7}$$

(We warn the reader that usually the parameter $-l'_\mathcal{I}$ is included in a different chamber than $\mathfrak{c}$, that is $Q^{\mathfrak{c}}_{-h,\mathcal{F}_\mathcal{I}\setminus\mathcal{G}_\mathcal{I}}(-l'_\mathcal{I})\neq Q_{-h,\mathcal{F}_\mathcal{I}\setminus\mathcal{G}_\mathcal{I}}(-l'_\mathcal{I})$; in (8.7) in both sides we use the same chamber $\mathfrak{c}$, and from the pair $l'_\mathcal{I}, -l'_\mathcal{I}$ only one of them can be in $\bar{\mathfrak{c}}$ provided that $l'_\mathcal{I}\neq 0$.)

A distinguished subset of facets $\mathcal{G}_\mathcal{I}\subset\mathcal{F}_\mathcal{I}$ is defined as the coordinate facets $\{\mathcal{P}^{(l')}_\mathcal{I}\cap\{y_i=0\}\}_{i=1}^n$. We denote it by $\mathcal{G}^{co}_\mathcal{I}$.

Then the theory above has the following consequences regarding the modified counting function $q_{h,\mathcal{I}}$ of $Tz_h(\mathbf{t}_\mathcal{I})$ defined in (8.3):

Corollary 8.1 *We fix h and $\mathcal{I}$.*

(a) $q_{h,\mathcal{I}}$ is a piecewise quasipolynomial, which can be written as

$$q_{h,\mathcal{I}}(l')=\sum_k \iota_k\cdot Q_{h-[b_k],\mathcal{F}_\mathcal{I}\setminus\mathcal{G}^{co}_\mathcal{I}}(l'_\mathcal{I}-b_k|_\mathcal{I}). \tag{8.8}$$

(b) For a fixed chamber $\mathfrak{c}$ of $\mathbb{R}^{|\mathcal{I}|}$ define the quasipolynomial

$$q^{\mathfrak{c}}_{h,\mathcal{I}}(l'):=\sum_k \iota_k\cdot Q^{\mathfrak{c}}_{h-[b_k],\mathcal{F}_\mathcal{I}\setminus\mathcal{G}^{co}_\mathcal{I}}(l'_\mathcal{I}-b_k|_\mathcal{I}). \tag{8.9}$$

Then the restriction of $q_{h,\mathcal{I}}$ to (the closure of) $\cap_k(b_k|_\mathcal{I}+\mathfrak{c})$ behaves as a quasipolynomial, that is, $q_{h,\mathcal{I}}(l')=q^{\mathfrak{c}}_{h,\mathcal{I}}(l')$ for $l'_\mathcal{I}\in\cap_k(b_k|_\mathcal{I}+\mathfrak{c})$.

(c) For any fixed chamber $\mathfrak{c}$ the modified counting function $q_{h,\mathcal{I}}$ admits a quasipolynomial in the sense of section 4.4, namely $\mathfrak{q}^{\mathfrak{c}}_{h,\mathcal{I}}$, which satisfies for $l'=l+r_h$ the identity $\mathfrak{q}^{\mathfrak{c}}_{h,\mathcal{I}}(l)=q^{\mathfrak{c}}_{h,\mathcal{I}}(l')$, and admits also periodic constant $\mathrm{pc}^{\mathfrak{c}}(q_{h,\mathcal{I}})=\mathfrak{q}^{\mathfrak{c}}_{h,\mathcal{I}}(0)=q^{\mathfrak{c}}_{h,\mathcal{I}}(r_h)$ associated with the chamber $\mathfrak{c}$.

This $\mathrm{pc}^{\mathfrak{c}}(q_{h,\mathcal{I}})$ will be denoted by $\mathrm{mpc}^{\mathfrak{c}}(Tz_h(\mathbf{t}_\mathcal{I}))$. We call it the *modified periodic constant* of $Tz_h(\mathbf{t}_\mathcal{I})$ associated with h, $\mathcal{I}$ and the chamber $\mathfrak{c}$. (The terminology and the notation emphasize the presence of different cuts in the counting functions.)

8.1.5 The duality theorem

In general, the computation of quasipolynomials and their periodic constants (either modified or not) is hard: it measures the asymptotic behaviour of the coefficients in a certain cones. The next result based on the equivariant Ehrhart–Macdonald–Stanley reciprocity (8.7) shows that (under some conditions) the modified periodic constant associated with a chamber $\mathfrak{c}$ equals a *finite sum of certain coefficients of the Taylor expansion at infinity*.

Theorem 8.1 *Fix h and $\mathcal{I}$ as above. Write the h–component of the Taylor expansion at infinity as $(T^\infty z)_h(\mathbf{t}_\mathcal{I}) = \sum_{\tilde{l}} r_\mathcal{I}^{(h)}(\tilde{l})\mathbf{t}_\mathcal{I}^{\tilde{l}}$. Assume that the chamber $\mathfrak{c}$ has the property that $b_k|_\mathcal{I} \in \mathfrak{c}$ for all k. Then*

$$\mathrm{mpc}^\mathfrak{c}(Tz_h(\mathbf{t}_\mathcal{I})) = \sum_{\tilde{l}\geq 0} r_\mathcal{I}^{(h)}(\tilde{l}). \tag{8.10}$$

Proof By Corollary 8.1 the function $q_{h,\mathcal{I}}$ on the set $\cap_k(b_k|_\mathcal{I}+\mathfrak{c})$ is $\sum_k \iota_k \cdot Q^\mathfrak{c}_{h-[b_k],\mathcal{F}_\mathcal{I}\setminus\mathcal{G}_\mathcal{I}^{co}}(l'_\mathcal{I} - b_k|_\mathcal{I})$. Hence, by definitions, $\mathrm{mpc}^\mathfrak{c}(Tz_h(\mathbf{t}_\mathcal{I})) = \mathrm{pc}^\mathfrak{c}(q_{h,\mathcal{I}})$ exists and equals

$$\sum_k \iota_k \cdot Q^\mathfrak{c}_{h-[b_k],\mathcal{F}_\mathcal{I}\setminus\mathcal{G}_\mathcal{I}^{co}}((r_h - b_k)|_\mathcal{I}).$$

First we claim that

$$Q^\mathfrak{c}_{h-[b_k],\mathcal{F}_\mathcal{I}\setminus\mathcal{G}_\mathcal{I}^{co}}((r_h - b_k)|_\mathcal{I}) = Q^\mathfrak{c}_{h-[b_k],\mathcal{F}_\mathcal{I}\setminus\mathcal{G}_\mathcal{I}^{co}}(-b_k|_\mathcal{I}).$$

Indeed, if $[\sum_i y_i a_i] = h - [b_k]$ and $(\sum_i y_i a_i)_v < (r_h - b_k)_v$ for all $v \in \mathcal{I}$ (where the strictness of the inequality is imposed by the boundary condition $\mathcal{F}_\mathcal{I} \setminus \mathcal{G}_\mathcal{I}^{co}$) then necessarily $(\sum_i y_i a_i)_v < (-b_k)_v$ also holds for $v \in \mathcal{I}$. That is, if $[a] = h$ and $a_v < (r_h)_v$ then $a_v < 0$. To see this write $a = r_h + l$ with $l \in L$, then $l_v < 0$ hence $l_v \leq -1$, and $a_v \leq (r_h)_v - 1 < 0$.

On the other hand, since $b_k \in \mathfrak{c}$, from (8.7) one has

$$(-1)^n \cdot Q^\mathfrak{c}_{-h+[b_k],\mathcal{G}_\mathcal{I}^{co}}(b_k|_\mathcal{I}) = Q^\mathfrak{c}_{h-[b_k],\mathcal{F}_\mathcal{I}\setminus\mathcal{G}_\mathcal{I}^{co}}(-b_k|_\mathcal{I}).$$

The expression $Q^\mathfrak{c}_{-h+[b_k],\mathcal{G}_\mathcal{I}^{co}}(b_k|_\mathcal{I})$ counts solutions of $\sum_i y_i a_{i,v} \leq b_{k,v}$ for all $v \in \mathcal{I}$ under the restrictions $[b_k - \sum_i y_i a_i] = h$ and $y_i > 0$ for all i. On the other hand, in the expansion at infinity,

$$(-1)^n \cdot \frac{\mathbf{t}^{b_k}}{\prod_i(1-\mathbf{t}^{a_i})} \stackrel{T^\infty}{\longrightarrow} \mathbf{t}^{b_k} \cdot \sum_{y_i>0} \mathbf{t}^{-\sum_i y_i a_i}.$$

Hence

$$\sum_k \iota_k \cdot (-1)^n \cdot Q^\mathfrak{c}_{-h+[b_k],\mathcal{G}_\mathcal{I}^{co}}(b_k|_\mathcal{I}) = \sum_{\tilde{l}\geq 0} r_\mathcal{I}^{(h)}(\tilde{l}).$$

8.2 Duality for topological Poincaré series

8.2.1 'Chamber property'

As we already mentioned, the **modified counting function** has some additional nice properties (compared with the original counting function). Regarding it, in the sequel we will use several results from [46] (where $q_{h,\mathcal{I}}$ associated with f is called the 'coefficient function', since $q_{h,\mathcal{I}}(l'_0)$ is the coefficient of $\mathbf{t}_{\mathcal{I}}^{l'_0}$ in the Taylor expansion of $f_h(\mathbf{t}_{\mathcal{I}})\cdot\prod_{v\in\mathcal{I}}\mathbf{t}_{\mathcal{I}}^{E_v}/(1-\mathbf{t}_{\mathcal{I}}^{E_v})$).

If we wish to apply Theorem 8.1 for $q_{h,\mathcal{I}}$, we need to find a chamber associated with the denominator of the rational function $f(\mathbf{t}_{\mathcal{I}})$, which contains all vectors of type $b_k|_{\mathcal{I}}$ where b_k are the exponents appearing in the numerator. The next proposition shows the existence of a chamber which contains the whole projected real Lipman cone. In order to give some intuition for this fact, we also provide an intermediate step of its proof.

For a subset $\mathcal{I}\subset\mathcal{V}$, $\mathcal{I}\neq\emptyset$ we define its closure $\overline{\mathcal{I}}$ as the set of vertices of that *connected* minimal full subgraph $\Gamma_{\overline{\mathcal{I}}}$ of Γ, which contains $\mathcal{I}$. We denote by $\delta_{v,\overline{\mathcal{I}}}$ the valency of a vertex $v\in\overline{\mathcal{I}}$ in the graph $\Gamma_{\overline{\mathcal{I}}}$.

In [46, Lemma 11] is proved that $f(\mathbf{t}_{\mathcal{I}})$ has a product decomposition of type

$$f(\mathbf{t}_{\mathcal{I}})=R(\mathbf{t}_{\mathcal{I}})\cdot\prod_{v\in\overline{\mathcal{I}}}\left(1-\mathbf{t}_{\mathcal{I}}^{E_v^*}\right)^{\delta_{v,\overline{\mathcal{I}}}-2},\tag{8.11}$$

where R is a polynomial supported on $\pi_{\mathcal{I}}(\mathcal{S}')$, in particular it has no pole. Hence, the possible poles of f via the $\mathcal{I}$–reduction are replaced from the set of poles of $\{1-\mathbf{t}^{E_v^*}\}_{v\in\mathcal{E}}$ to the set of poles of $\{1-\mathbf{t}_{\mathcal{I}}^{E_v^*}\}_{v\in\mathcal{E}_{\overline{\mathcal{I}}}}$. Here $\mathcal{E}_{\overline{\mathcal{I}}}$ is the set of end–vertices of $\Gamma_{\overline{\mathcal{I}}}$; note that $\mathcal{E}_{\overline{\mathcal{I}}}\subset\mathcal{I}$. Therefore, by the construction of the chamber decomposition of $\mathbb{R}^{|\mathcal{I}|}$ (cf. 8.1.4) the chambers associated with $f(\mathbf{t}_{\mathcal{I}})$ are determined by the bases selected from $\{E_v^*|_{\mathcal{I}},E_u|_{\mathcal{I}}\ :\ v\in\mathcal{E}(\Gamma_{\overline{\mathcal{I}}}),u\in\mathcal{I}\}$. These facts and a lattice–combinatorial argument provide

Proposition 8.1 *[46] For any $\mathcal{I}$ the interior of the projected Lipman cone* $\mathrm{int}\,(\pi_{\mathcal{I}}(\mathcal{S}'_{\mathbb{R}}))$ *is contained entirely in a (big) chamber $\mathfrak{c}$ of $f(\mathbf{t}_{\mathcal{I}})$.*

Remark 8.1 In [46, 43] is also proved the following 'convexity property': If $l'_0\in Z_K-E+\mathcal{S}'$ and $[l'_0]=h$ then $q_{h,\mathcal{I}}(l'_0)=q_{h,\overline{\mathcal{I}}}(l'_0)$.

8.2.2 Duality for counting functions and periodic constants

Now we are ready to prove the main theorem of this chapter: a duality/pairing between special evaluations of the counting functions associated with f and the periodic constants.

The duality is the upshot of *two 'symmetries'*, manifested at two different levels. The first one is the *equivariant Ehrhart–Macdonald–Stanley reciprocity of the polytopes*, while the second is a topological imprint of the Gorenstein duality present at the level of the topological Poincaré series: a $\{x\leftrightarrow Z_K-x\}$ symmetry.

Theorem 8.2 *Fix any $\mathcal{I}\subset\mathcal{V}$, $\mathcal{I}\neq\emptyset$ and $h\in H$. Then*
(a) $\mathrm{mpc}^{\pi_{\mathcal{I}}(\mathcal{S}'_{\mathbb{R}})}(Z_h(\mathbf{t}_{\mathcal{I}}))=q_{[Z_K]-h,\mathcal{I}}(Z_K-r_h)$;
(b) $\mathrm{pc}^{\pi_{\mathcal{I}}(\mathcal{S}'_{\mathbb{R}})}(Z_h(\mathbf{t}_{\mathcal{I}}))=Q_{[Z_K]-h,\mathcal{I}}(Z_K-r_h)$.

That is, (in principle, the hardly computable) periodic constant of the series Z_h can be determined as a precise finite sum of coefficients of the dual series $Z_{[Z_K]-h}$.

Proof By the inclusion–exclusion principle (8.4) (b) is implied by (a). Next we prove (a).

The substitution $x \mapsto Z_K - x$ in f, together with the identities $Z_K - E = \sum_{v\in\mathcal{V}}(\delta_v - 2)E^*_v$ from (2.4) and $-2 = \sum_{n\in\mathcal{V}}(\delta_n - 2)$ (since Γ is a tree) gives

$$f(\mathbf{t}_\mathcal{I}) = \mathbf{t}_\mathcal{I}^{Z_K-E}\cdot f(\mathbf{t}_\mathcal{I}^{-1}). \tag{8.12}$$

This on the Taylor expansion level transforms into the symmetry $T^\infty f(\mathbf{t}_\mathcal{I}) = \mathbf{t}_\mathcal{I}^{Z_K-E} Z(\mathbf{t}_\mathcal{I}^{-1}) = \sum_{l'\in\mathcal{S}'} z(l')\mathbf{t}_\mathcal{I}^{Z_K-E-l'}$. The corresponding h-equivariant parts are

$$(T^\infty f)_h(\mathbf{t}_\mathcal{I}) = \sum_{l'\in\mathcal{S}',[l']=[Z_K]-h} z(l')\mathbf{t}_\mathcal{I}^{Z_K-E-l'}. \tag{8.13}$$

Furthermore, (8.11) and the sentence after it shows that in the numerator of $f(\mathbf{t}_\mathcal{I})$ all the exponents are situated in the projected Lipman cone $\pi_\mathcal{I}(\mathcal{S}'_\mathbb{R})$. In particular, by Lemma 8.1, they are contained in a fixed chamber $\mathfrak{c}\subset\mathbb{R}^{|\mathcal{I}|}$ which contains $\pi_\mathcal{I}(\mathcal{S}'_\mathbb{R})$. Therefore Theorem 8.1 gives

$$\mathrm{mpc}^{\pi_\mathcal{I}(\mathcal{S}'_\mathbb{R})}(Z_h(\mathbf{t}_\mathcal{I})) = \sum_{l'|_\mathcal{I}\leq(Z_K-E)|_\mathcal{I},[l']=[Z_K]-h} z(l').$$

But the right–hand side is exactly the counting function $q_{[Z_K]-h,\mathcal{I}}(Z_K - r_h)$, since $l'|_\mathcal{I} \leq (Z_K - E)|_\mathcal{I}$ is equivalent with $l'|_\mathcal{I} \prec (Z_K - r_h)|_\mathcal{I}$ if $[l'] = [Z_K] - h$. □

Corollary 8.2 *$Q_{[Z_K]-h,\mathcal{N}}(Z_K - r_h) = \mathfrak{sw}^{norm}_h(M)$; that is, the Seiberg–Witten invariant can be expressed via the counting function as a finite sum of Z–coefficients.*

Proof Follows from Theorems 5.1 and 8.2. □

8.3 The polynomial part of the series $Z(\mathbf{t})$

8.3.1 The 'Polynomial–negative degree part' decomposition: motivation and history

Consider a one–variable rational function $z(t) = B(t)/A(t)$ with $A(t) = \prod_{i=1}^n(1 - t^{a_i})$ and $a_i > 0$. In [15, 7.0.2] is observed that any such function has a unique decomposition of the form $z(t) = P^+(t) + z^{neg}(t)$, where $P^+(t)$ is a polynomial (with non-negative exponents) and $z^{neg}(t)$ is a rational function of negative degree. Furthermore, $z(t)$ admits a periodic constant (associated with the Taylor expansion of z and the cone $\mathbb{R}_{\geq 0}$), which equals $P^+(1)$. The decomposition can be established using the Euclidean division algorithm. P^+ is called the *polynomial part* while the rational function z^{neg} is the *negative degree part* of the decomposition.

This one–variable periodic constant computation was useful in theorems of type 5.1 with only one node (Seifert 3–manifolds), or surgery formulas along one node, cf. [15]. Basically, in these cases, the concrete computation of the periodic constant was based on the computation of P^+.

Later, in the works of [40, 46, 45, 42] it turned out that the multivariable generalization is much more subtle. In this case the main questions were the following: (a) what are the universal properties of the parts P^+ and z^{neg}, which guarantee that a decomposition $z = P^+ + z^{neg}$ exists and it is unique; and, (b) what algorithm provides this decomposition. Additionally, the wished decomposition must satisfy (at least) the next basic property: (c) in the geometric context $z(\mathbf{t}) = f(\mathbf{t})$, cf. (5), $\mathrm{pc}(z) = P^+(1)$, where pc is associated with the Lipman cone (or, at least, with some subcone of it). Therefore, in this geometric situation, when all these are satisfied, whenever the Seiberg–Witten invariant is computable via such periodic constant, e.g. when $\mathcal{I}$ contains all the nodes, cf. Theorem 5.1, then P^+ is a polynomial generalization of the Seiberg–Witten invariant.

For functions with two variables in 4.10 a decomposition was already presented satisfying all the required properties, based on a 'two–variable division procedure'. Furthermore, for more variables, [46] constructed a candidate polynomial P_1^+ based on the combinatorics of Γ, Ehrhart theory and reduction to the one- and two–variable divisions. It satisfied (c), but it didn't answer (a) in a natural way. Later, [45] considered another polynomial P_2^+ (with $P_2^+ \neq P_1^+$ in general) constructed via an inductive multivariable Euclidean division. It answered (a)-(b), but (c) was not established, so it was not clear if P_2^+ is helpful at all in pc (or Seiberg–Witten) computations. In [45, 4.4] we not quite conjectured, but asked convincingly whether this P_2^+ is the right candidate for the polynomial part.

In the sequel we will construct this candidate using a multivariable Euclidean division algorithm, then using the duality results presented in this chapter we show that this is indeed a good polynomial part.

8.3.2 The polynomial part by division

First we review one of the main statement of [45] which constructs the 'candidate' for the polynomial part of $Z(\mathbf{t})$ by multivariable Euclidean division.

We start, similarly as in section 8.1, with a pair of free $\mathbb{Z}$–modules $L \subset L'$, and a general multivariable rational function (in variables $\mathbf{t}^{L'}$)

$$z(\mathbf{t}) = \frac{\sum_{k=1}^{r} \iota_k \mathbf{t}^{b_k}}{\prod_{i=1}^{n}(1-\mathbf{t}^{a_i})},$$

where $\{\iota_k\}_{k=1}^r \in \mathbb{Z}$, $\{b_k\}_{k=1}^r$, $\{a_i\}_{i=1}^n \in L'$ such that $b_k \not< 0$ for all k and $0 < a_i$ for all i. (In our application z will be $f_h(\mathbf{t}_\mathcal{I})$ for some $\mathcal{I} \neq \emptyset$.)

Proposition 8.2 *[45]*
(a) $z(\mathbf{t})$ can be written in the following form by a 'multivariable Euclidean division algorithm':

$$z(\mathbf{t}) = \sum_{S\subset\{1,\dots,n\}} \frac{\sum_j \iota_{S,j}\mathbf{t}^{b_{S,j}}}{\prod_{i\in S}(1-\mathbf{t}^{a_i})}, \tag{8.14}$$

where $\iota_{S,j} \in \mathbb{Z}$, $b_{S,j} \nless 0$ for all (S,j), and $b_{S,j} \prec a_i$ for all (S,j) and all $i \in S$ whenever $\mathcal{S} \neq \emptyset$.

(b) $z(\mathbf{t})$ has a decomposition of type $P^+(\mathbf{t}) + z^{neg}(\mathbf{t})$ with the next properties:

(i) $P^+(\mathbf{t})$ is a finite sum (polynomial) $\sum_j n_j \mathbf{t}^{c_j}$ with $c_j \nless 0$ for all j;

(ii) $z^{neg}(\mathbf{t})$ is a rational function with negative degree in all variables t_i.

Furthermore, a decomposition of $z(\mathbf{t})$ with properties (i)-(ii) is unique.

(c) In fact, the terms P^+ and z^{neg} of the decomposition from (b) are given via (8.14) as follows: P^+ (resp. z^{neg}) is the sum of terms from (8.14) over $S = \emptyset$ (resp. $S \neq \emptyset$).

Proof *(a)* Assume that we have an expression $g(\mathbf{t}) = \mathbf{t}^b / \prod_{i \in S}(1 - \mathbf{t}^{a_i})$ such that $b \nless 0$ and there exists some i_0 such that $b \nless a_{i_0}$. Then we replace g by $-\mathbf{t}^{b-a_{i_0}} / \prod_{i \in S \setminus \{i_0\}}(1 - \mathbf{t}^{a_i}) + \mathbf{t}^{b-a_{i_0}} / \prod_{i \in S}(1 - \mathbf{t}^{a_i})$. Note that $b - a_{i_0} \nless 0$, hence the new fractions have similar form. Starting from the original expression of z, by repeating the above step whenever is applicable, after finitely many steps we obtain (8.14).

(b) Define P^+ and z^{neg} as is indicated in (c). Then properties (i)–(ii) are automatically satisfied. Next, we prove the uniqueness of the decomposition. We need to show that if $P^+(\mathbf{t}) + z^{neg}(\mathbf{t}) = 0$ (†) and if P^+ and z^{neq} satisfy (i)–(ii), then both are zero. But, from (†), both P^+ and z^{neq} should be simultaneously polynomials and also rational functions of negative degrees in all the variables, a fact which happens only of they are zero since a rational function can be represented as a rational function in a unique way. □

8.3.3 The polynomial part of $f_h(\mathbf{t}_\mathcal{I})$ by duality

Assume again that we are in the situation of a plumbing graph and its rational function $f(\mathbf{t})$ as in Ssections 3.2, 5 and 8.2.

We fix $h \in H$. Then, by the proof of Theorem 8.2 we have $f_h(\mathbf{t}_\mathcal{I}) = \mathbf{t}_\mathcal{I}^{Z_K - E} \cdot f_{[Z_K]-h}(\mathbf{t}_\mathcal{I}^{-1})$ and $T^\infty f_h(\mathbf{t}_\mathcal{I}) = \mathbf{t}_\mathcal{I}^{Z_K-E} Z_{[Z_K]-h}(\mathbf{t}_\mathcal{I}^{-1})$. Write this expression $\mathbf{t}_\mathcal{I}^{Z_K-E} Z_{[Z_K]-h}(\mathbf{t}_\mathcal{I}^{-1})$ as $\sum_{l' \in Z_K - E - \mathcal{S}'} w(l') \mathbf{t}_\mathcal{I}^{l'}$, where $[l'] = h$ automatically whenever $w(l') \neq 0$. Define

$$P^+_{h,\mathcal{I}}(\mathbf{t}_\mathcal{I}) := \sum_{l'|_\mathcal{I} \nless 0|_\mathcal{I}} w(l') \mathbf{t}_\mathcal{I}^{l'}, \quad f^{neg}_{h,\mathcal{I}}(\mathbf{t}_\mathcal{I}) := \sum_{l'|_\mathcal{I} \prec 0|_\mathcal{I}} w(l') \mathbf{t}_\mathcal{I}^{l'}. \tag{8.15}$$

Write also $P^+_{h,\mathcal{I}}(\mathbf{1}) := P^+_{h,\mathcal{I}}(\mathbf{t}_\mathcal{I})|_{t_i=1, \forall i}$.

Theorem 8.3 *Consider the decomposition $f_h(\mathbf{t}_\mathcal{I}) = P^+_{h,\mathcal{I}}(\mathbf{t}_\mathcal{I}) + f^{neg}_{h,\mathcal{I}}(\mathbf{t}_\mathcal{I})$ from (8.15).*

(a) $P^+_{h,\mathcal{I}}(\mathbf{t}_\mathcal{I})$ and $f^{neg}_{h,\mathcal{I}}(\mathbf{t}_\mathcal{I})$ satisfy the requirements (i)–(ii) from Proposition 8.2(b). In particular, by the uniqueness of the decomposition, this decomposition agrees with the decomposition from 8.2(a)–(c) given by Euclidean division.

(b) $P^+_{h,\mathcal{I}}(\mathbf{1}) = Q_{[Z_K]-h,\mathcal{I}}(Z_K - r_h)$. In particular, $P^+_{h,\mathcal{I}}(\mathbf{1}) = \mathrm{pc}^{\pi_\mathcal{I}(S'_\mathbb{R})}(Z_h(\mathbf{t}_\mathcal{I}))$.

Proof *(a)* By its definition, $P^+_{h,\mathcal{I}}(\mathbf{t}_\mathcal{I})$ is a finite sum, hence $f^{neg}_{h,\mathcal{I}}(\mathbf{t}_\mathcal{I}) = f_h(\mathbf{t}_\mathcal{I}) - P^+_{h,\mathcal{I}}(\mathbf{t}_\mathcal{I})$ is a rational function. Since in its expansion all monomials $w(l')\mathbf{t}_\mathcal{I}^{l'}$ satisfy $l'|_\mathcal{I} \prec 0|_\mathcal{I}$, it has negative degree in all the variables.

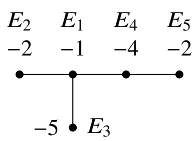
(b) By the above transformations $w(l') = z(Z_K - E - l')$, $[l'] = h$. Hence $l'|_{\mathcal{I}} \not< 0|_{\mathcal{I}}$ transforms into $(Z_K - E)|_{\mathcal{I}} \not< (Z_K - E - l')|_{\mathcal{I}}$, or $(Z_K - E - l')|_{\mathcal{I}} \not\geq (Z_K - r_h)|_{\mathcal{I}}$. This proves the first identity. For the second one use Theorem 8.2 *(b)*. □

Corollary 8.2 and Theorem 8.3 combined gives

Corollary 8.3 *For any $h \in H$ one has $P^+_{h,\mathcal{N}}(\mathbf{1}) = \mathfrak{sw}^{norm}_h(M)$.*

We illustrate the above duality results on the following example.

Example 8.4 Let us consider the Brieskorn sphere $M = \Sigma(2,5,7)$. Recall that the entries of E^*_v's are the corresponding columns of the matrix $\{-(E^*_i, E^*_j)\}_{i,j}$, which is in fact $-I^{-1}$, the negative of the inverse of the intersection matrix. In this case the graph and $-I^{-1}$ are the following:



$$-I^{-1} = \begin{bmatrix} 70 & 35 & 14 & 20 & 10 \\ 35 & 18 & 7 & 10 & 5 \\ 14 & 7 & 3 & 4 & 2 \\ 20 & 10 & 4 & 6 & 3 \\ 10 & 5 & 2 & 3 & 2 \end{bmatrix}$$

Therefore, the zeta-function reduced to the set of nodes $\mathcal{N} = \{v_0\}$ and its 'polynomial-negative degree part' decomposition (obtained by simple division of polynomials) can be written as

$$f(t) = \frac{1 - t^{70}}{(1-t^{35})(1-t^{14})(1-t^{10})} = t + t^{11} + \frac{1 - t + t^{15} + t^{21}}{(1-t^{14})(1-t^{10})}.$$

The same result follows by duality as well. Indeed, the Taylor expansion is $Z(t) = 1 + t^{10} + t^{14} + \ldots$ and one calculates $Z_K = (12, 6, 3, 4, 2)$ where we use short notation $(l_1, l_2, l_3, l_4, l_5)$ for $l = \sum_{i=1}^5 l_i E_i$. Hence, for $t^{Z_K - E} Z(t^{-1})$ (where $t^{Z_K - E} := t^{(Z_K)_{v_0} - 1}$, see the notation from 8.1.2) we get $t^{11} + t^1 + t^{-3} + \cdots$, which gives $P^+(t) = t^{11} + t$ again. □

8.4.1 Example. Plane curve singularities

Let us describe the analogue of (8.10) and of the 'polynomial-negative degree part' decomposition for the embedded situation of an irreducible plane curve singularity $g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. In this case the series $z(t)$ is the Taylor expansion at the origin of the monodromy zeta function $\zeta(t) = \Delta(t)/(1-t)$, cf. [1, 2]. A'Campo's formula constructs $\zeta(t)$ via (3.8), but now applied to the minimal embedded resolution graph Γ_g of the plane germ (see Figure 8.1), whose unique (-1)-vertex has an extra arrowhead–neighbour (where the arrow represents the strict transform of $\{g = 0\}$, thus the (-1)-vertex is considered to be a node of the graph), and this series is reduced to the variable of the (-1)-node. (Below the doted lines represent string subgraphs.)

Let $\mathcal{M} \subset \mathbb{Z}_{\geq 0}$ be the semigroup of g consisting of all intersection multiplicities of g with all possible analytic germs. By [22] $T\zeta(t) = \sum_{s \in \mathcal{M}} t^s$. Let $\mu = \deg(\Delta)$ be the Milnor number

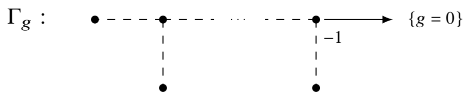


Fig. 8.1 The shape of the minimal embedded resolution graph of g

of g. We determine the polynomial part of $T\zeta(t)$ and we verify the analogue of (8.10). Here the needed Gorenstein duality reads as follows: $(\dagger)$ $s \notin \mathcal{M}$ if and only if $\mu - 1 - s \in \mathcal{M}$. This also shows that $\mathbb{Z} \setminus \mathcal{M}$ is finite, its cardinality is exactly $\mu/2$, and the largest element of $\mathbb{Z} \setminus \mathcal{M}$ is $\mu - 1$.

Since $T\zeta(t) = \sum_{s\in\mathcal{M}} t^s = \sum_{s\geq 0} t^s - \sum_{s\notin\mathcal{M}, s\geq 0} t^s$ we get that $\zeta^{neg}(t) = \sum_{s\geq 0} t^s = 1/(1-t)$, and $P^+(t) = -\sum_{s\notin\mathcal{M}, s\geq 0} t^s$. In particular, $\mathrm{pc}(\zeta) = P^+(1) = -\mu/2$. On the other hand, by duality $(\dagger)$ one has $T^\infty\zeta(t) = -t^{\mu-1} \cdot T\zeta(t^{-1}) = -t^{\mu-1}(\sum_{s\in\mathcal{M}, s<\mu} t^{-s} + \sum_{s\geq\mu} t^{-s}) \overset{(\dagger)}{=} -\sum_{s\notin\mathcal{M}, s\geq 0} t^s - (t^{-1} + t^{-2} + \cdots) = P^+(t) - (t^{-1} + t^{-2} + \cdots)$, whose part with positive exponents is exactly $P^+(t)$.

[In this 'relative case', the geometric object is not a 3–manifold, but a knot in S^3. By the above computation, the periodic constant of the corresponding 'relative' series is $-\mu/2$, where μ serves as the genus of the knot as well. $\mu/2$ is in fact the 'delta–invariant' of the algebraic knot, which is the low–dimensional analogue of the 'geometric genus' defined for surfaces.]

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