Adrian Magdaș

Contributions to fixed point theory for cyclic operators and applications

Presa Universitară Clujeană

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Foreword

This book addresses an important topic of fixed point theory, namely that of fixed point theory for cyclic operators (singlevalued and multivalued) defined on metric spaces, generalized metric spaces or special classes of Banach spaces. The book also studies the existence of the best proximity points for multivalued operators of cyclic type. As applications, the coupled fixed point problem is considered and, in particular, a study of the existence of the solution for certain classes of systems of nonlinear integral equations is performed. Throughout the book, a special class of cyclic operators is considered, namely those who satisfy a generalized contraction condition of Ćirić type.

The topic of this monograph is also approached in numerous papers, starting with the 1990s and continuing with the most recent years, when these research topics received an impetus development, the literature containing over 1700 papers in which the issue of cyclical operators is addressed. Of these, over 100 articles or books deal with the issue of the best proximity point.

The main quality of this book is to carry out a complete study (existence, uniqueness, approximation, data dependence, various types of stability) of the above mentioned problems. More specifically, the aim of the monograph "Contributions to the Fixed Point Theory for Cyclic Operators and Applications" is to study the fixed point theory for classes of cyclic operators (singlevalued and multivalued of Ćirić type), as well as the study of the best proximity problems, coupled fixed point problems for classes of cyclic operators that satisfy a generalized Ćirić type contraction condition. In order to illustrate the theoretical results, some applications to systems of integral equations are considered.

Overall, this monograph is clear, rigorous and well written, containing a remarkable number of results, as well as conclusive examples that accompany the main results and notions. The presented bibliographic list successfully covers the field considered for study.

This monograph is based on the doctoral thesis of the author, successfully defended by Mr. Adrian Magdaş at Babeș-Bolyai University, Cluj-Napoca, in April 2020.

Cluj-Napoca,
April 2021

Prof. dr. Adrian Petruşel

Introduction

The theory of fixed points has been revealed as a powerful tool for solving various problems arising in different fields of pure and applied mathematics. The cornerstone of the metric fixed point theory, S. Banach contraction principle [1], has been generalized in several directions. Most of these generalizations, see for example [54], [70], weaken the contractive nature of the operator but, in compensation, have conditions that enrich the metric space structure and / or have additional requirements on the operator.

In 1969, S.B. Nadler extended Banach contraction principle from single-valued to multi-valued mapping (see [40]). Nadler's Theorem has been generalized by many mathematicians, see for example the fixed point results for multi-valued mappings of generalized contractive type of H. Covitz, S.B. Nadler [13], L. Ćirić [9], N. Mizoguchi and W. Takahashi [38], S.B. Nadler [41], A. Petruşel [56], C. Vetro and F. Vetro [80].

Banach contraction principle was extended for single-valued contraction on spaces endowed with vector-valued metrics by A.I. Perov and A.V. Kibenko [45]. The case of multi-valued contractions on spaces endowed with vector-valued metrics is treated in A. Petruşel [53], I.R. Petre, A. Petruşel [46].

One of the consistent generalization of the Contraction Principle was given in 2003 by W.A. Kirk, P.S. Srinivasan and P. Veeramani, using the concept of cyclic operator (see [29]). This concept attracted the interest of many authors because of its potential in the study of differential and integral equations (see for example [2], [23], [61], [75]).

The concept of coupled fixed point was introduced by V.I. Opoitsev [43], but the issue gets a fast development due to the works of D. Guo and V. Lakshmikantham [20] and T.G. Bhaskar, V. Lakshmikantham [5]. A new research direction for the theory of coupled fixed points has been developed by many authors (see V. Lakshmikantham, L. Ćirić [31], A. Petruşel, G. Petruşel and B. Samet [57], B. Samet and C. Vetro [74]) using contractive type conditions.

A. Eldred and P. Veeramani opened in 2006 another research direction, searching conditions which ensure the existence of a best proximity point of cyclic operators in the framework of metric spaces (see [16]).

In the present work we develop a study regarding the existence, uniqueness, qualitative properties of fixed point, coupled fixed point, best proximity point for single-valued and multi-valued operators satisfying cyclic conditions. We support this study by presenting also some applications.

The study material is organized into three chapters connected to each other through various threads, each chapter containing several sections.

Chapter 1: Preliminaries

This chapter is a brief overview of the basic notions and results which are further considered in the next chapters of this work, allowing us to present the results of this thesis. We start by presenting standard notations and terminology of nonlinear analysis. The concept and related properties of comparison function which will be used throughout thesis are presented as well. Then basic metric fixed point theorems, starting with Banach contraction principle and some classical contractive operators are presented. The basic notion used in the development of this thesis, namely cyclic operator, is presented accompanied by some fixed point theorems given by Kirk, Srinivasan and Veeramani in [29]. Basic best proximity point theorems and coupled fixed point theorems useful for the development of the next chapters are presented in the last two sections.

Chapter 2: Single-valued generalized contractions on cyclic representations

In this chapter, we give some fixed point results for single-valued operators defined on cyclic representations in metric spaces and in spaces endowed with vector-valued metrics. This chapter has three sections.

In the **first section** we investigate the properties of some Ćirić type generalized contractions defined on cyclic representations in a metric space.

Ćirić type generalized contraction condition is one of the most general metrical ´ condition for which the set of fixed points is a singleton and the fixed points can be approximated by means of Picard iteration. Our results generalize fundamental metrical fixed point theorems in literature given for Banach, Kannan, Bianchini, Reich, Chatterjea, Zamfirescu, Ćirić type operators (see [52], [66]), in the case of a cyclic condition (see [47]). Also, the main result of this section (Theorem 2.1.5) is a generalization of the following results: Theorem 2.1.1 given by Petric and Zlatanov in [50] and Theorem 2.1.3 given by Păcurar and Rus in [44].

Throughout this section we develop a theory of the stated fixed point theorem, theory consisting of:

• the existence and uniqueness results for fixed points of single-valued cyclic φ-contraction of Ćirić type;

• convergence theorems for Picard iteration to these fixed points;

• continuous data dependence of the operator perturbation;

• well posedness of the fixed point problem;

• sequences of operators and fixed points.

Also, we state a Maia type theorem related to Ćirić type generalized contractions defined on cyclic representations.

The results presented in this section are included in the paper Magdaş [33].

In the **second section** we present a Perov type theorem for cyclic operators. Our approach is based on Perov's fixed point theorem (see Theorem 2.2.3), in spaces endowed with vector-valued metrics. Our main result in this section is Theorem 2.2.5, an extension of Theorem 1.3.1 and Theorem 1.3.12 in a space endowed with a vector-valued metric. We state two results regarding the data dependence and the well posedness of the fixed point problem. As applications, we study existence, uniqueness and data dependence of the solution of a system of Fredholm type of integral equations; the solution of the system can be obtained by the successive appoximation. Also we study existence and uniqueness of the solution of a system of Volterra type of integral equations. The results presented in this section are contained in the following paper: Magdaş [36].

In the **third section** we study the coupled fixed point problem for single-valued cyclic contraction type operators. The approach is based on fixed point results for appropriate operators generated by the initial problems.

Our main result in this section is Theorem 2.3.2 which is a generalization of several theorems such as Theorem 1.5.9, Theorem 1.5.11, Theorem 1.5.13, Theorem 1.5.15. We also provide an iterative method for approximating the strong coupled fixed point and we give some qualitative properties of the coupled fixed point set, such as data dependence, generalized Ulam-Hyers stability and well posedness. As applications, we study the existence and the uniqueness of the solution of a system of Fredholm type of integral equations; generalized Ulam-Hyers stability of the system is studied as well. Also we study existence and uniqueness of the solution of a system of Volterra type of integral equations. The results presented in this section are contained in the paper Magdaş [35].

Chapter 3: Multi-valued generalized contractions on cyclic representations

In this chapter, we give fixed point and best proximity point results for multi-valued operators defined on cyclic representations of a metric space (X, d). This chapter has three sections.

In the **first section** we investigate the properties of multi-valued φ-contractions of Ćirić type defined on cyclic representations in a metric space (X, d). We will study under which conditions such an operator T possesses fixed points, i.e., $x \in X$ satisfying the relation $x \in T(x)$. We construct a sequence of successive approximations of T that guarantees convergence from any starting point (x, y) from the graph of the operator to a point $x^* \in F_T$, the set of all fixed points of T. We also study data dependence and generalized Ulam-Hyers stability of the fixed point inclusion $x \in T(x)$. Our results extend metrical fixed point theorems in literature such as Nadler's Theorem (see [40]) or fixed point results of multi-valued Ćirić type operators (see [9]), in the case of a cyclic condition. Also, the main result Theorem 3.1.4 is a generalization of the Theorem 2.1 given by Neammanee and Kaewkhao in [42]. The results presented in this section are included in the paper Magdaş [34].

In the **second section** we study existence of the solutions and generalized Ulam-Hyers stability of the best proximity problem for cyclic multi-valued operators: If (X, d) is a metric space, $A, B \in P(X)$, $T : A \cup B \to P(X)$ is a multi-valued operator satisfying the cyclic condition $T(A) \subseteq B, T(B) \subseteq A$, then we are interested to find $x^* \in A \cup B$ such that $D(x^*, T(x^*)) = D(A, B)$, where D is the gap functional. x^* is said to be a best proximity point of T.

Several authors studied the existence of best proximity points for cyclic operators on metric spaces, see e.g. [17], [19], [24], [25], [26], [28], [49], [51]. The first main result of this section extends Theorem 1.4.5 (Suzuki, Kikkawa, C. Vetro, [77]) and Theorem 1.4.6 (Neammanee, Kaewkhao [42]) to the case of multi-valued Ćirić type cyclic operator which takes proximinal values, in the framework of metric spaces with the property UC. The results presented in this section are contained in the paper Magdaş [37].

In the **third section** we study the coupled fixed point problem and the coupled best proximity point problem for cyclic multi-valued operators. The approach is based on fixed point results for appropriate operators generated by the initial problems. The first result Theorem 3.3.5 states a coupled fixed point result for cyclic coupled φ-contraction of Ćirić type multi-valued operator. The generalized Ulam-Hyers stability of the coupled fixed point problem is studied as well. Theorem 3.3.10 studies the existence of the coupled best proximity point of a cyclic coupled Ćirić type multi-valued operator which takes proximinal values, in the framework of metric spaces with the property UC. The results presented in this section are contained in the paper Magdaş [35].

This book is concluded by the references used in the text and a list of published papers.

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Chapter 1

Preliminaries

The aim of this chapter is to present the basic concepts and results which are further considered in the next chapters.

1.1 Basic notations and notions

We present some standard notations and terminology of nonlinear analysis which will be used throughout this work.

Let (X,d) be a metric space, Y be a nonempty subset of X. We denote:

$$P(X) := \{A \subseteq X \mid A \text{ is nonempty}\};\ P_b(X) := \{A \in P(X) \mid A \text{ is bounded}\};$$

$$P_{cl}(X) := \{A \in P(X) \mid A \text{ is closed}\};\ P_{cp}(X) := \{A \in P(X) \mid A \text{ is compact}\};$$

$$P_{cv}(X) := \{A \in P(X) \mid A \text{ is convex}\};\ P_{cl,cv}(X) := P_{cl}(X) \cap P_{cv}(X).$$

Let us define the following (generalized) functionals used in this paper:

• the diameter functional

$$\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ \delta(A,B) = \sup\{d(a,b) \mid a \in A,\ b \in B\};$$

• the gap functional

$$D : P(X) \times P(X) \to \mathbb{R}_+,\ D(A,B) = \inf\{d(a,b) \mid a \in A,\ b \in B\};$$

• the generalized excess functional

$$\rho : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ \rho(A,B) = \sup\{D(a,B) \mid a \in A\};$$

- the generalized Pompeiu-Hausdorff functional

$$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ \ H(A,B) = \max\{\rho(A,B), \rho(B,A)\}.$$

We recall now the following notions and results.

Lemma 1.1.1. *Let (X, d) be a metric space, $A, B \in P(X)$. Then for any $\varepsilon > 0$ and for any $a \in A$ there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \varepsilon.$$

Definition 1.1.2. (Fletcher, Moors [18]) Let (X, d) be a metric space and let $Y \in P(X)$. We denote

$$P_Y(x) = \{y \in Y \mid d(x, y) = D(x, Y)\} \text{ for } x \in X.$$

The set Y is called proximinal if for any $x \in X$, $P_Y(x)$ is nonempty. If for any $x \in X$, $P_Y(x)$ is singleton, then Y is called Chebyshev set.

Obviously, any Chebysev set is proximinal.

We denote $P_{prox}(X) = \{Y \in P(X) \mid Y \text{ is proximinal}\}$.

Remark 1.1.3. Let (X, d) be a metric space. Then

$$P_{cp}(X) \subset P_{prox}(X) \subset P_{cl}(X).$$

Remark 1.1.4. (Deutsch [15]) A Banach space X is reflexive if and only if every nonempty closed convex subset of X is proximinal.

Remark 1.1.5. (Cobzaş [14]) If Y is a nonempty complete convex subset of a uniformly convex normed space X, then Y is a Chebyshev set in X.

For details concerning the above notions see [40], [63], [70], [76].

1.2 Comparison functions

There are several conditions regarding the notion of comparison function that have been considered in literature. Throughout this paper we shall refer only to the following notion.

Definition 1.2.1. (Rus, Şerban [72]) A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a comparison function if it satisfies:

$(i)_\varphi$ φ is increasing;

$(ii)_\varphi$ $(\varphi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$, for all $t \in \mathbb{R}_+$.

If the condition $(ii)_\varphi$ is replaced by the condition:

$(iii)_\varphi$ $\sum_{k=0}^{\infty} \varphi^k(t) < \infty$, for any $t > 0$,

then φ is called a strong comparison function.

Lemma 1.2.2. (Rus, A. Petruşel, G. Petruşel [70]) *If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, then the following hold:*

(i) $\varphi(t) < t$, for any $t > 0$;

(ii) $\varphi(0) = 0$;

(iii) φ is continuous at 0.

Lemma 1.2.3. (Păcurar, Rus [44], Rus, Şerban [72]) *If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strong comparison function, then the following hold:*

(i) φ is a comparison function;

(ii) the function $s : \mathbb{R}_+ \to \mathbb{R}_+$, defined by

$$s(t) = \sum_{k=0}^{\infty} \varphi^k(t), \quad t \in \mathbb{R}_+, \tag{1.2.1}$$

is increasing and continuous at 0;

(iii) there exist $k_0 \in \mathbb{N}$, $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\varphi^{k+1}(t) \le a\varphi^k(t) + v_k, \text{ for } k \ge k_0 \text{ and any } t \in \mathbb{R}_+.$$

Remark 1.2.4. Some authors use the notion of (c)-comparison function defined by the statements (i) and (iii) from Lemma 1.2.3. Actually, the concept of (c)-comparison function coincides with that of strong comparison function.

Example 1.2.5. The following functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ are comparison functions:

(1) $\varphi(t) = at$, where $a \in [0,1)$.

(2) $\varphi(t) = \begin{cases} at, & \text{for } t \in [0,1] \\ t + a - 1, & \text{for } t > 1 \end{cases}$, where $a \in [0,1)$.

(3) $\varphi(t) = at + \frac{1}{2}[t]$, where $a \in (0, \frac{1}{2})$.

(4) $\varphi(t) = \begin{cases} \frac{t}{a}, & \text{for } t \in [0, a] \\ a, & \text{for } t > a \end{cases}$, where $a \in (1, \infty)$.

(5) $\varphi(t) = \frac{t}{t+a}$, where $a \in [1, \infty)$.

The first four examples are strong comparison functions. The fifth example is a strong comparison function iff $a \in (1, \infty)$. For more considerations on comparison functions see [69], [70] and the references therein.

1.3 Basic metric fixed point theorems

If $f : Y \to X$ is a single-valued operator, then
$\text{Graph}(f) := \{(x, f(x)) \mid x \in Y\}$ denotes the graph of f and
$F_f := \{x \in Y \mid f(x) = x\}$ denotes the fixed point set of f.

If $T : Y \to P(X)$ is a multi-valued operator, then
$\text{Graph}(T) := \{(x, y) \mid x \in Y, y \in T(x)\}$ denotes the graph of T and
$F_T := \{x \in Y \mid x \in T(x)\}$ denotes the fixed point set of T.

A sequence $(x_n)_{n \in \mathbb{N}}$ satisfying the following conditions:

(i) $x_0 = x,\ x_1 = y$;

(ii) $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}$;

is called a sequence of successive approximations of T starting from $(x, y) \in \text{Graph}(T)$.

Banach contraction principle is one of the most useful results in nonlinear analysis. In a metric space setting, the statement of the contraction principle was given in 1922.

Theorem 1.3.1. (Banach [1]) *Let (X, d) be a complete metric space and let $f : X \to X$ be a contraction operator, that is there exists a constant $a \in [0, 1)$ such that for any $x, y \in X$,*

$$d(f(x), f(y)) \leq ad(x, y).$$

Then:

(1) *f has a unique fixed point $x^* \in X$;*

(2) *the Picard iteration* $(x_n)_{n\geq 0}$ *defined by*

$$x_n = f(x_{n-1}),\ \ n \geq 1 \tag{1.3.1}$$

converges to x^* *for any starting point* $x_0 \in X$*;*

(3) *the following estimate holds:*

$$d(x_{n+k-1}, x^*) \leq \frac{a^k}{1-a} d(x_n, x_{n-1}), \forall n, k \in \mathbb{N}^*;$$

(4) *the rate of convergence of Picard iteration is given by:*

$$d(x_n, x^*) \leq a d(x_{n-1}, x^*), \forall n \geq 1.$$

In 1969, Nadler [40] extended the Banach contraction principle from single-valued to multi-valued operator. The existence of fixed points for various multi-valued contractive operators has been studied by many authors under different conditions, see Ćirić [9], [10], Mizoguchi, Takahashi [38], Rhoades [67].

We recall now Nadler's fixed point theorem.

Theorem 1.3.2. (Nadler [40]) *Let* (X, d) *be a complete metric space and let* $T : X \to P_{b,cl}(X)$ *be a multi-valued a-contraction, that is there exists a constant* $a \in [0, 1)$ *such that for any* $x, y \in X$*,*

$$H(T(x), T(y)) \leq a \cdot d(x, y).$$

Then T *has a fixed point.*

In the last decades, authors gave many generalization of the Banach contraction principle, a way of generalization being the weakening of the contraction condition.

We present some of such conditions existing in the literature. Let (X, d) be a complete metric space and let $f : X \to X$ be an operator.

(i) (Kannan, [22]) there exists a constant $a \in [0, \frac{1}{2})$ such that

$$d(f(x), f(y)) \leq a[d(x, f(x)) + d(y, f(y))], \forall x, y \in X;$$

(ii) (Chatterjea, [7]) there exists a constant $a \in [0, \frac{1}{2})$ such that

$$d(f(x), f(y)) \leq a[d(x, f(y)) + d(y, f(x))], \forall x, y \in X;$$

(iii) (Zamfirescu, [81]) there exist the real numbers $a \in [0,1), b, c \in [0, \frac{1}{2})$ such that for any $x, y \in X$, at least one of the following holds:

(z1) $d(f(x), f(y)) \leq ad(x, y)$;

(z2) $d(f(x), f(y)) \leq b[d(x, f(x)) + d(y, f(y))]$;

(z3) $d(f(x), f(y)) \leq c[d(x, f(y)) + d(y, f(x))]$.

(iv) (Bianchini, [6]) there exists a constant $a \in [0,1)$ such that

$$d(f(x), f(y)) \leq a \max\{d(x, f(x)), d(y, f(y))\}, \forall x, y \in X;$$

(v) (Reich [64], Rus [69]) there exist the real numbers $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that for any $x, y \in X$,

$$d(f(x), f(y)) \leq ad(x, y) + bd(x, f(x)) + cd(y, f(y)).$$

Ljubomir Ćirić weakened the above conditions introducing in 1971 the notion of generalized contraction.

Definition 1.3.3. (Ćirić [8]) Let (X, d) be a metric space. An operator $f : X \to X$ is said to be a *lambda*-generalized contraction iff for every $x, y \in X$ there are non-negative numbers $q(x, y)$, $r(x, y)$, $s(x, y)$ and $t(x, y)$ with

$$\sup\{q(x, y) + r(x, y) + s(x, y) + 2t(x, y) \mid x, y \in X\} = \lambda < 1$$

such that $d(f(x), f(y)) \leq q(x, y)d(x, y) + r(x, y)d(x, f(x)) +$

$$+ s(x, y)d(y, f(y)) + t(x, y)(d(x, f(y)) + d(y, f(x))).$$

Remark 1.3.4. (Ćirić [9]) f is a generalized contraction if and only if there exists $q \in [0, 1)$ such that for any $x, y \in X$,

$$d(f(x), f(y)) \leq qM(x, y),$$

where $M(x, y) =$

$$= \max\{d(x, y), d(x, f(x)), d(y), f(y)), \frac{1}{2}[d(x, f(y)) + d(y, f(x))]\}. \quad (1.3.2)$$

Example 1.3.5. (Ćirić [8]) Let $X = [0, 2]$, and let $f : X \to X$, $f(x) = \frac{x}{9}$, for $0 \leq x \leq 1$; $f(x) = \frac{x}{10}$, for $1 < x \leq 2$. The operator f is not a contraction but is a generalized contraction.

Definition 1.3.6. [9] Let (X, d) be a metric space and let $T : X \to P_{cl}(X)$ be a multi-valued operator. T is said to be a generalized multi-valued q-contraction if there exists $q \in (0, 1)$ such that

$$H(T(x), T(y)) \leq$$
$$\leq q \cdot \max\left\{d(x, y), D(x, T(x)), D(y, T(y)), \frac{1}{2}[D(x, T(y)) + D(y, T(x))]\right\},$$

holds for every $x, y \in X$.

Definition 1.3.7. [8] Let (X, d) be a metric space and let $f : X \to X$ be a single-valued operator. X is said to be f-orbitally complete if every Cauchy sequence $(f^{n_i}(x))_{i\in\mathbb{N}}, x \in X$, has a limit point in X.

Definition 1.3.8. [9] Let (X, d) be a metric space and let $T : X \to P(X)$ be a multi-valued operator. X is said to be T-orbitally complete if every Cauchy sequence $(x_{n_i})_{i\in\mathbb{N}}$ with $x_{n_i} \in T(x_{n_i-1})$ converges in X.

Theorem 1.3.9. [8] *Let f be a λ-generalized contraction of f-orbitally complete metric space (X, d) into itself. Then*

(1) *f has a unique fixed point $x^* \in X$;*

(2) *the Picard iteration $(x_n)_{n\in\mathbb{N}}$ defined by*

$$x_n = f(x_{n-1}),\ n \geq 1, \tag{1.3.3}$$

converges to x^* for any starting point $x_0 \in X$;

(3) *the following estimate holds:*

$$d(x_n, x^*) \leq \frac{\lambda^n}{1-\lambda} \cdot d(x_0, x_1), \forall n \geq 0.$$

Theorem 1.3.10. [9] *Let $T : X \to P_{cl}(X)$ be a generalized multi-valued q-contraction and let (X, d) be a T-orbitally complete metric space.*

Then the following statements hold:

(1) *$F_T \neq \emptyset$;*

(2) *there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations of T starting from any point $(x, y) \in$ Graph(T), that converges to a fixed point $x^*(x, y) \in X$;*

(3) *the following estimate holds:*

$$d(x_n, x^*(x, y)) \leq \frac{q^{an}}{1-q^a} \cdot d(x_0, x_1), \forall a \in (0, 1), \forall n \geq 0.$$

Another consistent way to generalize Banach contraction principle was presented in 2003 by Kirk, Srinivasan and Veeramani, using the concept of cyclic operator.

Definition 1.3.11. (Kirk, Srinivasan, Veeramani [29]) Let A and B be two nonempty sets. An operator $f : A \cup B \to A \cup B$ is called cyclic if $f(A) \subseteq B$ and $f(B) \subseteq A$.

They prove the following results.

Theorem 1.3.12. [29] *Let A and B be two nonempty subsets of a complete metric space (X, d) and suppose $f : X \to X$ satisfies the following conditions:*

(1) $f(A) \subseteq B$ *and* $f(B) \subseteq A$;

(2) $d(f(x), f(y)) \leq kd(x, y),\ \forall x \in A,\ \forall y \in B,$ *where* $k \in (0, 1)$.

Then f has a unique fixed point.

Theorem 1.3.13. [29] *Let $\{A_i\}_{i=1}^m$ be nonempty subsets of a complete metric space and suppose $f : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ satisfies the following conditions (where $A_{m+1} = A_1$):*

(1) $f(A_i) \subseteq A_{i+1}$ *for* $1 \leq i \leq m$;

(2) $d(f(x), f(y)) \leq \psi(d(x, y)),\ \forall x \in A_i,\ \forall y \in A_{i+1},$ *for* $1 \leq i \leq m$, *where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is upper semi-continuous from the right and satisfies* $0 \leq \psi(t) < t$ *for* $t > 0$.

Then f has a unique fixed point.

This results suggested the introduction of the following definition.

Definition 1.3.14. Let X be a nonempty set, m a positive integer and $T : X \to P(X)$ a multi-valued operator. By definition, $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T if:

(i) $X = \bigcup_{i=1}^{m} A_i$, with $A_i \in P(X)$, for $1 \leq i \leq m$;

(ii) $T(A_i) \subseteq A_{i+1}$, for $1 \leq i \leq m$, where $A_{m+1} = A_1$.

For the particular case of a single-valued operator see Rus [68].

1.4 Basic best proximity point theorems

The best proximity problem for a cyclic multi-valued operator is defined as follows:

If (X, d) is a metric space, $A, B \in P(X)$, $T : A \cup B \to P(X)$ is a multi-valued operator satisfying the cyclic condition $T(A) \subseteq B, T(B) \subseteq A$, then we are interested to find

$$x^* \in A \cup B \text{ such that } D(x^*, T(x^*)) = D(A, B). \tag{1.4.1}$$

x^* is said to be a best proximity point of T.

In particular, if the operator is single-valued then we get the following best proximity problem for a cyclic single-valued operator:

If (X, d) is a metric space, $A, B \in P(X)$, $f : A \cup B \to X$ is a single-valued operator satisfying the cyclic condition $f(A) \subseteq B, f(B) \subseteq A$, then we are interested to find

$$x^* \in A \cup B \text{ such that } d(x^*, f(x^*)) = D(A, B). \tag{1.4.2}$$

x^* is said to be a best proximity point of f.

Eldred and Veeramani proved in 2006 the following theorem which ensures the existence of a best proximity point of cyclic contractions in the framework of uniformly convex Banach spaces.

Theorem 1.4.1. (Eldred, Veeramani [16])

Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space. Suppose $f : A \cup B \to A \cup B$ is a cyclic contraction map, that is f satisfies to following conditions:

(1) $f(A) \subseteq B$ *and* $f(B) \subseteq A$;

(2) $\|f(x) - f(y)\| \leq k\,\|x - y\| + (1 - k)D(A, B),\ \forall x \in A,\ \forall y \in B,$ *where* $k \in (0, 1)$.

Then there exists a unique best proximity point in A. Further, if $x_0 \in A$ and $x_{n+1} = f(x_n)$, then $(x_{2n})_{n\in\mathbb{N}}$ converges to the best proximity point.

In 2009, Suzuki, Kikkawa and C. Vetro introduced the property UC and extended Theorem 1.4.1 to metric spaces with the property UC.

Definition 1.4.2. (Suzuki, Kikkawa, C. Vetro [77]) Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) is said to satisfy the property UC if for $(x_n)_{n\in\mathbf{N}}$ and $(z_n)_{n\in\mathbf{N}}$ sequences in A and $(y_n)_{n\in\mathbf{N}}$ a sequence in B such that $d(x_n, y_n) \to D(A, B)$ and $d(z_n, y_n) \to D(A, B)$ as $n \to \infty$, then $d(x_n, z_n) \to 0$ as $n \to \infty$.

The following are examples of pairs of nonempty subsets of a metric space satisfying the property UC.

Proposition 1.4.3. *Any pair of nonempty subsets (A, B) of a metric space (X, d) with $D(A, B) = 0$ has the property UC.*

Proposition 1.4.4. (Eldred, Veeramani [16]) *Any pair of nonempty subsets (A, B) of a uniformly convex Banach space with A convex has the property UC.*

Theorem 1.4.5. [77] *Let (X, d) be a metric space and let A and B be nonempty subsets of X such that (A, B) satisfies the property UC. Assume that A is complete. Let $f : A \cup B \to X$ be a cyclic mapping, that is $f(A) \subseteq B$ and $f(B) \subseteq A$.*

Assume that there exists $k \in (0, 1)$ such that for each $x \in A$ and $y \in B$,

$$d(f(x), f(y)) \le k \max\{d(x, y), d(x, f(x)), d(y, f(y))\} + (1 - k)D(A, B).$$

Then the following hold:

(i) f has a unique best proximity point $z \in A$.

(ii) z is a unique fixed point of f^2 in A.

(iii) $(f^{2n}(x))_{n\in\mathbb{N}}$ converges to z for every $x \in A$.

(iv) f has at least one best proximity point in B.

(v) If (B, A) satisfies the property UC, then $f(z)$ is a unique best proximity point in B and $(f^{2n}(y))_{n\in\mathbb{N}}$ converges to $f(z)$ for every $y \in B$.

Theorem 1.4.6. (Neammanee, Kaewkhao [42]) *Let A and B be nonempty subsets of a metric space (X, d) such that (A, B) satisfies the property UC and A is complete. Let $T : A \cup B \to P(X)$ with closed bounded valued, be a multi-valued cyclic contraction, that is:*

(i) $T(A) \subseteq B$ and $T(B) \subseteq A$;

(ii) there exists $k \in (0, 1)$ such that for any $x \in A$, $y \in B$,

$$H(T(x), T(y)) \le kd(x, y) + (1 - k)D(A, B).$$

Then T has a best proximity point in A.

1.5 Basic coupled fixed point theorems

A very useful concept in many applications, especially to the theory of integral and differential equations and inclusions, is the coupled fixed point theory. Opoitsev in [43] considered, for the first time, the coupled fixed point problem, but the issue gets a fast development by the seminal works of D. Guo and V. Lakshmikantham [20] and T.G. Bhaskar, V. Lakshmikantham [5]. A new research direction for the theory of coupled fixed points was developed by many authors (see [4], [21], [31], [57], [58], [60], [73]) using contractive type conditions.

We give the notion of coupled fixed point in terms of single-valued, respectively multi-valued operators.

Definition 1.5.1. Let X be a nonempty set. A pair $(x, y) \in X \times X$ is called coupled fixed point of the single-valued operator $F : X \times X \to X$ if

$$\begin{cases} F(x, y) = x \\ F(y, x) = y. \end{cases} \tag{1.5.1}$$

If $F(x, x) = x$ then x is called strong coupled fixed point of F (also called, in several papers, fixed point of F).

Definition 1.5.2. Let X be a nonempty set. A pair $(x, y) \in X \times X$ is called coupled fixed point of the multi-valued operator $F : X \times X \to P(X)$ if

$$\begin{cases} x \in F(x, y) \\ y \in F(y, x). \end{cases} \tag{1.5.2}$$

If $x \in F(x, x)$ then x is called strong coupled fixed point of F.

In order to state the main result in [5], we need the following notion.

Definition 1.5.3. Let $(X, \leq)$ be a partially ordered set. We say that $F : X \times X \to X$ has the mixed monotone property if $F(x, y)$ is monotone increasing in x and is monotone decreasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y),$$

respectively,

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

Theorem 1.5.4. (Bhaskar, Lakshmikantham [5])

Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous operator having the mixed monotone property on X.

Assume that there exists a constant $k \in [0, 1)$ with

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} \cdot [d(x, u)) + d(y, v)], \forall x \geq u, \forall y \leq v. \quad (1.5.3)$$

If there exists $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \text{ and } y_0 \geq F(y_0, x_0), \quad (1.5.4)$$

then there exist $x, y \in X$ such that

$$x = F(x, y) \text{ and } y = F(y, x).$$

Also, Bhaskar and Lakshmikantham established in [5] uniqueness results of the coupled fixed point under an additional assumption on the metric space, as well as existence results of the strong coupled fixed point.

Remark 1.5.5. If (X, d) is a complete metric space without a partially order and (1.5.3) is supposed to hold for any pairs $(x, y), (u, v) \in X \times X$, then we can get existence and uniqueness of the strong coupled fixed point without the continuity and monotonicity conditions and without the assumption (1.5.4).

A more general result was given by A. Petruşel et al. in [59] for symmetric multi-valued contractions:

Theorem 1.5.6. (A. Petruşel, G. Petruşel, Samet, Yao [59]) *Let $(X, \preceq, d)$ be an ordered b-metric space with constant $s \geq 1$ such that the b-metric d is complete. Let $G : X \times X \to P_{cl}(X)$ be a multi-valued operator having the strict mixed monotone property with respect to "$\preceq$". Assume:*

(i) there exists $k \in (0, \frac{1}{s})$ such that

$$H_d(G(x, y), G(u, v)) + H_d(G(y, x), G(v, u)) \leq k[d(x, u) + d(y, v)], \forall x \preceq u, y \succeq v;$$

(ii) there exist $(x_0, y_0) \in X \times X$ and $(x_1, y_1) \in G(x_0, y_0) \times G(y_0, x_0)$ such that $x_0 \preceq x_1$ and $y_0 \succeq y_1$.

Then, there exist $x^*, y^* \in X$ and there exist two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X, with $x_{n+1} \in G(x_n, y_n)$ and $y_{n+1} \in G(y_n, x_n)$ for all $n \in \mathbb{N}$, such that $(x_n)_{n\in\mathbb{N}} \to x^*$, $(y_n)_{n\in\mathbb{N}} \to y^*$ as $n \to \infty$ and

$$\begin{cases} x^* \in G(x^*, y^*) \\ y^* \in G(y^*, x^*). \end{cases}$$

If, in addition, the b-metric d is continuous, then, for the above mentioned two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, the following estimation holds:

$$d(x_n, x^*) + d(y_n, y^*) \leq \frac{sk^n}{1-sk}[d(x_0, x_1) + d(y_0, y_1)], \forall n \in \mathbb{N}^*.$$

We present now the concept of cyclic coupled single-valued operator.

Definition 1.5.7. (Choudhury, Maity [11]) Let A and B two nonempty subsets of a given set X. An operator $F : X \times X \to X$ having the property that for any $x \in A$ and $y \in B$, $F(x, y) \in B$ and $F(y, x) \in A$, is called a cyclic operator with respect to A and B.

Definition 1.5.8. [11] Let A and B two nonempty subsets of a metric space (X, d). An operator $F : X \times X \to X$ is called a cyclic coupled Kannan type contraction if F is cyclic with respect to A and B, satisfying for some $k \in (0, \frac{1}{2})$ the inequality:

$$d(F(x, y), F(u, v)) \leq k \cdot [d(x, F(x, y)) + d(u, F(u, v))],$$

where $x, v \in A$, $y, u \in B$.

Theorem 1.5.9. [11] *Let A and B two nonempty closed subsets of a completed metric space (X, d). Let $F : X \times X \to X$ be a cyclic coupled Kannan type contraction with respect to A and B and $A \cap B \neq \emptyset$. Then F has a strong coupled fixed point in $A \cap B$.*

Definition 1.5.10. (Choudhury, Maity, Konar [12]) Let A and B two nonempty subsets of a metric space (X, d). An operator $F : X \times X \to X$ is called a Banach type coupling if F is cyclic with respect to A and B, and if it satisfies the following inequality:

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} \cdot [d(x, u) + d(y, v)],$$

where $x, v \in A$, $y, u \in B$, and $k \in (0, 1)$.

Theorem 1.5.11. [12] *Let A and B two nonempty closed subsets of a completed metric space (X, d). Let $F : X \times X \to X$ be a Banach type coupling with respect to A and B.Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.*

Definition 1.5.12. [12] Let A and B two nonempty subsets of a metric space (X, d). An operator $F : X \times X \to X$ is called a Chatterjea type coupling if F is cyclic with respect to A and B, and if it satisfies the following inequality:

$$d(F(x, y), F(u, v)) \leq k \cdot [d(x, F(u, v)) + d(u, F(x, y))],$$

where $x, v \in A$, $y, u \in B$, and $k \in (0, \frac{1}{2})$.

Theorem 1.5.13. [12] *Let A and B two nonempty closed subsets of a completed metric space (X, d). Let $F : X \times X \to X$ be a Chatterjea type coupling with respect to A and B.Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.*

Definition 1.5.14. (Udo-utun [78]) Let A and B two nonempty subsets of a metric space (X, d). An operator $F : X \times X \to X$ is called a cyclic Ćirić operator with respect to A and B if F is cyclic with respect to A and B and for some constant $q \in (0, 1)$, F satisfies the following condition:

$$d(F(x, y), F(u, v)) \leq q \cdot M(x, v, y, u),$$

where $x, v \in A$, $y, u \in B$, and

$$\begin{aligned} M(x, v, y, u) = \max \Big\{ & d(x, u), \frac{1}{2} d(u, F(x, y)), \frac{1}{2} d(x, F(u, v)), \\ & \frac{1}{2} [d(x, F(x, y)) + d(u, F(u, v))] \Big\} \end{aligned}$$

Theorem 1.5.15. [78] *Let A and B be two nonempty closed subsets of a complete metric space (X, d), $F : X \times X \to X$ a cyclic Ćirić type operator with respect to A and B, with $A \cap B \neq \emptyset$. Then F has a strong coupled fixed point in $A \cap B$.*

Chapter 2

Single-valued generalized contractions on cyclic representations

In this chapter, we present fixed point results for single-valued operators defined on cyclic representations in metric spaces and in spaces endowed with vector-valued metrics. This chapter has three sections.

In the **first section** we investigate properties of some Ćirić type generalized contractions defined on cyclic representations in a metric space.

The Ćirić type generalized contraction condition is one of the most general metrical ´ condition for which the set of fixed points is a singleton and the fixed points can be approximated by means of Picard iteration. Our results generalize fundamental metrical fixed point theorems in literature given for Banach, Kannan, Bianchini, Reich, Chatterjea, Zamfirescu, Ćirić type operators (see [52], [66]), in the case of a cyclic condition (see [47]). Also, the main result Theorem 2.1.5 is a generalization of the following results: Theorem 2.1.1 given in [50] and Theorem 2.1.3 given in [44].

In this section we will present an extended study of the fixed point equation $x = f(x)$ with a cyclic operator of Ćirić type. More precisely, existence and uniqueness results for fixed points of single-valued cyclic φ-contraction of Ćirić type, as well as convergence theorems for Picard iteration to these fixed points are proved. This study also deals with data dependence of the fixed point, well posedness of the fixed point problem and sequences of operators and fixed points. We will state a Maia type theorem regarding Ćirić type generalized contractions defined on cyclic representations.

The original contributions in the first section are the following results:

• Theorem 2.1.5 extends fixed point results for contractive operators defined on cyclic representation of the space;

• Theorem 2.1.7 is a result concerning the well posedness of the fixed point equation;

• Theorem 2.1.8 studies the data dependence of the fixed point equation;

• Theorem 2.1.9 is a convergence result of a sequence of fixed points of a sequence of operators uniformly convergent to the given Ćirić type generalized contraction;

• Theorem 2.1.10 is a Maia type fixed point theorem for cyclic φ-contraction of Ćirić type.

The results presented in the first section are included in the following paper: Magdaş [33].

In the **second section** we present a Perov type theorem for cyclic operators. Our approach is based on Perov's fixed point theorem (see Theorem 2.2.3), in spaces endowed with vector-valued metrics.

The original contributions in the second section are the following results:

• Theorem 2.2.5 is the main result, an extension of Theorem 1.3.1 and Theorem 1.3.12 in a space endowed with a vector-valued metric;

• Theorem 2.2.6 states a result regarding the data dependence of the fixed point equation;

• Theorem 2.2.7 studies the well posedness of the fixed point equation;

• Theorem 2.2.8 studies the existence and the uniqueness of the solution of a system of Fredholm type of integral equations; the solution of the system can be obtained by the successive appoximation;

• Theorem 2.2.9 is a result concerning the data dependence of the solution of the given system of Fredholm type of integral equations;

• Theorem 2.2.11 studies the existence and the uniqueness of the solution of a system of Volterra type of integral equations.

The results presented in this section are contained in the following paper: Magdaş [36].

In the **third section** we study the coupled fixed point problem for single-valued contraction type operators defined on cyclic representations of the space.

The original contributions in the third section are the following results:

• Theorem 2.3.2 is the main result which generalize theorems 1.5.9, 1.5.11, 1.5.13, 1.5.15; our result provides an iterative method for approximating the strong coupled fixed point and estimations which allow us to study qualitative properties of the coupled fixed point set;

• Theorem 2.3.4 studies the well posedness property of the coupled fixed point problem;

• Theorem 2.3.5 studies the data dependence of the coupled fixed point problem;

• Theorem 2.3.6 is a convergence result of a sequence of strong coupled fixed points of a sequence of operators uniformly convergent to the given cyclic coupled φ-contraction of Ćirić type;

• Theorem 2.3.8 studies the generalized Ulam-Hyers stability for the coupled fixed point problem;

• Theorem 2.3.9 studies the existence and the uniqueness of the solution of a system of Fredholm type of integral equations;

• Theorem 2.3.11 studies the generalized Ulam-Hyers stability of the given system;

• Theorem 2.3.12 studies the existence and the uniqueness of the solution of a system of Volterra type of integral equations.

The results presented in this section are contained in the following paper: Magdaş [35].

2.1 A study of the fixed point problem for Ćirić type single-valued operators satisfying a cyclic condition

The purpose of this section is to investigate the properties of some Ćirić type generalized contractions defined on cyclic representations in a metric space.

Following the work of Kirk, Srinivasan and Veeramani in [29], many authors studied the existence, uniqueness and qualitative properties of the fixed point of a cyclic operator.

Zamfirescu's theorem (see [81]) is a generalization of Banach's, Kannan's and Chatterjea's fixed point theorems. Petric and Zlatanov asserted the following result for cyclic operators, generalizing Zamfirescu's fixed point theorem.

Theorem 2.1.1. (Petric, Zlatanov [50]) *Let (X, d) be a metric space, m a positive integer, $A_1, \ldots, A_m \in P_{cl}(X)$, and let $f : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ be a cyclical operator, that is $f(A_i) \subseteq A_{i+1}$, for $1 \leq i \leq m$, where $A_{m+1} = A_1$. Suppose that there exist real numbers $a \in [0, 1), b, c \in [0, \frac{1}{2})$ such that for each $x \in A_i$, $y \in A_{i+1}$ at least one of the following is true:*

(z1) $d(f(x), f(y)) \leq ad(x, y)$;

(z2) $d(f(x), f(y)) \leq b[d(x, f(x)) + d(y, f(y))]$;

(z3) $d(f(x), f(y)) \leq c[d(x, f(y)) + d(y, f(x))]$.

Then:

(1) *f has a unique fixed point $x^* \in \bigcap_{i=1}^{m} A_i$ and the Picard iteration $(x_n)_{n \in \mathbb{N}}$ given by* (1.3.1) *converges to x^* for any starting point $x_0 \in \bigcup_{i=1}^{m} A_i$.*

(2) *the following estimates hold:*

$$d(x_n, x^*) \leq \frac{\lambda^n}{1 - \lambda} d(x_0, x_1), \; n \geq 0;$$

$$d(x_{n+1}, x^*) \leq \frac{\lambda}{1 - \lambda} d(x_n, x_{n+1}), \; n \geq 0;$$

(3) *the rate of convergence of Picard iteration is given by:*

$$d(x_n, x^*) \leq \lambda d(x_{n-1}, x^*), \; n \geq 1$$

where $\lambda = \max \left\{ a, \frac{b}{1-b}, \frac{c}{1-c} \right\}$.

Păcurar and Rus presented in [44] a fixed point theorem for cyclic φ-contractions.

Definition 2.1.2. (Păcurar, Rus [44]) Let (X, d) be a metric space, m a positive integer, $A_1, \ldots, A_m \in P_{cl}(X)$, $Y \in P(X)$ and $f : Y \to Y$ an operator. If

(i) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to f;

(ii) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d(f(x), f(y)) \leq \varphi(d(x, y)),$$

for any $x \in A_i$, $y \in A_{i+1}$, $1 \leq i \leq m$, where $A_{m+1} = A_1$, then f is a cyclic φ-contraction.

Theorem 2.1.3. [44] *Let (X, d) be a complete metric space, m a positive integer, $A_1, \ldots, A_m \in P_{cl}(X)$, $Y \in P(X)$, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a (c)-comparison function, and $f : Y \to Y$ be an operator. Assume that:*

(i) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to f;

(ii) f is a cyclic φ-contraction.

Then:

(1) *f has a unique fixed point $x^* \in \bigcap_{i=1}^{m} A_i$ and the Picard iteration $(x_n)_{n \in \mathbb{N}}$ given by* (1.3.1) *converges to x^* for any starting point $x_0 \in Y$.*

(2) *the following estimates hold:*

$$d(x_n, x^*) \leq s(\varphi^n(d(x_0, x_1))), \ n \geq 1;$$

$$d(x_n, x^*) \leq s(d(x_n, x_{n+1})), \ n \geq 1;$$

(3) *for any $x \in Y$:*

$$d(x, x^*) \leq s(d(x, f(x))),$$

where s is given by (1.2.1) *in Lemma 1.2.3.*

Further on we present the notion of cyclic φ-contraction of Ćirić type.

Definition 2.1.4. (Magdaş [33]) Let (X, d) be a metric space, $Y \in P(X)$, $f : Y \to Y$ be an operator, $m \in \mathbb{N}^*$, $A_1, \ldots, A_m \in P_{cl}(X)$. If

(i) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to f;

(ii) there exists a strong comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d(f(x), f(y)) \leq \varphi(M(x, y)),$$

for any $x \in A_i$, $y \in A_{i+1}$, $1 \leq i \leq m$, where $A_{m+1} = A_1$ and $M(x, y)$ is given by (1.3.2), then f is said to be a cyclic φ-contraction of Ćirić type.

The main result of this section is the following theorem which generalizes some similar results for Ćirić type operators (see Petruşel [52], Rhoades [66]), in the case of a cyclic condition (see Petric [47]). Also, the following theorem generalizes Theorem 2.1.1 and Theorem 2.1.3.

Hereinafter we present an extended study of this theorem, study in connection with data dependence, well posedness of the fixed point problem, limit shadowing property and sequences of operators and fixed points.

Theorem 2.1.5. (Magdaş [33]) *Let (X, d) be a complete metric space, m be a positive integer, $A_1, \dots, A_m \in P_{cl}(X)$, $Y \in P(X)$, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strong comparison function, and $f : Y \to Y$ be an operator such that $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to f. Assume that f is a cyclic φ-contraction of Ćirić type.*

Then:

(1) *f has a unique fixed point $x^* \in \bigcap_{i=1}^{m} A_i$ and the Picard iteration $(x_n)_{n\geq 0}$ given by* (1.3.1) *converges to x^* for any starting point $x_0 \in Y$;*

(2) *the following estimates hold:*

$$d(x_n, x^*) \leq s(\varphi^n(d(x_0, x_1))),\ n \geq 0;$$

$$d(x_n, x^*) \leq s(d(x_n, x_{n+1})),\ n \geq 0;$$

(3) *for any $x \in Y$,*

$$d(x, x^*) \leq s(d(x, f(x))),$$

where s is given by (1.2.1) *in Lemma 1.2.3;*

(4) $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$*, i.e., f is a good Picard operator;*

(5) $\sum_{n=0}^{\infty} d(x_n, x^*) < \infty$*, i.e., f is a special Picard operator.*

Proof. (1) Let $x_0 \in Y$, $x_n = f(x_{n-1})$, for $n \geq 1$. Then we have:

$$d(f(x_{n-1}), f(x_n)) \leq \varphi\left(\max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}d(x_{n-1}, x_{n+1})\right\}\right). \tag{2.1.1}$$

Using the triangle inequality,

$$\frac{1}{2}d(x_{n-1},x_{n+1}) \le \frac{1}{2}[d(x_{n-1},x_n)+d(x_n,x_{n+1})]$$
$$\le \max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}$$

the inequality (2.1.1) becomes:

$$d(x_n,x_{n+1}) \le \varphi(\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}).$$

Supposing that there exists $p \in \mathbb{N}$, $p \ge 1$ such that

$$d(x_{p-1},x_p) \le d(x_p,x_{p+1}),$$

and taking in consideration that φ is a comparison function, from (2.1.1) we have:

$$d(x_p,x_{p+1}) \le \varphi(d(x_p,x_{p+1})) < d(x_p,x_{p+1}),$$

which is a contradiction.

It follows that $d(x_{n-1},x_n) > d(x_n,x_{n+1})$, for any $n \ge 1$, thus (2.1.1) becomes

$$d(x_n,x_{n+1}) \le \varphi(d(x_{n-1},x_n)). \tag{2.1.2}$$

Using the monotonicity of φ, for any $n \ge 0$ we have

$$d(x_n,x_{n+1}) \le \varphi^n(d(x_0,x_1)). \tag{2.1.3}$$

For $p \ge 1$ and $n \ge 0$ we infer

$$d(x_n,x_{n+p}) \le \varphi^n(d(x_0,x_1))+\varphi^{n+1}(d(x_0,x_1))+\ldots+\varphi^{n+p-1}(d(x_0,x_1)), \tag{2.1.4}$$

and denoting $S_n := \sum_{k=0}^{n} \varphi^k(d(x_0,x_1))$,

$$d(x_n,x_{n+p}) \le S_{n+p-1} - S_{n-1}. \tag{2.1.5}$$

As φ is a strong comparison function,

$$\sum_{k=0}^{\infty} \varphi^k(d(x_0,x_1)) < \infty,$$

so there is $S \in \mathbb{R}_+$ such that $\lim_{n\to\infty} S_n = S$.

Using (2.1.5), $d(x_n, x_{n+p}) \to 0$ as $n \to \infty$, which means that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete subspace Y. So the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to a $x^* \in Y$.

The sequence $(x_n)_{n\in\mathbb{N}}$ has an infinite number of terms in each A_i, $i = \overline{1,m}$, so from each A_i one, we can extract a subsequence of $(x_n)_{n\in\mathbb{N}}$ which converges to $x^* = \lim\limits_{n\to\infty} x_n$.

Because A_i are closed, it follows $x^* \in \bigcap\limits_{i=1}^{m} A_i$.

We have to show that $x^* = f(x^*)$.

Assuming the contrary, for n sufficiently large,
$\max\left\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, f(x^*)), \frac{1}{2}[d(x_n, f(x^*))+d(x^*, x_{n+1})]\right\}$
$= d(x^*, f(x^*))$.

It follows that

$$d(x^*, f(x^*)) \le d(x^*, x_{n+1}) + d(x_{n+1}, f(x^*)) \le d(x^*, x_{n+1}) +$$
$$+\varphi\left(\max\left\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, f(x^*)), \frac{1}{2}[d(x_n, f(x^*))+d(x^*, x_{n+1})]\right\}\right)$$
$$= d(x^*, x_{n+1}) + \varphi(d(x^*, f(x^*)).$$

By letting $n \to \infty$, we deduce that $d(x^*, f(x^*)) \le \varphi(d(x^*, f(x^*)))$, which is a contradiction.

It remains to study the uniqueness of the fixed point. If x^* and y^* are two fixed point of f then

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \le \varphi(d(x^*, y^*)).$$

φ is a comparison function so we obtain $d(x^*, y^*) = 0$.

We still have to prove that the Picard iteration converges to x^* for any initial guess $x \in Y$. Note that

$$d(x_{n+1}, x^*) = d(f(x_n), f(x^*))$$
$$\le\varphi\left(\max\left\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, f(x^*)), \frac{1}{2}[d(x_n, f(x^*))+d(x^*, x_{n+1})]\right\}\right).$$

If we denote $a_n = d(x_n, x^*)$, $n \in \mathbb{N}$, the above relation becomes

$$a_{n+1} \le \varphi\left(\max\left\{a_n, d(x_n, x_{n+1}), 0, \frac{1}{2}(a_n + a_{n+1})\right\}\right).$$

Using the fact that $\frac{1}{2}(a_n + a_{n+1}) \leq \max\{a_n, a_{n+1}\}$, we get

$$a_{n+1} \leq \varphi(\max\{a_n, a_{n+1}, d(x_n, x_{n+1})\}).$$

But $\max\{a_n, a_{n+1}, d(x_n, x_{n+1})\} \neq a_{n+1}$, otherwise we would have $a_{n+1} \leq \varphi(a_{n+1})$, contradicting the assumption that $\varphi(t) < t$, for any $t > 0$.

Consequently,

$$a_{n+1} \leq \varphi(\max\{a_n, d(x_n, x_{n+1})\}), \text{ for any } n \in \mathbb{N}. \tag{2.1.6}$$

The following cases need to be analysed:

a) There exists $k \in \mathbb{N}$ such that $a_k < d(x_k, x_{k+1})$.

For $n = k$, the inequality (2.1.6) becomes

$$a_{k+1} \leq \varphi(d(x_k, x_{k+1})).$$

For $n = k + 1$, using (2.1.2), the inequality (2.1.6) becomes

$$\begin{aligned} a_{k+2} &\leq \varphi(\max\{a_{k+1}, d(x_{k+1}, x_{k+2})\}) \\ &\leq \varphi(\max\{a_{k+1}, \varphi(d(x_k, x_{k+1}))\}) \\ &\leq \varphi^2(d(x_k, x_{k+1})). \end{aligned}$$

By induction, we obtain

$$a_{k+p} \leq \varphi^p(d(x_k, x_{k+1})) \tag{2.1.7}$$

and by letting $p \to \infty$, the sequence $\{a_n\}_{n \geq 0}$ converges to 0.

b) For any $n \in \mathbb{N}$, $a_n \geq d(x_n, x_{n+1})$.

The inequality (2.1.6) becomes

$$a_{n+1} \leq \varphi(a_n), \text{ for any } n \in \mathbb{N},$$

so $a_n \leq \varphi^n(a_0)$, which implies again $a_n \to 0$, as $n \to \infty$.

(2) By letting $p \to \infty$ in (2.1.4), we obtain the a priori estimate

$$d(x_n, x^*) \leq s(\varphi^n(d(x_0, x_1))), \text{ for any } n \geq 0.$$

Using (2.1.2) and the monotonicity of φ, we have

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}) \\ &\leq \sum_{k=0}^{p-1} \varphi^k(d(x_n, x_{n+1})), \end{aligned}$$

and letting $p \to \infty$,

$$d(x_n, x^*) \le \sum_{k=0}^{\infty} \varphi^k(d(x_n, x_{n+1})), \; n \ge 0. \tag{2.1.8}$$

Considering the definition of s, this yields the a posteriori estimate

$$d(x_n, x^*) \le s(d(x_n, x_{n+1})), \text{ for any } n \ge 0.$$

(3) Let $x \in Y$. From (2.1.8), for $x_0 := x$ we have:

$$d(x, x^*) \le \sum_{k=0}^{\infty} \varphi^k(d(x, f(x))).$$

(4) Using the inequality (2.1.3),

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \le \sum_{n=0}^{\infty} \varphi^n(d(x_0, x_1)) = s(d(x_0, x_1)) < \infty.$$

(5) We use the inequality (2.1.6), i.e.,

$$a_{n+1} \le \varphi(\max\{a_n, d(x_n, x_{n+1})\}),$$

for any $n \in \mathbb{N}$, where $a_n := d(x_n, x^*)$. We need to discuss two cases.

a) If there exists $k \in \mathbb{N}$ such that $a_k < d(x_k, x_{k+1})$, then the inequality (2.1.7), i.e.,

$$a_{k+p} \le \varphi^p(d(x_k, x_{k+1}))$$

holds for any $p \in \mathbb{N}$. Then

$$\sum_{n=k+1}^{\infty} a_k \le \sum_{n=1}^{\infty} \varphi^n(d(x_k, x_{k+1})) < \infty,$$

so

$$\sum_{n=0}^{\infty} d(x_n, x^*) < \infty.$$

b) If $a_n \ge d(x_n, x_{n+1})$, for any $n \in \mathbb{N}$, then (2.1.6) becomes

$$a_{n+1} \le \varphi(a_n), \text{ for any } n \in \mathbb{N},$$

which implies $a_n \le \varphi^n(a_0)$.

Then

$$\sum_{n=0}^{\infty} a_n \le \sum_{n=0}^{\infty} \varphi^n(a_0) < \infty,$$

so again

$$\sum_{n=0}^{\infty} d(x_n, x^*) < \infty.$$

□

Remark 2.1.6. For a related result obtained by a different method, concerning the existence and uniqueness of the fixed point, we mention the paper [27]. Our results extend the above mentioned theorem for an extensive study of the fixed point problem.

The next result gives the well posedness property for the fixed point problem. For the concept of well posedness for the fixed point problems see Reich, Zaslavski [65].

Theorem 2.1.7. (Magdaş [33]) *Let $f : Y \to Y$ be as in Theorem 2.1.5. Then the fixed point problem for f is well posed, that is, assuming there exist $z_n \in Y$, $n \in \mathbb{N}$ such that*

$$d(z_n, f(z_n)) \to 0 \text{ as } n \to \infty,$$

this implies that

$$z_n \to x^* \text{ as } n \to \infty,$$

where $F_f = \{x^*\}$.

Proof. Using the inequality $d(x, x^*) \leq s(d(x, f(x)))$ from Theorem 2.1.5, for $x := z_n$, we have:

$$d(z_n, x^*) \leq s(d(z_n, f(z_n))),\ n \in \mathbb{N},$$

and letting $n \to \infty$ we obtain

$$d(z_n, x^*) \to 0,\ n \to \infty.$$

□

Theorem 2.1.8. (Magdaş [33]) *Let $f : Y \to Y$ be as in Theorem 2.1.5, and $g : Y \to Y$ be such that:*

(i) g has at least one fixed point $x_g^* \in F_g$;

(ii) there exists $\eta > 0$ such that

$$d(f(x), g(x)) \leq \eta, \text{ for any } x \in Y.$$

Then $d(x_f^*, x_g^*) \leq s(\eta)$, where $F_f = \{x_f^*\}$ and s is defined in Lemma 1.2.3.

Proof. By letting $x := x_g^*$ in the inequality $d(x, x^*) \leq s(d(x, f(x)))$, we have

$$d(x_f^*, x_g^*) \leq s(d(x_g^*, f(x_g^*))) = s(d(g(x_g^*), f(x_g^*))).$$

Using the monotonicity of s we obtain $d(x_f^*, x_g^*) \leq s(\eta)$. □

Theorem 2.1.9. (Magdaş [33]) *Let $f : Y \to Y$ be as in Theorem 2.1.5 and $f_n : Y \to Y$, $n \in \mathbb{N}$ be such that:*

(i) for each $n \in \mathbb{N}$ there exists $x_n^* \in F_{f_n}$;

(ii) $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

Then $x_n^* \to x^*$ as $n \to \infty$, where $F_f = \{x^*\}$.

Proof. As $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f, there exists $\eta_n \in \mathbb{R}_+$, $n \in \mathbb{N}$ such that $\eta_n \to 0$, $n \to \infty$ and $d(f_n(x), f(x)) \leq \eta_n$, for any $x \in Y$.

Using Theorem 2.1.7 for $g := f_n$, $n \in \mathbb{N}$, we have

$$d(x_n^*, x^*) \leq s(\eta_n),\ \ n \in \mathbb{N}.$$

By letting $n \to \infty$ above, we get $d(x_n, x^*) \to 0$. □

The following theorem is a Maia type result regarding Ćirić type generalized contractions defined on cyclic representations.

Theorem 2.1.10. (Magdaş [33]) *Let X be a nonempty set, d and ρ be two metrics on X, m a positive integer, $A_1, \ldots, A_m \in P_{cl}(X)$, $Y \in P(X)$ and $f : Y \to Y$ be an operator. Assume that:*

(i) there exists $c > 0$ such that $d(x, y) \leq c \cdot \rho(x, y)$, for any $x, y \in Y$;

(ii) (Y, d) is a complete metric space;

(iii) $f : (Y, d) \to (Y, d)$ is continuous;

(iv) $f : (Y, \rho) \to (Y, \rho)$ is a cyclic φ-contraction of Ćirić type.

Then f has a unique fixed point $x^* \in \bigcap_{i=1}^{m} A_i$ and the Picard iteration $(x_n)_{n\in\mathbb{N}}$ given by (1.3.1) *converges to x^* for any starting point $x_0 \in Y$.*

Proof. By the same reasoning as in Theorem 2.1.5, using condition (iv), we obtain that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, ρ).

Using condition (i) it follows that it is Cauchy in (X, d) as well.

By (ii) and (iii) it is easy to prove that $(x_n)_{n\in\mathbb{N}}$ converges to the unique fixed point of f. □

Remark 2.1.11. It is an open problem to find conditions under which the operator $f : Y \to Y$ defined as in Theorem 2.1.5 has the Ostrowski's stability property that is, if $F_f = \{x^*\}$ and for any sequence $(z_n)_{n\in\mathbb{N}} \subset Y$, with the property $d(z_{n+1}, f(z_n)) \to 0$ as $n \to \infty$, we have

$$z_n \to x^* \text{ as } n \to \infty.$$

2.2 Perov type theorems for cyclic contractions

The aim of this section is to prove a fixed point theorem of Perov type for cyclic contractions on complete generalized metric spaces. Then, as applications, we will study the existence, uniqueness and approximation of the solution for a system of Fredholm type of integral equations, as well as the continuous dependence phenomenon of the given system. Also, we will study the existence and uniqueness of the solution for a system of Volterra type of integral equations.

The matrices convergent to zero were used by Perov and Kibenko [45] to generalize the contraction principle in the case of metric spaces with a vector-valued distance.

Theorem 2.2.1. (Varga [79], Rus, A. Petruşel, G. Petruşel [70])

Let $S \in \mathcal{M}_p(\mathbb{R}_+)$. The following statements are equivalent:

(i) S is a matrix convergent to zero, that is $S^k \to 0$ as $k \to +\infty$;

(ii) $S^k x \to 0$ as $k \to +\infty$, $\forall\, x \in \mathbb{R}^p$;

(iii) $I_p - S$ is non-singular and

$$(I_p - S)^{-1} = I_p + S + S^2 + \dots \tag{2.2.1}$$

(iv) $I_p - S$ is non-singular and $(I_p - S)^{-1}$ has nonnegative elements;

(v) $\lambda \in \mathbb{C}$, $\det(S - \lambda I_p) = 0$ implies $|\lambda| < 1$.

Definition 2.2.2. (Rus, A. Petruşel, G. Petruşel [70]) Let (X, d) be a metric space with $d : X \times X \to \mathbb{R}^p_+$ a vector-valued distance and $f : X \to X$. The operator f is called an S-contraction if there exists a matrix $S \in \mathcal{M}_p(\mathbb{R}_+)$ such that:

(i) S is a matrix convergent to zero;

(ii) $d(f(x), f(y)) \leq Sd(x, y), \ \forall \ x, y \in X.$

Theorem 2.2.3. (Perov, Kibenko [45]) *Let (X, d) be a complete metric space with $d : X \times X \to \mathbb{R}^p_+$ a vector-valued distance and $f : X \to X$ be an S-contraction. Then:*

(i) f has a unique fixed point $x^* \in X$;

(ii) $f^k(x) \xrightarrow{d} x^*$ as $k \to +\infty$, for all $x \in X$;

(iii) $d(f^k(x), x^*) \leq S^k(I_p - S)^{-1}d(x, f(x))$, for all $x \in X$ and $k \in \mathbb{N}$;

(iv) $d(x, x^*) \leq (I_p - S)^{-1}d(x, f(x))$ for all $x \in X$.

We recall the following notion, introduced in [36], suggested by the considerations in [29].

Definition 2.2.4. (Magdaş [36]) Let (X, d) be a metric space with $d : X \times X \to \mathbb{R}^p_+$ a vector-valued distance, $A_1, \ldots, A_m \in P_{cl}(X)$ and $f : X \to X$ be an operator. If:

(i) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to f;

(ii) there exists a matrix $S \in \mathcal{M}_p(\mathbb{R}_+)$ convergent to zero such that

$$d(f(x), f(y)) \leq S \cdot d(x, y), \text{ for any } x \in A_i, \ y \in A_{i+1}, \text{ where } A_{m+1} = A_1,$$

then, by definition, we say that f is a cyclic S-contraction.

The main result of this section is the following theorem which generalizes the Perov fixed point Theorem 2.2.3, in spaces endowed with vector-valued metrics.

Theorem 2.2.5. (Magdaş [36]) *Let (X, d) be a complete metric space with $d : X \times X \to \mathbb{R}^p_+$ a vector-valued distance, $A_1, A_2, \ldots, A_m \in P_{cl}(X)$. If $f : X \to X$ is a cyclic S-contraction then the following statements hold:*

(1) f has a unique fixed point $x^* \in \bigcap_{i=1}^{m} A_i$ and the Picard iteration $(x_n)_{n \in \mathbb{N}}$ given by

$$x_n = f(x_{n-1}), \ n \geq 1,$$

converges to x^* for any starting point $x_0 \in X$;

(2) the following estimates hold:

$$d(x_n, x^*) \leq S^n(I_p - S)^{-1}d(x_0, x_1), \ n \geq 1; \qquad (2.2.2)$$

$$d(x_n, x^*) \leq (I_p - S)^{-1} d(x_n, x_{n+1}),\ \ n \geq 1; \tag{2.2.3}$$

(3) for any $x \in X$,

$$d(x, x^*) \leq (I_p - S)^{-1} d(x, f(x)). \tag{2.2.4}$$

Proof. (1) $d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \leq Sd(x_{n-1}, x_n)$

$$\leq \ldots \leq S^n d(x_0, x_1)$$

For $k \geq 1$ we have

$$\begin{aligned} d(x_n, x_{n+k}) &\leq S^n d(x_0, x_1) + S^{n+1} d(x_0, x_1) + \ldots + S^{n+k-1} d(x_0, x_1) \\ &= S^n (I_p + S + S^2 + \ldots + S^{k-1}) d(x_0, x_1) \\ &\leq S^n (I_p + S + S^2 + \ldots) d(x_0, x_1) \\ &= S^n (I_p - S)^{-1} d(x_0, x_1) \to 0 \text{ as } n \to \infty, \end{aligned} \tag{2.2.5}$$

which means that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.
(X, d) is a complete metric space, so the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to a $q \in X$.

The sequence $(x_n)_{n\geq 0}$ has an infinite number of terms in each A_i, $i = \overline{1, m}$, so from each A_i one we can extract a subsequence of $(x_n)_{n\geq 0}$ which converges to $q = \lim\limits_{n\to\infty} x_n$.

Because A_i are closed, $q \in \bigcap\limits_{i=1}^{m} A_i$, so $\bigcap\limits_{i=1}^{m} A_i \neq \emptyset$.

Let be the restriction $f\Big|_{\bigcap\limits_{i=1}^{m} A_i} : \bigcap\limits_{i=1}^{m} A_i \to \bigcap\limits_{i=1}^{m} A_i$.

$\bigcap\limits_{i=1}^{m} A_i$ is also complete. Applying Perov's theorem, $f\Big|_{\bigcap\limits_{i=1}^{m} A_i}$ has a unique fixed point, which can be obtained by means of the Picard iteration starting from any initial point.

It remains to prove that the Picard iteration converges to x^*, for any initial guess $x \in X$.

$$\begin{aligned} d(x_{n+1}, x^*) &= d(f(x_n), f(x^*)) \leq Sd(x_n, x^*) \\ &\leq \ldots \leq S^n d(x_0, x^*) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Then for any $n \in \mathbb{N},\ k \geq 1$, we have

$$\begin{aligned} d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+k-1}, x_{n+k}) \\ &\leq d(x_n, x_{n+1}) + Sd(x_n, x_{n+1}) + \ldots + S^{k-1} d(x_n, x_{n+1}) \\ &= (I_p + S + \ldots + S^{k-1}) d(x_n, x_{n+1}) \\ &\leq (I_p - S)^{-1} d(x_n, x_{n+1}). \end{aligned} \tag{2.2.6}$$

Using the statement (iii) from Theorem 2.2.1, by letting $k \to \infty$ in (2.2.5) and (2.2.6) we obtain the estimates (2.2.2) and (2.2.3).
(3) Let $x \in X$. For $n = 0$, $x_0 := x$, the a posteriori estimate (2.2.3) becomes

$$d(x, x^*) \leq (I_p - S)^{-1} d(x, f(x)).$$

□

The conclusions of Theorem 2.2.5 are useful to study the data dependence and the well posedness of the fixed point of a cyclic S-contraction.

Theorem 2.2.6. (Magdaş [36]) *Let $f : X \to X$ be as in Theorem 2.2.5 with $F_f = \{x_f^*\}$. Let $g : X \to X$ be an operator such that:*
(i) g has at least one fixed point x_g^*;
(ii) there exists $\eta > 0$ such that

$$d(f(x), g(x)) \leq \eta, \ \textit{for any } x \in X.$$

Then $d(x_f^*, x_g^*) \leq \eta (I_p - S)^{-1}$.

Proof. By letting $x := x_g^*$ in the inequality (2.2.4), we have

$$\begin{aligned} d(x_g^*, x_f^*) &\leq (I_p - S)^{-1} d(x_g^*, f(x_g^*)) = (I_p - S)^{-1} d(g(x_g^*), f(x_g^*)) \\ &\leq (I_p - S)^{-1} \eta. \end{aligned}$$

□

Theorem 2.2.7. (Magdaş [36]) *Let $f : X \to X$ be as in Theorem 2.2.5. Then the fixed point problem for f is well posed, that is, assuming there exist $z_n \in X$, $n \in \mathbb{N}$ such that $d(z_n, f(z_n)) \to 0$, as $n \to \infty$, this implies that $z_n \to x^*$, as $n \to \infty$, where $F_f = \{x^*\}$.*

Proof. By letting $x := z_n$ in the inequality (2.2.4), we have

$$d(z_n, x^*) \leq (I_p - S)^{-1} d(z_n, Tz_n),\ n \in \mathbb{N}$$

and letting $n \to \infty$ we obtain $d(z_n, x^*) \to 0$, $n \to \infty$. □

Further on we apply the results given by Theorem 2.2.5 to study existence and uniqueness of the solutions of the following system of Fredholm type integral equations:

$$\begin{cases} x_1(t) = \displaystyle\int_a^b G_1(t,s) f_1(s, x_1(s), x_2(s)) ds \\ x_2(t) = \displaystyle\int_a^b G_2(t,s) f_2(s, x_1(s), x_2(s)) ds \end{cases},\ t \in [a,b] \tag{2.2.7}$$

where $a, b \in \mathbb{R}$, $a < b$,

$$G_1, G_2 \in C([a,b] \times [a,b], [0,\infty)),$$

$$f_1, f_2 \in C([a,b] \times \mathbb{R} \times \mathbb{R}).$$

Theorem 2.2.8. (Magdaş [36]) *We suppose that:*

(i) there exist $\alpha_k, \beta_k \in C[a,b]$, $m_k, M_k \in \mathbb{R}, m_k \leq \alpha_k(t) \leq \beta_k(t) \leq M_k$, for any $t \in [a,b]$, such that for $k \in \{1,2\}$,

$$\begin{cases} \alpha_k(t) \leq \displaystyle\int_a^b G_k(t,s) f_k(s, \beta_1(s), \beta_2(s)) ds \\ \beta_k(t) \geq \displaystyle\int_a^b G_k(t,s) f_k(s, \alpha_1(s), \alpha_2(s)) ds \end{cases},\ \textit{for any } t \in [a,b]. \tag{2.2.8}$$

(ii) there exist $a_1, b_1, a_2, b_2 \in \mathbb{R}_+$ such that

$$\begin{aligned} |f_1(s, u_1, u_2) - f_1(s, v_1, v_2)| &\leq a_1|u_1 - v_1| + a_2|u_2 - v_2|, \\ |f_2(s, u_1, u_2) - f_2(s, v_1, v_2)| &\leq b_1|u_1 - v_1| + b_2|u_2 - v_2|, \end{aligned} \tag{2.2.9}$$

for any $s \in [a,b]$ and $u_k, v_k \in \mathbb{R}$, with

$$\begin{cases} u_k \leq M_k \\ v_k \geq m_k \end{cases} \quad \textit{or} \quad \begin{cases} u_k \geq m_k \\ v_k \leq M_k \end{cases} \quad \textit{for } k \in \{1,2\};$$

(iii) $\displaystyle\sup_{t\in[a,b]} \int_a^b G_k(t,s) ds \leq 1$ *for $k \in \{1,2\}$;*

(iv) f_k *is decreasing in each of the last two variables, that is,*

$$u_1, u_2, v_1, v_2 \in \mathbb{R},\ u_1 \leq v_1,\ u_2 \leq v_2 \Rightarrow f_k(s,u,v) \geq f_k(s,u_2,v_2),$$

for any $s \in [a,b]$*, and* $k \in \{1,2\}$*;*

(v) the matrix $S = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ *converges to zero.*

Then the system (2.2.7) has a unique solution

$x^* = (x_1^*, x_2^*) \in C([a,b],\mathbb{R}^2)$*, with* $\alpha_k \leq x_k^* \leq \beta_k$*, for* $k \in \{1,2\}$*.*

This solution can be obtained by the successive approximations method, starting at any element $x^0 \in C([a,b],\mathbb{R}^2)$*. Moreover, if* x^n *is the* n^{th} *successive approximation, then we have the following estimation:*

$$\|x^* - x^n\| \leq S^n (I_2 - S)^{-1} \|x^0 - x^1\|,$$

where

$$\|x\| = \begin{pmatrix} |x_1|_\infty \\ |x_2|_\infty \end{pmatrix} \quad \text{and} \quad |x|_\infty = \max_{t \in [a,b]} |x(t)|.$$

Proof. Let us denote

$$X := (C[a,b], |\cdot|_\infty),\ Z = X \times X,$$

$$\|\cdot\| : Z \to \mathbb{R}^2,\ \|x\| = \|(x_1,x_2)\| = \begin{pmatrix} |x_1|_\infty \\ |x_2|_\infty \end{pmatrix}.$$

Then $(Z, \|\cdot\|)$ is a generalized Banach space.

We consider the following closed subsets of X:

$$A_1 = \{(x_1,x_2) \in Z \mid x_k \leq \beta_k,\ k \in \{1,2\}\},$$

$$A_2 = \{(x_1,x_2) \in Z \mid x_k \geq \alpha_k,\ k \in \{1,2\}\},$$

and the operator $T : Z \to Z$,

$$(x_1,x_2) = x \mapsto Tx = (T_1 x, T_2 x),$$

$$T_k x(t) := \int_a^b G_k(t,s) f_k(s, x_1(s), x_2(s)) ds, \text{ for } k \in \{1,2\}. \tag{2.2.10}$$

The system (2.2.7) is equivalent with the equation $Tx = x$.

We will prove that $A_1 \cup A_2$ is a cyclic representation of Z with respect to T.

Taking $x = (x_1, x_2) \in A_1$ we have $x_k(s) \le \beta_k(s),\ \forall\, s \in [a,b]$, for $k \in \{1,2\}$. Using the monotonicity of f_k we have

$$G_k(t,s)f_k(s, x_1(s), x_2(s)) \ge G_k(t,s)f_k(s, \beta_1(s), \beta_2(s)),\ \text{for } k \in \{1,2\}$$

and from (i), by integration,

$$\int_a^b G_k(t,s)f_k(s, x_1(s), x_2(s))ds \ge \alpha_k(t),$$

which means that

$$T_k x(t) \ge \alpha_k(t),\ \forall\, t \in [a,b],\ \text{for } k \in \{1,2\} \Rightarrow Tx \in A_2.$$

So $TA_1 \subseteq A_2$. In a similar way we have $TA_2 \subseteq A_1$.

Using the conditions (ii) and (iii) we have

$$\begin{aligned}|T_k x(t) - T_k y(t)| &\le \int_a^b G_k(t,s)|f_k(s, x_1(s), x_2(s)) - f_k(s, y_1(s), y_2(s))|ds \\ &\le \int_a^b G_k(t,s)(a_k|x_1(s) - y_1(s)| + b_k|x_2(s) - y_2(s)|)ds \\ &\le \int_a^b G_k(t,s)(a_k|x_1 - y_1|_\infty + b_k|x_2 - y_2|_\infty)ds \\ &\le a_k|x_1 - y_1|_\infty + b_k|x_2 - y_2|_\infty,\ \forall\, t \in [a,b].\end{aligned}$$

We infer

$$|T_k x - T_k y|_\infty \le a_k|x_1 - y_1|_\infty + b_k|x_2 - y_2|_\infty,$$

which implies

$$\begin{pmatrix} |T_1 x - T_1 y|_\infty \\ |T_2 x - T_2 y|_\infty \end{pmatrix} \le S \begin{pmatrix} |x_1 - y_1|_\infty \\ |x_2 - y_2|_\infty \end{pmatrix},$$

so we have

$$\|Tx - Ty\| \le S\|x - y\|,\ \text{for any } (x,y) \in A_1 \times A_2,$$

and by the condition (v) it results that the operator T is a cyclic S-contraction.

All the conditions of Theorem 2.2.5 are satisfied, so T has a unique fixed point

$$x^* = (x_1^*, x_2^*) \in A_1 \cap A_2,\ \text{with } \alpha_k \le x_k^* \le \beta_k,\ \text{for } k \in \{1,2\}.$$

This finishes the proof. □

Further on we study the continuous dependence phenomenon for the system (2.2.7).

We consider the perturbed system of integral equations

$$\begin{cases} y_1(t) = \int_a^b H_1(t,s)g_1(s,y_1(s),y_2(s))ds \\ y_2(t) = \int_a^b H_2(t,s)g_2(s,y_1(s),y_2(s))ds \end{cases} \tag{2.2.11}$$

where

$$H_1, H_2 \in C([a,b]\times[a,b],[0,\infty)), \quad g_1, g_2 \in C([a,b]\times\mathbb{R}\times\mathbb{R}).$$

Theorem 2.2.9. (Magdaş [36]) *We suppose that the conditions of Theorem 2.2.5 are satisfied and we denote by x^* the unique solution of the system of integral equations (2.2.7).*

If $y^* \in C([a,b],\mathbb{R}^2)$ is a solution of the perturbed system of integral equations (2.2.17), and

$$\sup_{t\in[a,b]} \int_a^b H_k(t,s)ds \leq 1,$$

then we have the following estimation:

$$\|x^* - y^*\|_{\mathbb{R}^2} \leq (I_2 - S)^{-1}(\eta + \tau), \tag{2.2.12}$$

where $\eta = (\eta_1,\eta_2)$, $\tau = (\tau_1,\tau_2)$ and

$$\begin{cases} \eta_k = \sup\{|f_k(s,u,v)| \ \mid s\in[a,b],\ u,v\in\mathbb{R}\}, \\ \tau_k = \sup\{|g_k(s,u,v)| \ \mid s\in[a,b],\ u,v\in\mathbb{R}\}, \end{cases} \quad \textit{for } k\in\{1,2\}.$$

Proof. We consider the operator $T: Z\to Z$ attached to the system (2.2.7), defined by the relation (2.2.16).

Let $U: Z\to Z$ be an operator attached to the perturbed system (2.2.17) and defined by the relation:

$$(y_1,y_2) = y \mapsto Uy = (U_1y, U_2y),$$

$$U_ky(t) := \int_a^b H_k(t,s)g_k(s,y_1(s),y_2(s))ds, \text{ for } k\in\{1,2\}.$$

We have

$$|T_kx(t) - U_kx(t)| \leq \int_a^b G_k(t,s)|f_k(s,x_1(s),x_2(s))ds$$

$$+\int_a^b H_k(t,s)|g_k(s,x_1(s),x_2(s))|ds$$

$$\leq \eta_k \int_a^b G_k(t,s)ds + \tau_k \int_a^b H_k(t,s)ds \leq \eta_k + \tau_k,\ \forall\ t \in [a,b], \quad \text{for } k \in \{1,2\},$$

which implies

$$|T_k x - U_k x|_\infty \leq \eta_k + \tau_k, \quad \text{for } k \in \{1,2\}.$$

We conclude

$$\|Tx - Ux\| \leq \eta + \tau,\ \forall\ x \in Z.$$

The conditions of Theorem 2.2.6 are satisfied, so the estimation (2.2.18) is proved. □

Remark 2.2.10. A similar approach can be achieved for a system of Volterra type integral equations using, instead of the supremum norm, the Bielecki type norm approach. For example, we have the following result.

Theorem 2.2.11. *Considering the following system of Volterra type integral equations:*

$$\begin{cases} x_1(t) = \displaystyle\int_a^t G_1(t,s)f_1(s,x_1(s),x_2(s))ds \\ x_2(t) = \displaystyle\int_a^t G_2(t,s)f_2(s,x_1(s),x_2(s))ds \end{cases},\ t \in [a,b], \tag{2.2.13}$$

where $a, b \in \mathbb{R}$, $a < b$,

$$G_1, G_2 \in C([a,b]\times[a,b],[0,\infty)),$$

$$f_1, f_2 \in C([a,b]\times\mathbb{R}\times\mathbb{R}),$$

we suppose that:

(i) there exist $\alpha_k, \beta_k \in C[a,b]$, $m_k, M_k \in \mathbb{R}, m_k \leq \alpha_k(t) \leq \beta_k(t) \leq M_k$, *for any* $t \in [a,b]$, *such that for* $k \in \{1,2\}$,

$$\begin{cases} \alpha_k(t) \leq \displaystyle\int_a^t G_k(t,s)f_k(s,\beta_1(s),\beta_2(s))ds \\ \beta_k(t) \geq \displaystyle\int_a^t G_k(t,s)f_k(s,\alpha_1(s),\alpha_2(s))ds \end{cases}, \text{ for any } t \in [a,b]. \tag{2.2.14}$$

(ii) there exist $a_1, b_1, a_2, b_2 \in \mathbb{R}_+$ *such that*

$$\begin{aligned} |f_1(s,u_1,u_2) - f_1(s,v_1,v_2)| &\le a_1|u_1 - v_1| + a_2|u_2 - v_2|, \\ |f_2(s,u_1,u_2) - f_2(s,v_1,v_2)| &\le b_1|u_1 - v_1| + b_2|u_2 - v_2|, \end{aligned} \tag{2.2.15}$$

for any $s \in [a,b]$ *and* $u_k, v_k \in \mathbb{R}$, *with*

$$\begin{cases} u_k \le M_k \\ v_k \ge m_k \end{cases} \quad or \quad \begin{cases} u_k \ge m_k \\ v_k \le M_k \end{cases} \quad for\ k \in \{1,2\};$$

(iii) f_k *is decreasing in each of the last two variables, that is,*

$$u_1, u_2, v_1, v_2 \in \mathbb{R},\ u_1 \le v_1,\ u_2 \le v_2 \Rightarrow f_k(s,u,v) \ge f_k(s,u_2,v_2),$$

for any $s \in [a,b]$, *and* $k \in \{1,2\}$.
Then the system (2.2.13) has a unique solution
$x^* = (x_1^*, x_2^*) \in C([a,b],\mathbb{R}^2)$, *with* $\alpha_k \le x_k^* \le \beta_k$, *for* $k \in \{1,2\}$.

Proof. Let us denote

$$X := (C[a,b], |\cdot|_B),\ Z = X \times X,$$

$$\|\cdot\|_B : Z \to \mathbb{R}^2,\ \|x\|_B = \|(x_1,x_2)\| = \begin{pmatrix} |x_1|_B \\ |x_2|_B \end{pmatrix},$$

where

$$|x|_B = \max_{t\in[a,b]}(|x(t)|e^{-\tau(t-a)}), \tau > 0.$$

Then $(Z, \|\cdot\|_B)$ is a generalized Banach space.

We consider the following closed subsets of X:

$$A_1 = \{(x_1,x_2) \in Z \mid x_k \le \beta_k,\ k \in \{1,2\}\},$$

$$A_2 = \{(x_1,x_2) \in Z \mid x_k \ge \alpha_k,\ k \in \{1,2\}\},$$

and the operator $T : Z \to Z$,

$$(x_1,x_2) = x \mapsto Tx = (T_1x, T_2x),$$

$$T_k x(t) := \int_a^t G_k(t,s) f_k(s,x_1(s),x_2(s))ds,\ \text{for } k \in \{1,2\}. \tag{2.2.16}$$

The system (2.2.13) is equivalent with the equation $Tx = x$.
We will prove that $A_1 \cup A_2$ is a cyclic representation of Z with respect to T.
Let $x = (x_1, x_2) \in A_1 \Rightarrow x_k(s) \leq \beta_k(s),\ \forall\ s \in [a, b]$, for $k \in \{1, 2\}$.
Using the monotonicity of f_k we have

$$G_k(t,s) f_k(s, x_1(s), x_2(s)) \geq G_k(t,s) f_k(s, \beta_1(s), \beta_2(s)),\ \text{for } k \in \{1, 2\}$$

and from (i), by integration,

$$\int_a^t G_k(t,s) f_k(s, x_1(s), x_2(s)) ds \geq \alpha_k(t),$$

we have

$$T_k x(t) \geq \alpha_k(t),\ \forall\ t \in [a, b],\ \text{for } k \in \{1, 2\} \Rightarrow$$

which means that $Tx \in A_2$. So $TA_1 \subseteq A_2$.

In a similar way we have $TA_2 \subseteq A_1$.

Using the condition (ii), for $k \in \{1, 2\}$ we have

$$|T_k x(t) - T_k y(t)| \leq \int_a^t G_k(t,s) |f_k(s, x_1(s), x_2(s)) - f_k(s, y_1(s), y_2(s))| ds$$

$$\leq \int_a^t G_k(t,s)(a_k |x_1(s) - y_1(s)| + b_k |x_2(s) - y_2(s)|) ds$$

$$\leq \int_a^t G_k(t,s)(a_k |x_1 - y_1|_B + b_k |x_2 - y_2|_B) e^{\tau(s-a)} ds$$

$$\leq (a_k |x_1 - y_1|_B + b_k |x_2 - y_2|_B) \int_a^t G_k(t,s) e^{\tau(s-a)} ds$$

$$\leq M_G (a_k |x_1 - y_1|_B + b_k |x_2 - y_2|_B) \int_a^t e^{\tau(s-a)} ds$$

$$= M_G (a_k |x_1 - y_1|_B + b_k |x_2 - y_2|_B) \frac{1}{\tau} [e^{\tau(t-a)} - 1]$$

$$\leq \frac{M_G}{\tau} (a_k |x_1 - y_1|_B + b_k |x_2 - y_2|_B) e^{\tau(t-a)},\ \forall\ t \in [a, b],$$

$$\text{where} \quad M_G = \max_{t,s \in [a,b]} \{G_1(t,s), G_2(t,s)\},$$

which implies

$$\| T_k x(t) - T_k y(t) | e^{-\tau(t-a)} \leq \frac{M_G}{\tau} (a_k |x_1 - y_1|_B + b_k |x_2 - y_2|_B),\ \forall\ t \in [a, b].$$

We infer

$$|T_k x - T_k y|_B \leq \frac{M_G}{\tau} (a_k |x_1 - y_1|_B + b_k |x_2 - y_2|_B),$$

which means

$$\begin{pmatrix} |T_1x - T_1y|_B \\ \\ |T_2x - T_2y|_B \end{pmatrix} \leq \frac{M_G}{\tau} \cdot S \cdot \begin{pmatrix} |x_1 - y_1|_B \\ \\ |x_2 - y_2|_B \end{pmatrix}, \text{where } S = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}.$$

So we have

$$\|Tx - Ty\|_B \leq \frac{M_G}{\tau} \cdot S \cdot \|x - y\|_B, \text{ for any } (x, y) \in A_1 \times A_2. \qquad (2.2.17)$$

Hereinafter we will study for which values of $\tau > 0$ the matrix $\frac{M_G}{\tau} \cdot S$ converges to zero.

According to Theorem 2.2.1, this condition is equivalent to

$$\det(\frac{M_G}{\tau} \cdot S - \lambda I_2) = 0, \lambda \in \mathbb{C} \qquad (2.2.18)$$

imply $|\lambda| < 1$.

The equation (2.2.18) is equivalent to

$$\lambda^2 - \frac{M_G}{\tau} tr(S)\lambda + \left(\frac{M_G}{\tau}\right)^2 \cdot det(S) = 0.$$

We have to discuss two cases:

1) $\Delta < 0 \Leftrightarrow (tr(S))^2 - 4det(S) < 0$.

$|\lambda_{1,2}| < 1 \Leftrightarrow \left(\frac{M_G}{\tau}\right)^2 \cdot |det(S)| < 1 \Leftrightarrow \tau > M_G\sqrt{det(S)}$.

2) $\Delta \geq 0 \Leftrightarrow (tr(S))^2 - 4det(S) \geq 0$.

$|\lambda_{1,2}| < 1 \Leftrightarrow \frac{M_G}{\tau} \left| \frac{tr(S) \pm \sqrt{(tr(S))^2 - 4det(S)}}{2} \right| < 1$

$\Leftrightarrow \tau > \frac{M_G}{2} \left| tr(S) \pm \sqrt{(tr(S))^2 - 4det(S)} \right|$.

Thus, if we choose

$$\tau > \frac{M_G}{2} \max \left\{ \sqrt{4det(S)}, \left| tr(S) \pm \sqrt{(tr(S))^2 - 4det(S)} \right| \right\},$$

then the matrix $\frac{M_G}{\tau} \cdot S$ converges to zero.

By (2.2.17), the operator T is a $\frac{M_G}{\tau} \cdot S$ - contraction.

All the conditions of Theorem 2.2.5 are satisfied, so T has a unique fixed point

$$x^* = (x_1^*, x_2^*) \in A_1 \cap A_2, \text{ with } \alpha_k \leq x_k^* \leq \beta_k, \text{ for } k \in \{1, 2\}.$$

□

Remark 2.2.12. Similar with the apriori estimation achieved in Theorem 2.2.8 for a system of Fredholm type integral equations using the supremum norm approach, in the framework of a system of Volterra type integral equation using the Bielecki type norm approach can be achieved an estimation depending of τ.

2.3 Coupled fixed point theorems for single-valued cyclic contraction type operators

The purpose of this section is to study the coupled fixed point problem for single-valued cyclic contraction type operators:

If (X, d) is a metric space, $A, B \in P(X)$, $F : X \times X \to X$ is a single-valued operator satisfying the cyclic condition $F(A \times B) \subseteq B, F(B \times A) \subseteq A$, then we are interested to find $(x^*, y^*) \in X \times X$ such that

$$\begin{cases} F(x^*, y^*) = x^* \\ F(y^*, x^*) = y^*. \end{cases} \tag{2.3.1}$$

The pair (x^*, y^*) is called coupled fixed point of the single-valued operator $F : X \times X \to X$. If $x^* = y^*$ then x^* is said to be strong coupled fixed point of F.

The approach is based on fixed point results for appropriate operators generated by the initial problems.

The first aim of this section is to generalize Theorem 1.5.9, Theorem 1.5.11, Theorem 1.5.13 and Theorem 1.5.15, weakening the contractive condition. Also, we may observe that the assumption $A \cap B \neq \emptyset$ from Theorem 1.5.9 and Theorem 1.5.15 is not necessary. We also prove the uniqueness of the strong coupled fixed point and we provide an iterative method for approximating the strong coupled fixed point. On the other hand, some qualitative properties of the coupled fixed point set, such as data dependence, generalized Ulam-Hyers stability and well posedness are studied. Our approach is based on the following idea inspired by the work of A. Petruşel in [55]: we transform the coupled fixed point problem into a fixed point problem for an appropriate operator defined on a cartesian product of the spaces. In this way, many coupled fixed point results can be obtained using classical fixed point theorems.

We introduce now the following concept.

Definition 2.3.1. (Magdaş [35])

Let (X, d) be a metric space, $A, B \in P_{cl}(X)$, $Y = A \cup B$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a strong comparison function. By definition, an operator $F : Y \times Y \to Y$ is called a cyclic coupled φ-contraction of Ćirić type if the following statements hold:

(i) F is cyclic with respect to A and B;

(ii)

$$d(F(x, y), F(u, v)) \leq \varphi(M(x, v, y, u)), \tag{2.3.2}$$

for any $x, v \in A$ and $y, u \in B$, where

$$\begin{aligned} M(x, v, y, u) = \max \Big\{ & d(x, u), d(v, y), d(x, F(x, y)), d(u, F(u, v)), d(v, F(v, u)), \\ & d(y, F(y, x)), \frac{1}{2}[d(x, F(u, v)) + d(u, F(x, y))], \\ & \frac{1}{2}[d(y, F(v, u)) + d(v, F(y, x))] \Big\}. \end{aligned}$$

The main result of this section is the following theorem which generalizes Theorem 1.5.9, Theorem 1.5.11, Theorem 1.5.13 and Theorem 1.5.15.

Theorem 2.3.2. (Magdaş [35]) *Let (X, d) be a complete metric space, $A, B \in P_{cl}(X)$, $Y = A \cup B$ and $F : Y \times Y \to Y$ a cyclic coupled φ-contraction of Ćirić type. Then:*

(1) *F has a unique strong coupled fixed point $x^* \in A \cap B$;*

(2) *for any $(x_0, y_0) \in A \times B$, there exists a sequence $((x_n, y_n))_{n \in \mathbb{N}} \subset X \times X$ defined by*

$$\begin{cases} x_n = F(y_{n-1}, x_{n-1}) \\ y_n = F(x_{n-1}, y_{n-1}) \end{cases}, \text{ for } n \geq 1,$$

that converges to (x^*, x^*);

(3) *the following estimates hold for any $n \in \mathbb{N}$:*

$$\max\{d(x_n, x^*), d(y_n, x^*)\} \leq s(\varphi^n(\max\{d(x_0, F(x_0, y_0)), d(y_0, F(y_0, x_0))\})),$$

$$\max\{d(x_n, x^*), d(y_n, x^*)\} \leq s(\max\{d(x_n, x_{n+1}), d(y_n, y_{n+1})\});$$

(4) *for any* $x, y \in Y$, $d(x, x^*) \leq s(\max\{d(x, F(x,y)), d(y, F(y,x))\})$, *where* s *is given by Lemma 1.2.3.*

Proof. $(1) - (2)$ Changing the roles between x and v and similarly for y and u, the inequality (2.3.2) becomes:

$$d(F(v,u), F(y,x)) \leq \varphi(M(v,x,u,y)), \text{ for } x, v \in A \text{ and } y, u \in B. \tag{2.3.3}$$

Obviously, $M(x,v,y,u) = M(v,x,u,y)$. From the inequalities (2.3.2) and (2.3.3) we obtain

$$\max\{d(F(x,y), F(u,v)), d(F(y,x), F(v,u))\} \leq \varphi(M(x,v,y,u)). \tag{2.3.4}$$

For $z = (x,y) \in A \times B$, $w = (u,v) \in B \times A$, denote

$$d^*(z,w) = \max\{d(x,u), d(y,v)\}. \tag{2.3.5}$$

Then $(X \times X, d^*)$ is a complete metric space.
Let $f : Y \times Y \to Y \times Y$ be defined by $f(x,y) = (F(x,y), F(y,x))$. We have:

$$\begin{aligned}
&\frac{1}{2}[d^*(z, f(w)) + d^*(w, f(z))] = \\
&= \frac{1}{2}\max\{d(x, F(u,v)), d(y, F(v,u))\} + \frac{1}{2}\max\{d(u, F(x,y)), d(v, F(y,x))\} \\
&\geq \max\left\{\frac{1}{2}[d(x, F(u,v)) + d(u, F(x,y))], \frac{1}{2}[d(y, F(v,u)) + d(v, F(y,x))]\right\}.
\end{aligned}$$

Using the above relation, from (2.3.4) we get

$$\begin{aligned}
d^*(f(z), f(w)) \leq \varphi\Big(\max\Big\{&d^*(z,w), d^*(z, f(z)), d^*(w, f(w)), \\
&\frac{1}{2}[d^*(z, f(w)) + d^*(w, f(z))]\Big\}\Big),
\end{aligned} \tag{2.3.6}$$

for any $z \in A \times B$, $w \in B \times A$.
Because $F(A \times B) \subseteq B$ and $F(B \times A) \subseteq A$, we have

$$f(A \times B) \subseteq B \times A \text{ and } f(B \times A) \subseteq A \times B. \tag{2.3.7}$$

(2.3.6) and (2.3.7) means that the operator f is a cyclic φ-contraction of Ćirić type. Applying Theorem 2.1.5, there exists a unique $z^* = (x^*, y^*) \in (A \times B) \cap$

$(B \times A)$ such that $f(z^*) = z^*$ and the Picard iteration $z_n = f(z_{n-1}), n \in \mathbb{N}^*$, converges to z^* for any starting point $z_0 \in Y$. So

$$\begin{cases} F(x^*, y^*) = x^* \\ F(y^*, x^*) = y^* \end{cases}, \tag{2.3.8}$$

where $x^*, y^* \in A \cap B$.
From unicity of the pair (x^*, y^*) and the symmetry with respect to x^* and y^* of the system (2.3.8) we conclude $x^* = y^*$.
Then F has a unique strict fixed point $x^* \in A \cap B$ and for any starting point $(x_0, y_0) \in A \times B$ there exists a sequence $((x_n, y_n))_{n\in\mathbb{N}} \subset Y \times Y$ with

$$\begin{cases} x_n = F(y_{n-1}, x_{n-1}) \\ y_n = F(x_{n-1}, y_{n-1}) \end{cases}, \ n \geq 1$$

that converges to (x^*, x^*).

(3) By the second conclusion of Theorem 2.1.5,

$$d^*(z_n, (x^*, x^*)) \leq s(\varphi^n(d^*(z_0, z_1)))$$

and

$$d^*(z_n, (x^*, x^*)) \leq s(d^*(z_n, z_{n+1})), \ n \geq 0.$$

Hence

$$\max\{d(x_n, x^*), d(y_n, x^*)\} \leq s(\varphi^n(\max\{d(x_0, F(x_0, y_0)), d(y_0, F(y_0, x_0))\}))$$
$$\max\{d(x_n, x^*), d(y_n, x^*)\} \leq s(\max\{d(x_n, x_{n+1}), d(y_n, y_{n+1}\})), \ n \geq 0.$$

(4) Using (3) from Theorem 2.1.5, for any $(x, y) \in Y \times Y$,

$$d^*((x, y), (x^*, x^*)) \leq s(d^*((x, y), T(x, y))).$$

Hence

$$\max\{d(x, x^*), d(y, x^*)\} \leq s(\max\{d(x, F(x, y)), d(y, F(y, x))\}).$$

□

Example 2.3.3. (Magdaş [35]) Let $X = \mathbb{R}, d(x, y) = |x - y|$, for any $x, y \in \mathbb{R}$,

$$A = [0, 2], B = [0, 1], Y = A \cup B, F : Y \times Y \to Y, F(x, y) = \frac{x + 3y}{9}.$$

It is easy to verify that F is cyclic with respect to A and B.

For any $x, v \in A$ and $y, u \in B$,

$$\begin{aligned} d(F(x,y), F(u,v)) &= \left|\frac{x+3y}{9} - \frac{u+3v}{9}\right| \\ &= \left|\frac{x-u}{9} + \frac{y-v}{3}\right| \\ &\leq \left|\frac{1}{9}(x-u) + \frac{10}{27}(y-v)\right| \\ &= \frac{1}{3}\left|y - \frac{v+3u}{9} + \frac{y+3x}{9} - v\right| \\ &\leq \frac{1}{3}\Big(\left|y - F(v,u)\right| + \left|v - F(y,x)\right|\Big) \\ &\leq \frac{2}{3} \cdot \frac{1}{2}[d(y, F(v,u)) + d(v, F(y,x))]. \end{aligned}$$

Then F is a cyclic coupled φ-contraction of Ćirić type, where $\varphi(t) = \frac{2}{3} \cdot t$.

The hypotheses of Theorem 2.3.2 are satisfied, so by Theorem 2.3.2, F has a unique strong coupled fixed point $x^* \in A \cap B$. By calculation we get:

$$F(x^*, x^*) = x^* \Leftrightarrow x^* = 0.$$

Our next theorem gives the well posedness property for the coupled fixed point problem.

Theorem 2.3.4. (Magdaş [35]) *Let $F : Y \times Y \to Y$ be as in Theorem 2.3.2. Then the coupled fixed point problem is well posed, that is, if there exists a sequence $((a_n, b_n))_{n\in\mathbb{N}} \subset Y \times Y$ such that*

$$\begin{cases} d(a_n, F(a_n, b_n)) \to 0 \\ d(b_n, F(b_n, a_n)) \to 0 \end{cases} \quad \textit{as } n \to \infty,$$

then $a_n \to x^*$ and $b_n \to x^*$, as $n \to \infty$.

Proof. Using the inequality

$$d(x, x^*) \leq s(\max\{d(x, F(x,y)), d(y, F(y,x))\})$$

from Theorem 2.3.2 for $x := a_n$ and next for $x := b_n$, we have:

$$\begin{cases} d(a_n, x^*) \leq s(\max\{d(a_n, F(a_n, b_n)), d(b_n, F(b_n, a_n))\}) \\ d(b_n, x^*) \leq s(\max\{d(b_n, F(b_n, a_n)), d(a_n, F(a_n, b_n))\}) \end{cases}, \; n \in \mathbb{N},$$

and letting $n \to \infty$ we obtain

$$\begin{cases} d(a_n, x^*) \to 0 \\ d(b_n, x^*) \to 0 \end{cases}, \ n \to \infty.$$

□

For the data dependence problem we have the following result.

Theorem 2.3.5. (Magdaş [35]) *Let $F : Y \times Y \to Y$ be as in Theorem 2.3.2. Let $G : Y \times Y \to Y$ be such that:*

(i) G has at least one strong coupled fixed point x_G^*;

(ii) there exists $\eta > 0$ such that

$$d(F(x,x), G(x,x)) \leq \eta, \ \text{for any } x \in Y.$$

Then $d(x_F^*, x_G^*) \leq s(\eta)$, where x_F^* is the unique strong coupled fixed point of F and

$$s(t) = \sum_{k=0}^{\infty} \varphi^k(t), \ t \in \mathbb{R}_+.$$

Proof. By letting $x := x_G^*$ and $y := x_G^*$ in the inequality

$$d(x, x^*) \leq s(\max\{d(x, F(x,y)), d(y, F(y,x))\}),$$

we have

$$d(x_G^*, x_F^*) \leq s(d(x_G^*, F(x_G^*, x_G^*))) = s(d(G(x_G^*, x_G^*), F(x_G^*, x_G^*))),$$

and using the monotonicity of s we obtain

$$d(x_F^*, x_G^*) \leq s(\eta).$$

□

Theorem 2.3.6. (Magdaş [35]) *Let $F : Y \times Y \to Y$ be as in Theorem 2.3.2 and $F_n : Y \times Y \to Y$, $n \in \mathbb{N}$, be such that:*

(i) for each $n \in \mathbb{N}$ there exists a strong coupled fixed point x_n^* of F_n;

(ii) $(F_n)_{n\in\mathbb{N}}$ converges uniformly to F.

Then $\lim\limits_{n\to\infty} x_n = x^*$, where x^* is the unique strong coupled fixed point of F.

Proof. The sequence $(F_n)_{n\in\mathbb{N}}$ converges uniformly to F.
Then there exist $\eta_n \in \mathbb{R}_+$, $n \in \mathbb{N}$ such that $\eta_n \to 0$ as $n \to \infty$ and

$$d(F_n(x,y), F(x,y)) \leq \eta_n \text{ for any } (x,y) \in Y \times Y.$$

Using Theorem 2.3.2 for $G := F_n$, $n \in \mathbb{N}$, we have

$$d(x_n, x^*) \leq s(\eta_n) \text{ as } n \to \infty.$$

□

We will discuss Ulam-Hyers stability for the coupled fixed point problem corresponding to a cyclic operator.

Definition 2.3.7. (Magdaş [35]) Let (X,d) be a metric space, $Y \in P(X)$ and let $F : Y \times Y \to Y$ be an operator. The coupled fixed point problem

$$\begin{cases} F(x,y) = x \\ F(y,x) = y \end{cases}, \quad x, y \in Y \tag{2.3.9}$$

is called generalized Ulam-Hyers stable if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous at 0 and $\psi(0) = 0$ such that for any $\varepsilon_1 > 0, \varepsilon_2 > 0$ and for any solution $(x,y) \in Y \times Y$ of the system

$$\begin{cases} d(x, F(x,y)) \leq \varepsilon_1 \\ d(y, F(y,x)) \leq \varepsilon_2 \end{cases},$$

there exists a solution (x^*, y^*) of the coupled fixed point problem such that

$$\begin{cases} d(x, x^*) \leq \psi(\varepsilon) \\ d(y, y^*) \leq \psi(\varepsilon) \end{cases}, \quad \text{where } \varepsilon = \max\{\varepsilon_1, \varepsilon_2\}.$$

In particular, if $x^* = y^*$, then we have generalized Ulam-Hyers stability for the strong coupled fixed point problem $F(x,x) = x, x \in Y$.

Theorem 2.3.8. (Magdaş [35]) *Suppose that all the hypotheses of Theorem 2.3.2 hold. Then the coupled fixed point problem (2.3.9) is generalized Ulam-Hyers stable.*

Proof. By Theorem 2.3.2 we have a unique $x^* \in Y$ such that $F(x^*, x^*) = x^*$. Let $\varepsilon_1 > 0, \varepsilon_2 > 0$ and $(\widetilde{x}, \widetilde{y}) \in Y \times Y$ such that

$$\begin{cases} d(\widetilde{x}, F(\widetilde{x}, \widetilde{y})) \leq \varepsilon_1 \\ d(\widetilde{y}, F(\widetilde{y}, \widetilde{x})) \leq \varepsilon_2. \end{cases}$$

We know that

$$d(x, x^*) \leq s(\max\{d(x, F(x, y)), d(y, F(y, x))\}), \ \forall \ (x, y) \in Y \times Y.$$

Then for

$$\begin{cases} x := \widetilde{x} \\ y := \widetilde{y} \end{cases}$$

and next for

$$\begin{cases} x := \widetilde{y} \\ y := \widetilde{x} \end{cases}$$

using the monotonicity of s, we obtain that

$$\begin{aligned} \max\{d(\widetilde{x}, x^*), d(\widetilde{y}, x^*)\} &\leq s(\max\{d(\widetilde{x}, F(\widetilde{x}, \widetilde{y})), d(\widetilde{y}, F(\widetilde{y}, \widetilde{x}))\}) \\ &\leq s(\max\{\varepsilon_1, \varepsilon_2\}). \end{aligned}$$

As a conclusion, the coupled fixed point problem (2.3.9) is generalized Ulam-Hyers stable with $\psi = s$. □

We apply the results given by Theorem 2.3.2 to study existence and uniqueness of the solutions of the following system of integral equations:

$$\begin{cases} x(t) = \displaystyle\int_a^b G(t, s) f(s, x(s), y(s)) ds \\ y(t) = \displaystyle\int_a^b G(t, s) f(s, y(s), x(s)) ds \end{cases}, \ t \in [a, b] \tag{2.3.10}$$

where $a, b \in \mathbb{R}$, $a < b$,

$$G \in C([a, b] \times [a, b], [0, \infty)),$$

$$f \in C([a, b] \times \mathbb{R} \times \mathbb{R}).$$

Theorem 2.3.9. (Magdaş [35]) *Suppose that:*

(i) there exist $\alpha, \beta \in C[a,b]$, *with* $\alpha(t) \leq \beta(t)$, *for any* $t \in [a,b]$, *such that*

$$\begin{cases} \alpha(t) \leq \int_a^b G(t,s) f(s, \beta(s), \alpha(s)) ds \\ \beta(t) \geq \int_a^b G(t,s) f(s, \alpha(s), \beta(s)) ds \end{cases} \quad \text{for any } t \in [a,b]; \tag{2.3.11}$$

(ii) there exists a strong comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$|f(s, u_1, u_2) - f(s, v_1, v_2)| \leq \varphi(\max\{|u_1 - v_1|, |u_2 - v_2|\}),$$

for any $s \in [a,b]$ *and* $u_1, u_2, v_1, v_2 \in \mathbb{R}$;

(iii) $\sup_{t \in [a,b]} \int_a^b G(t,s) ds \leq 1$;

(iv) $f(s, \cdot, \ y)$ *is monotone decreasing for any* $s \in [a,b]$ *and any* $y \in \mathbb{R}$;

(v) $f(s, x, \cdot\ )$ *is monotone increasing for any* $s \in [a,b]$ *and any* $x \in \mathbb{R}$.

Then the system (2.3.10) has a unique solution $(x^*, x^*) \in C([a,b], \mathbb{R}^2)$, *with* $\alpha \leq x^* \leq \beta$.

Proof. Let us consider $X := C[a,b]$ and the Chebyshev norm $|x|_\infty = \max_{t \in [a,b]} |x(t)|$. Then $(X, |\cdot|_\infty)$ is a Banach space.

We consider the following closed subsets of X:

$$A = \{x \in X \mid x \leq \beta\}, B = \{x \in X \mid x \geq \alpha\},$$

$Y = A \cup B$ and the operator $F : Y \times Y \to Y$,

$$F(x,y)(t) := \int_a^b G(t,s) f(s, x(s), y(s)) ds.$$

The system (2.3.10) is equivalent to

$$\begin{cases} F(x,y) = x \\ F(y,x) = y \end{cases}, x, y \in Y.$$

We will prove that F is cyclic with respect to A and B, that is

$$F(A \times B) \subseteq B \text{ and } F(B \times A) \subseteq A.$$

Let $x \in A$ and $y \in B \Rightarrow x(s) \leq \beta(s), y(s) \geq \alpha(s), \ \forall\ s \in [a,b]$.

Using the monotonicity of f we have

$$G(t,s)f(s,x(s),y(s)) \geq G(t,s)f(s,\beta(s),\alpha(s)),$$

and from (i), by integration,

$$\int_a^b G(t,s)f(s,x(s),y(s))ds \geq \alpha(t),$$

which means that

$$F(x,y)(t) \geq \alpha(t),\ \forall\ t \in [a,b] \Rightarrow F(x,y) \in B.$$

So $F(A \times B) \subseteq B$. In a similar way we have $F(B \times A) \subseteq A$.

Using the conditions (ii) and (iii), and the monotonicity of φ, for any $x, v \in A$ and $y, u \in B$, we have

$$\begin{aligned} |f(s,x(s),y(s)) - f(s,u(s),v(s))| &\leq \varphi(\max_{s\in[a,b]}\{|x(s)-u(s)|, |y(s)-v(s)|\}) \\ &\leq \varphi(\max\{|x-u|_\infty, |y-v|_\infty\}). \end{aligned}$$

We infer

$$\begin{aligned} |F(x,y)(t) - F(u,v)(t)| &\leq \int_a^b G(t,s)|f(s,x(s),y(s)) - f(s,u(s),v(s))|ds \\ &\leq \varphi(\max\{|x-u|_\infty, |y-v|_\infty\}) \int_a^b G(t,s)ds \\ &\leq \varphi(\max\{|x-u|_\infty, |y-v|_\infty\}),\ \forall\ t \in [a,b]. \end{aligned}$$

We conclude

$$|F(x,y)-F(u,v)|_\infty \leq \varphi(\max\{|x-u|_\infty, |y-v|_\infty\}) \text{ for any } x,v\in A \text{ and } y,u\in B,$$

so the operator F is a cyclic coupled φ-contraction of Ćirić type.

All the conditions of Theorem 2.3.2 are satisfied, so T has a unique strong coupled fixed point $(x^*, x^*) \in A \cap B$, with $\alpha(t) \leq x^*(t) \leq \beta(t)$, for any $t \in [a,b]$. □

Definition 2.3.10. (Magdaş [35]) The system (2.3.10) is said to be generalized Ulam-Hyers stable if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous at 0 and

$\psi(0) = 0$ such that for any $\varepsilon_1 > 0, \varepsilon_2 > 0$ and for any solution $(x, y) \in C([a, b], \mathbb{R}^2)$, of the system

$$\begin{cases} |x(t) - \int_a^b G(t,s) f(s, x(s), y(s)) ds| \le \varepsilon_1 \\ |y(t) - \int_a^b G(t,s) f(s, y(s), x(s)) ds| \le \varepsilon_2 \end{cases}$$

there exists a solution $(x^*, y^*) \in C([a, b], \mathbb{R}^2)$ of the system (2.3.10) such that for any $t \in [a, b]$,

$$\begin{cases} |x(t) - x^*(t)| \le \psi(\varepsilon) \\ |y(t) - y^*(t)| \le \psi(\varepsilon) \end{cases}, \quad \text{where } \varepsilon = \max\{\varepsilon_1, \varepsilon_2\}.$$

Theorem 2.3.11. (Magdaş [35]) *Suppose that the hypotheses of the Theorem 2.3.9 hold. Then the system (2.3.10) is generalized Ulam-Hyers stable.*

Proof. By Theorem 2.3.9, the system (2.3.10) has a unique solution $(x^*, x^*) \in C([a, b], \mathbb{R}^2)$, with $\alpha \le x^* \le \beta$. Applying Theorem 2.3.9 to the operator $F : Y \times Y \to Y$,

$$F(x, y)(t) := \int_a^b G(t, s) f(s, x(s), y(s)) ds,$$

in the same setting as in the proof of Theorem 2.3.8, we get the conclusion. □

Similar with the approach in Theorem 2.2.11, if we consider the following system of Volterra type of integral equations:

$$\begin{cases} x(t) = \int_a^t f(s, x(s), y(s)) ds \\ y(t) = \int_a^t f(s, y(s), x(s)) ds \end{cases}, \ t \in [a, b], \tag{2.3.12}$$

where $a, b \in \mathbb{R}$, $a < b, f \in C([a, b] \times \mathbb{R} \times \mathbb{R})$, then an existence and uniqueness result can be obtained working with a Bielecki type norm.

More precisely, we consider $C[a, b]$ endowed with the following Bielecki type norm

$$|x|_B = \max_{t \in [a,b]} (|x(t)| e^{-\tau(t-a)}), \tau > 0.$$

Then $(C[a, b], |\cdot|_B)$ is a Banach space.

Theorem 2.3.12. *Consider the system (2.3.12). We suppose that:*

(i) there exist $\alpha, \beta \in C[a,b]$, *with* $\alpha(t) \leq \beta(t)$, *for any* $t \in [a,b]$, *such that*

$$\begin{cases} \alpha(t) \leq \int_a^t f(s, \beta(s), \alpha(s))ds \\ \beta(t) \geq \int_a^t f(s, \alpha(s), \beta(s))ds \end{cases}, \text{ for any } t \in [a,b];$$

(ii) there exists a strong comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *having the properties:*

$(i)_\varphi$ *there exists* $M > e^{b-a}$ *such that for any* $q \in (1, M)$ *and* $t > 0$,

$$\varphi(qt) \leq q \cdot \varphi(t);$$

$(ii)_\varphi$ *for any* $s \in [a,b]$ *and* $u_1, u_2, v_1, v_2 \in \mathbb{R}$,

$$|f(s, u_1, u_2) - f(s, v_1, v_2)| \leq \varphi(\max\{|u_1 - v_1|, |u_2 - v_2|\});$$

(iii) $f(s, \cdot, y)$ *is monotone decreasing for any* $s \in [a,b]$ *and any* $y \in \mathbb{R}$;

(iv) $f(s, x, \cdot)$ *is monotone increasing for any* $s \in [a,b]$ *and any* $x \in \mathbb{R}$.

Then the system (2.3.12) has a unique solution $(x^*, x^*) \in C([a,b], \mathbb{R}^2)$, *with* $\alpha \leq x^* \leq \beta$.

Proof. We consider the following closed subsets of $C[a,b]$:

$$A = \{x \in C[a,b] \mid x \leq \beta\}, B = \{x \in C[a,b] \mid x \geq \alpha\},$$

$Y = A \cup B$ and the operator $F : Y \times Y \to Y$,

$$F(x,y)(t) := \int_a^t f(s, x(s), y(s))ds.$$

The system (2.3.12) is equivalent to

$$\begin{cases} F(x,y) = x \\ F(y,x) = y \end{cases}, x, y \in Y.$$

We will prove that F is cyclic with respect to A and B, that is

$$F(A \times B) \subseteq B \text{ and } F(B \times A) \subseteq A.$$

Let $x \in A$ and $y \in B \Rightarrow x(s) \leq \beta(s), y(s) \geq \alpha(s), \ \forall \ s \in [a,b]$.

Using the monotonicity of f we have

$$f(s, x(s), y(s)) \geq f(s, \beta(s), \alpha(s)),$$

and from (i), by integration,

$$\int_a^t f(s, x(s), y(s)) ds \geq \alpha(t),$$

which means that

$$F(x, y)(t) \geq \alpha(t), \ \forall \ t \in [a, b] \Rightarrow F(x, y) \in B.$$

So $F(A \times B) \subseteq B$. In a similar way we have $F(B \times A) \subseteq A$.
If we choose $\tau \in \left(1, \dfrac{\ln M}{b-a}\right)$ then for any $s \in [a, b]$,

$$e^{\tau(b-a)} < M \Rightarrow e^{\tau(s-a)} \in (1, M),$$

and using the assumption (ii_φ) we have

$$\varphi(e^{\tau(s-a)} \cdot t) \leq e^{\tau(s-a)} \cdot \varphi(t), \forall t > 0. \tag{2.3.13}$$

Using the condition (ii) and the monotonicity of φ, for any $s \in [a, b]$, $x, v \in A$ and $y, u \in B$, we have

$$|f(s, x(s), y(s)) - f(s, u(s), v(s))| \leq \varphi(\max\{|x(s) - u(s)|, |y(s) - v(s)|\}),$$

and by integration we obtain

$$\begin{aligned}
|F(x, y)(t) - F(u, v)(t)| &\leq \int_a^t |f(s, x(s), y(s)) - f(s, u(s), v(s))| ds \\
&\leq \int_a^t \varphi(\max\{|x(s) - u(s)|, |y(s) - v(s)|\}) ds \\
&\overset{(2.3.13)}{\leq} \int_a^t e^{\tau(s-a)} \varphi(\max\{|x(s) - u(s)| e^{-\tau(s-a)}, |y(s) - v(s)| e^{-\tau(s-a)}\}) ds \\
&\leq \varphi(\max\{|x - u|_B, |y - v|_B\}) \int_a^t e^{\tau(s-a)} ds \\
&= \varphi(\max\{|x - u|_B, |y - v|_B\}) \frac{1}{\tau} [e^{\tau(t-a)} - 1] \\
&\leq \varphi(\max\{|x - u|_B, |y - v|_B\}) e^{\tau(t-a)}, \ \forall \ t \in [a, b],
\end{aligned}$$

which implies

$$|F(x, y) - F(u, v)|_B \leq \varphi(\max\{|x - u|_B, |y - v|_B\}) \text{ for any } x, v \in A \text{ and } y, u \in B,$$

so the operator F is a cyclic coupled φ-contraction of Ćirić type.

All the conditions of Theorem 2.3.2 are satisfied, so T has a unique strong coupled fixed point $(x^*, x^*) \in A \cap B$, with $\alpha(t) \leq x^*(t) \leq \beta(t)$, for any $t \in [a, b]$. $\square$

Chapter 3

Multi-valued generalized contractions on cyclic representations

In this chapter, we present fixed point and best proximity point results for multi-valued operators defined on cyclic representations in metric spaces. This chapter has three sections.

The purpose of the **first section** is to investigate the properties of multi-valued cyclic φ-contractions of Ćirić type. In this situation, such operators T possess fixed points, i.e., $x \in X$ satisfying the relation $x \in T(x)$. Also, we will study data dependence and generalized Ulam-Hyers stability of the fixed point inclusion $x \in T(x)$.

The original contributions in the first section are the following results:

• Theorem 3.1.4 is the main result of this section, an extension of other fixed point results for multi-valued contractive operators defined on cyclic representation of the space (see for example Theorem 3.1.6);

• Theorem 3.1.8 is a result concerning data dependence of the fixed point inclusion;

• Theorem 3.1.9 studies the generalized Ulam-Hyers stability of the fixed point inclusion.

The results presented in the first section are included in the following paper: Magdaş [34].

The purpose of the **second section** is to study existence of the solutions and generalized Ulam-Hyers stability of the best proximity problem for multi-valued Ćirić type cyclic operators.

The original contributions in the second section are the following results:

• Theorem 3.2.4, the first main result of this section, extends Theorem 1.4.5 (Suzuki, Kikkawa, Vetro, [77]) and Theorem 1.4.6 (Neammanee, Kaewkhao [42]) to the case of multi-valued Ćirić type cyclic operator which takes proximinal values, in the framework of metric spaces with the property UC;

• Theorem 3.2.8, the second main result of this section, proves that if φ is a subadditive strong comparison function, then the condition that the multi-valued operator takes proximinal values can be removed;

• Theorem 3.2.10 studies the generalized Ulam-Hyers stability of the best proximity problem for a cyclic multi-valued operator.

The results presented in this section are contained in the following paper: Magdaş [37].

In the **third section** we study the coupled fixed point problem and the coupled best proximity point problem for multi-valued cyclic contraction type operators.

The original contributions in the third section are the following results:

• Theorem 3.3.5 states a coupled fixed point result for cyclic coupled φ-contraction of Ćirić type multi-valued operator;

• Theorem 3.3.7 is a result concerning the generalized Ulam-Hyers stability of the coupled fixed point problem;

• Theorem 3.3.10 studies the existence of the coupled best proximity point of a cyclic coupled Ćirić type multi-valued operator which takes proximinal values, in the framework of metric spaces with the property UC.

The results presented in this section are contained in the following paper: Magdaş [35].

3.1 A study of the fixed point problem for Ćirić type multi-valued operators satisfying a cyclic condition

The aim of this section is to study the properties of multi-valued cyclic φ-contraction of Ćirić type. In this situation, such operators T possess fixed points, i.e., $x \in X$ satisfying the relation $x \in T(x)$. We construct a sequence of successive approximations of T that guarantees convergence from any starting point $(x, y) \in \text{Graph}(T)$ to a point $x^* \in F_T$, the set of all fixed points of T. We also study data dependence and generalized Ulam-Hyers stability of the fixed point inclusion $x \in T(x)$.

Definition 3.1.1. (Magdaş [34]) Let (X, d) be a metric space, m a positive integer, $A_1, \ldots, A_m \in P_{cl}(X)$, $Y := \bigcup_{i=1}^{m} A_i$ and $T : Y \to P(Y)$ a multi-valued operator. If:

(i) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T;

(ii) there exists a strong comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$H(T(x), T(y)) \leq$$

$$\leq \varphi\left(\max\left\{d(x, y), D(x, T(x)), D(y, T(y)), \frac{1}{2}[D(x, T(y)) + D(y, T(x))]\right\}\right),$$

for any $x \in A_i$, $y \in A_{i+1}$, where $A_{m+1} = A_1$,

then T is called a multi-valued cyclic φ-contraction of Ćirić type.

For the following notions see [53], [69] and [71].

Definition 3.1.2. Let (X, d) be a metric space. Then $T : X \to P(X)$ is called multi-valued weakly Picard operator (briefly MWP operator) if for each $(x, y) \in \text{Graph}(T)$ there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in X such that:

(i) $x_0 = x,\ x_1 = y$;

(ii) $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}$;

(iii) the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent and its limit is a fixed point of T.

If $T : X \to P(X)$ is a MWP operator, then we define

$$T^\infty : \text{Graph}(T) \to P(F_T)$$

by the formula

$T^{\infty}(x,y) := \{z \in F_T \mid$ there exists a sequence of successive approximation of T starting from (x,y) that converges to $z\}$.

Definition 3.1.3. (Lazăr [32]) Let (X,d) be a metric space and $T : X \to P(X)$ a MWP operator. Then T is called a ψ-multi-valued weakly Picard operator (ψ-MWP operator) if the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and continuous at 0 such that $\psi(0) = 0$, and there exists a selection t^{∞} of T^{∞} such that

$$d(x, t^{\infty}(x,y)) \leq \psi(d(x,y)), \text{ for all } (x,y) \in \text{Graph}(T).$$

In particular, if $\psi(t) := ct$, with $c > 0$, then T is called a c-MWP operator (see [69]).

The main result of this section is the following theorem.

Theorem 3.1.4. (Magdaş [34]) *Let (X,d) be a complete metric space, m be a positive integer, $A_1, A_2, \ldots, A_m \in P_{cl}(X)$, $Y := \bigcup_{i=1}^{m} A_i$, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strong comparison function and $T : Y \to P_{prox}(Y)$ be a multi-valued operator. Assume that:*

(i) $\bigcup_{i=1}^{m} A_i$ *is a cyclic representation of Y with respect to T;*

(ii) *T is a multi-valued cyclic φ-contraction of Ćirić type.*

Then the following statements hold:

(1) $F_T \neq \emptyset$;

(2) *for each $(x,y) \in \text{Graph}(T)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations of T starting from any point $(x,y) \in \text{Graph}(T)$, that converges to a fixed point $x^*(x,y) \in \bigcap_{i=1}^{m} A_i$, thus T is a MWP operator;*

(3) *the following estimations hold:*

$$d(x_n, x^*(x,y)) \leq s(\varphi^n(d(x,y))), \text{ for any } (x,y) \in \text{Graph}(T),\ n \geq 1,$$

$$d(x_n, x^*(x,y)) \leq s(d(x_n, x_{n+1})), \text{ for any } (x,y) \in \text{Graph}(T),\ n \geq 1;$$

(4) *for any $(x,y) \in \text{Graph}(T)$,*

$$d(x, x^*(x,y)) \leq s(d(x,y)), \text{ i.e. } T \text{ is an } s - MWP \text{ operator},$$

where s is given by Lemma 1.2.3;

(5) $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$, *i.e. T is a good MWP operator.*

Proof. (1)+(2) Let $(x, y) \in \mathrm{Graph}(T)$ be arbitrary. We construct a sequence of successive approximations of T starting from (x, y) in the following way:

$$x_0 = x \text{ that lie in a subset } A_i \text{ of } Y;$$
$$x_1 = y \in T(x) \subseteq T(A_i), \text{ so } x_1 \in A_{i+1};$$
$$x_{n+1} \in T(x_n) \text{ such that } d(x_n, x_{n+1}) = D(x_n, T(x_n)), \text{ for } n \geq 1, \quad (3.1.1)$$

the existence of x_{n+1} being assured by the proximinality of $T(x_n)$. Then

$$d(x_n, x_{n+1}) = D(x_n, T(x_n)) \leq H(T(x_{n-1}), T(x_n))$$
$$\leq \varphi\Big(\max\Big\{d(x_{n-1}, x_n), D(x_{n-1}, T(x_{n-1})), D(x_n, T(x_n)),$$
$$\frac{1}{2}[D(x_{n-1}, T(x_n)) + D(x_n, T(x_{n-1}))]\Big\}\Big) \text{ for } n \geq 1. \quad (3.1.2)$$

Notice that

$$D(x_{n-1}, T(x_{n-1})) \leq d(x_{n-1}, x_n) \text{ and } D(x_n, T(x_{n-1})) = 0.$$

Using the triangle inequality,

$$D(x_{n-1}, T(x_n)) \leq d(x_{n-1}, x_n) + D(x_n, T(x_n))$$
$$= d(x_{n-1}, x_n) + d(x_n, x_{n+1}), \ n \geq 1.$$

So

$$\frac{1}{2}[D(x_{n-1}, T(x_n)) + D(x_n, T(x_{n-1}))] \leq \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$
$$\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}, \ n \geq 1.$$

Using the monotonicity of φ, the relation (3.1.2) becomes

$$d(x_n, x_{n+1}) \leq \varphi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}), \text{ for } n \geq 1.$$

The following cases need to be discussed:

- there exists $k \geq 1$ such that

$$d(x_{k-1}, x_k) < d(x_k, x_{k+1}). \quad (3.1.3)$$

In this case,

$$d(x_k, x_{k+1}) \le \varphi(d(x_k, x_{k+1})).$$

Because $\varphi(t) < t$, for any $t > 0$, it is necessary $d(x_k, x_{k+1}) = 0$ which is a contradiction with (3.1.3).

- $d(x_n, x_{n+1}) \le d(x_{n-1}, x_n)$, for any $n \ge 1$.

Thus, for any $n \ge 1$,

$$d(x_n, x_{n+1}) \le \varphi(d(x_{n-1}, x_n)) \le \ldots \le \varphi^n(d(x_0, x_1)). \tag{3.1.4}$$

For $p \ge 1$, we have

$$d(x_n, x_{n+p}) \le \varphi^n(d(x_0, x_1)) + \ldots + \varphi^{n+p-1}(d(x_0, x_1)) \tag{3.1.5}$$

and denoting $S_n := \sum_{k=0}^{n-1} \varphi^k(d(x_0, x_1))$, we immediately get that

$$d(x_n, x_{n+p}) \le S_{n+p} - S_n. \tag{3.1.6}$$

As φ is a strong comparison function,

$$\sum_{k=0}^{\infty} \varphi^k(d(x_0, x_1)) < \infty,$$

thus there exists $S \in \mathbb{R}_+$ such that $\lim\limits_{n\to\infty} S_n = S$.

By (3.1.6) we get that $(x_n)_{n\ge1}$ is a Cauchy sequence in the complete metric space Y. Hence, the sequence $(x_n)_{n\ge1}$ converges to an $x^*(x, y) \in Y$. But $(x_n)_{n\ge1}$ has an infinite number of terms in each A_i for $i \in \{1, 2, \cdots, m\}$, so from each A_i one we can extract a subsequence convergent to $x^*(x, y)$. Because A_i are closed, we obtain that $x^*(x, y) \in \bigcap\limits_{i=1}^{m} A_i$.

We will prove that $x^* \in T(x^*)$. We have:

$$\begin{aligned} D(x^*, T(x^*)) &\le d(x^*, x_{n+1}) + D(x_{n+1}, T(x^*)) \\ &\le d(x^*, x_{n+1}) + H(T(x_n), T(x^*)) \\ &\le d(x^*, x_{n+1}) + \varphi\Big(\max\Big\{d(x_n, x^*), D(x_n, T(x_n)), D(x^*, T(x^*)), \\ &\qquad \frac{1}{2}(D(x^*, T(x_n)) + D(x_n, T(x^*)))\Big\}\Big), \text{ for } n \ge 0. \end{aligned} \tag{3.1.7}$$

We observe that

$$D(x^*, T(x_n)) \le d(x^*, x_n) + D(x_n, T(x_n)) = d(x^*, x_n) + d(x_n, x_{n+1}),$$

and

$$D(x_n, T(x^*)) \leq d(x_n, x^*) + D(x^*, T(x^*)), \text{ for } n \geq 0.$$

By summing and using again the monotonicity of φ, the inequality (3.1.7) becomes

$$D(x^*, T(x^*)) \leq d(x^*, x_{n+1}) + \varphi\Big(\max\Big\{d(x_n, x^*), d(x_n, x_{n+1}), D(x^*, T(x^*)), d(x^*, x_n) + \frac{1}{2}(d(x_n, x_{n+1}) + D(x^*, T(x^*)))\Big\}\Big).$$

Passing to the limit as $n \to \infty$, we get

$$D(x^*, T(x^*)) \leq \varphi(D(x^*, T(x^*))). \tag{3.1.8}$$

If $D(x^*, T(x^*)) > 0$, according to Lemma 1.2.2, we have

$$\varphi(D(x^*, T(x^*)) < D(x^*, T(x^*))$$

which is a contradiction with (3.1.8). Therefore, $D(x^*, T(x^*)) = 0$.

(3) By letting $p \to \infty$ in (3.1.5), we obtain an a priori estimation:

$$d(x_n, x^*) \leq s(\varphi^n(d(x_0, x_1))), \text{ for any } n \geq 1.$$

Using (3.1.4) and the monotonicity of φ,

$$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}) \leq \sum_{k=0}^{p-1} \varphi^k(d(x_n, x_{n+1})), \text{ for any } n \geq 0,$$

and letting $p \to \infty$,

$$d(x_n, x^*) \leq \sum_{k=0}^{\infty} \varphi^k(d(x_n, x_{n+1})), \; n \geq 0 \tag{3.1.9}$$

which considering the definition of s yields an a posteriori estimation

$$d(x_n, x^*) \leq s(d(x_n, x_{n+1})), \text{ for any } n \geq 1.$$

(4) For $n = 0$, (3.1.9) becomes:

$$d(x_0, x^*) \leq s(d(x_0, x_1)).$$

(5) Using the inequality (3.1.4),

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=0}^{\infty} \varphi^n(d(x_0, x_1)) = s(d(x_0, x_1)) < \infty.$$

□

Remark 3.1.5. If we choose $\varphi(t) = kt$, for $k \in (0,1)$, then we have a generalization of the following theorem (Theorem 2.1 in [42]), where the multi-valued operator T takes closed and bounded values.

Theorem 3.1.6. (Neammanee, Kaewkhao [42]) *Let A and B be nonempty closed subsets of a metric space (X,d). Suppose $T : A \cup B \to P(X)$ is a multi-valued mapping with closed and bounded valued, satisfying the conditions:*

(i) $T(A) \subseteq B, T(B) \subseteq A$;

(ii) there exists $k \in (0,1)$ such that for any $x \in A$, $y \in B$,

$$H(T(x), T(y)) \leq kd(x,y).$$

Then T has at least one fixed point in $A \cap B$.

Remark 3.1.7. If the strong comparison function φ is subadditive, then the proximinality condition can be less restrictive: the values of the multi-valued operator T should be closed. The proof runs in the same manner.

A data dependence theorem for the stated problem is:

Theorem 3.1.8. (Magdaş [34]) *Let $T : Y \to P_{prox}(Y)$ be as in Theorem 3.1.4 and $U : Y \to P(Y)$ such that:*

(i) $F_U \neq \emptyset$;

(ii) there exists $\eta > 0$ such that

$$\rho(T(x), U(x)) \leq \eta, \text{ for any } x \in Y.$$

Then $\rho(F_U, F_T) \leq s(\eta)$, where s is given by Lemma 1.2.3.

Proof. Let x_U^* be a fixed point of U. Since $T(x_U^*)$ is proximinal, there exists $y_U \in T(x_U^*)$ such that

$$D(x_U^*, T(x_U^*)) = d(x_U^*, y_U).$$

Then, from Theorem 3.1.4, for arbitrary $(x,y) \in \text{Graph}(T)$, we have:

$$d(x, x_T^*(x,y)) \leq s(d(x,y)),$$

where $x_T^*(x,y)$ is a fixed point of T.
Choosing $x := x_U^*$ and $y := y_U$,

$$\begin{aligned} d(x_U^*, x_T^*) &\leq s(d(x_U^*, y_U)) = s(D(x_U^*, T(x_U^*)) \\ &\leq s(\rho(T(x_U^*), U(x_U^*))) \leq s(\eta) \end{aligned}$$

Thus we obtain

$$D(x_U^*, F_T) \leq d(x_U^*, x_T^*) \leq s(\eta), \text{ for any } x_U^* \in F_U.$$

It follows $\rho(F_U, F_T) \leq s(\eta)$. □

Theorem 3.1.9. (Magdaş [34]) *(Generalized Ulam-Hyers stability of the inclusion $x \in T(x)$) Let $T : Y \to P_{prox}(Y)$ be as in Theorem 3.1.4, $\varepsilon > 0$ and $x \in Y$ be such that $D(x, T(x)) \leq \varepsilon$. Then there exists $x^* \in F_T$ such that $d(x, x^*) \leq s(\varepsilon)$, where s is given by Lemma 1.2.3.*

Proof. Using the monotonicity of s and the inequality $d(x, x^*(x, y)) \leq s(d(x, y))$ by Theorem 3.1.4, for $(x, y) \in \text{Graph}(T)$ with $d(x, y) = D(x, T(x))$ (the existence of y is assured by the proximinality of $T(x)$), we have:

$$d(x, x^*) \leq s(d(x, y)) = s(D(x, T(x))) \leq s(\varepsilon).$$

□

Remark 3.1.10. Many open problems related to the multi-valued cyclic φ-contraction of Ćirić type operators can be discussed. We present here two such open questions:

1) Is the fixed point problem for a multi-valued operator $T : Y \to P_{prox}(Y)$ satisfying the conditions of Theorem 3.1.4 well-posed with respect to D ?, that is, assuming there exists a sequence $(z_n)_{n\in\mathbb{N}} \subset Y$ such that

$$D(z_n, T(z_n)) \to 0 \text{ as } n \to \infty,$$

it follows that $(z_n)_{n\in\mathbb{N}}$ converges to a fixed point of T.

2) In which conditions the operator $T : Y \to P_{prox}(Y)$ satisfying the assumptions in Theorem 3.1.4 has the limit shadowing property ?, that is, assuming that there exists a sequence $(z_n)_{n\in\mathbb{N}} \subset Y$ such that $D(z_{n+1}, T(z_n)) \to 0$ as $n \to \infty$, then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset Y$ of successive approximations for T, such that $d(x_n, z_n) \to 0$ as $n \to \infty$.

3.2 Best proximity point theorems for multi-valued operators

The purpose of this section is to study existence of the solutions and generalized Ulam-Hyers stability of the following best proximity problem for a cyclic multi-valued operator:

If (X, d) is a metric space, $A, B \in P(X)$, $T : A \cup B \to P(X)$ is a multi-valued operator satisfying the cyclic condition $T(A) \subseteq B, T(B) \subseteq A$, then we are interested to find

$$x^* \in A \cup B \text{ such that } D(x^*, T(x^*)) = D(A, B). \tag{3.2.1}$$

x^* is said to be a best proximity point of T.

The concept of multi-valued Ćirić type cyclic operator is as follows.

Definition 3.2.1. (Magdaş [37]) Let (X, d) be a metric space, $A, B \in P(X)$, and $T : A \cup B \to P(X)$ be a multi-valued operator. If:

(i) $T(A) \subseteq B, T(B) \subseteq A$;

(ii) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $x \in A$, $y \in B$,

$$H(T(x), T(y)) \leq \varphi(M(x, y) - D(A, B)) + D(A, B),$$

where

$$M(x, y) = \max\left\{d(x, y), D(x, T(x)), D(y, T(y)), \frac{1}{2}[D(x, T(y)) + D(y, T(x))]\right\},$$

then T is called a multi-valued Ćirić type cyclic operator.

Example 3.2.2. The following operators are multi-valued Ćirić type cyclic operators:

(1) A multi-valued cyclic contraction (see [42]) i.e. a multi-valued cyclic operator $T : A \cup B \to P(X)$ satisfying the condition:

there exists $k \in (0, 1)$ such that for any $x \in A$, $y \in B$,

$$H(T(x), T(y)) \leq kd(x, y) + (1 - k)D(A, B).$$

(2) A multi-valued cyclic operator $T : A \cup B \to P(X)$ satisfying a Kannan type condition (for the single-valued case see [48]):

there exists $k \in (0, \frac{1}{2})$ such that for any $x \in A$, $y \in B$,

$$H(T(x), T(y)) \leq k(D(x, T(x)) + D(y, T(y))) + (1 - 2k)D(A, B).$$

(3) A multi-valued cyclic operator $T : A \cup B \to P(X)$ satisfying a Bianchini type condition (for the single-valued case see [49]):

there exists $k \in (0, 1)$ such that for any $x \in A$, $y \in B$,

$$H(T(x), T(y)) \leq k \cdot \max\{D(x, T(x)), D(y, T(y))\} + (1 - k)D(A, B).$$

The following lemma will be used to prove our results.

Lemma 3.2.3. [42] *Let be (A, B) a pair of nonempty subsets of a metric space (X, d), satisfying the property UC, and let be a sequence $(x_n)_{n\in\mathbf{N}}$ in A. If there exists a sequence $(y_n)_{n\in\mathbf{N}}$ in B such that $d(x_n, y_n) \to D(A, B)$ and $d(x_{n+1}, y_n) \to D(A, B)$, then $(x_n)_{n\in\mathbf{N}}$ is a Cauchy sequence.*

Our first main result extends Theorem 1.4.5 to multi-valued Ćirić type cyclic operator in the framework of metric spaces with the property UC. More than that, it extends Theorem 1.4.6 to the case of multi-valued Ćirić type cyclic operator in the setting of proximinal values.

Theorem 3.2.4. (Magdaş [37])

Let (X, d) be a complete metric space, $A \in P_{cl}(X), B \in P(X)$, such that (A, B) satisfies the property UC. If $T : A \cup B \to P_{prox}(X)$ is a multi-valued Ćirić type cyclic operator, then the following statements hold:

(i) T has a best proximity point $x_A^* \in A$;

(ii) there exists a sequence $(x_n)_{n\in\mathbf{N}}$ with $x_0 \in A$ and $x_{n+1} \in T(x_n)$, such that $(x_{2n})_{n\in\mathbf{N}}$ converges to x_A^*.

Proof. (i)+(ii) We construct a sequence of successive approximations of T starting from an arbitrary $x \in A$ in the following way:

$x_0 = x \in A$;

$x_{n+1} \in T(x_n)$ such that $d(x_n, x_{n+1}) = D(x_n, T(x_n))$, for $n \geq 0$,

the existence of x_{n+1} being assured by the proximinality of $T(x_n)$.

Then, for $n \geq 1$,

$$\begin{aligned} d(x_n, x_{n+1}) &= D(x_n, T(x_n)) \leq H(T(x_{n-1}), T(x_n)) \\ &\leq \varphi(M(x_{n-1}, x_n) - D(A, B)) + D(A, B), \end{aligned} \tag{3.2.2}$$

where

$$M(x_{n-1}, x_n) = \max\Big\{d(x_{n-1}, x_n), D(x_{n-1}, T(x_{n-1})), D(x_n, T(x_n)), \frac{1}{2}[D(x_{n-1}, T(x_n)) + D(x_n, T(x_{n-1}))]\Big\}\Big).$$

Notice that

$$D(x_{n-1}, T(x_{n-1})) = d(x_{n-1}, x_n) \text{ and } D(x_n, T(x_{n-1})) = 0.$$

Using the triangle inequality,

$$\begin{aligned} D(x_{n-1}, T(x_n)) &\leq d(x_{n-1}, x_n) + D(x_n, T(x_n)) \\ &= d(x_{n-1}, x_n) + d(x_n, x_{n+1}),\ n \geq 1. \end{aligned}$$

So

$$\frac{1}{2}[D(x_{n-1}, T(x_n)) + D(x_n, T(x_{n-1}))] \leq \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})],$$

and

$$M(x_{n-1}, x_n) \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\},\ n \geq 1.$$

Denoting $z_n = d(x_n, x_{n+1}) - D(A, B)$ and using the monotonicity of φ, (3.2.2) becomes

$$z_n \leq \varphi(\max\{z_{n-1}, z_n\}),\ \text{for } n \geq 1.$$

Because $\varphi(t) < t$, for any $t > 0$, we get

$$z_n \leq \varphi(z_{n-1}),\ \text{for any } n \geq 1.$$

Thus

$$z_n \leq \varphi^{n-1}(z_1) \to 0,\ \text{so } d(x_n, x_{n+1}) \to D(A, B) \text{ when } n \to \infty.$$

Since

$$(x_{2n})_{n\in\mathbf{N}} \subset A, (x_{2n+2})_{n\in\mathbf{N}} \subset A,\ \text{and } (x_{2n+1})_{n\in\mathbf{N}} \subset B,$$

by Lemma 3.2.3, $(x_{2n})_{n\in\mathbf{N}}$ is a Cauchy sequence in the complete metric space X. Hence, the Cauchy sequence $(x_{2n})_{n\in\mathbf{N}}$ converges to a point x_A^* which lies in A because $(x_{2n})_{n\in\mathbf{N}} \subset A$ and A is closed.

For $n \geq 1$, we have

$$D(A,B) \leq d(x_A^*, x_{2n-1}) \leq d(x_A^*, x_{2n}) + d(x_{2n}, x_{2n-1}),$$
$$\text{so } d(x_A^*, x_{2n-1}) \to D(A,B) \text{ when } n \to \infty.$$

$$\begin{aligned} D(A,B) &\leq D(x_{2n}, T(x_A^*)) \\ &\leq H(T(x_{2n-1}), T(x_A^*)) \\ &\leq \varphi(M(x_{2n-1}, x_A^*) - D(A,B)) + D(A,B) \\ &< M(x_{2n-1}, x_A^*) \\ &= \max\Big\{d(x_{2n-1}, x_A^*), D(x_{2n-1}, T(x_{2n-1})), D(x_{2n}, T(x_{2n})), \\ &\frac{1}{2}[D(x_{2n-1}, T(x_{2n})) + D(x_{2n}, T(x_{2n-1}))]\Big\} \end{aligned}$$

Each term from maximum's expression tends to $D(A,B)$:

$$\begin{aligned} d(x_{2n-1}, x_A^*) &\to D(A,B); \\ D(x_{2n-1}, T(x_{2n-1})) &= d(x_{2n-1}, x_{2n}) \to D(A,B); \\ D(x_{2n}, T(x_{2n})) &= d(x_{2n}, x_{2n+1}) \to D(A,B); \\ D(x_{2n}, T(x_{2n-1})) &= 0; \\ \frac{1}{2}[D(x_{2n-1}, T(x_{2n}))] &\leq \frac{1}{2}[d(x_{2n-1}, x_{2n}) + D(x_{2n}, T(x_{2n}))] \to D(A,B) \end{aligned}$$

Thus

$$D(x_{2n}, T(x_A^*)) \to D(A,B).$$

Then we have

$$D(A,B) \leq D(x_A^*, T(x_A^*)) \leq d(x_A^*, x_{2n}) + D(x_{2n}, T(x_A^*)) \to D(A,B).$$

Therefore

$$D(x_A^*, T(x_A^*)) = D(A,B).$$

□

Remark 3.2.5. If in Theorem 3.2.4 $D(A,B) = 0$, then we obtain a fixed point result similar to Theorem 3.1.4 for $m = 2$.

Theorem 3.2.6. (Magdaş [37])

Let (X,d) be a complete metric space, $A, B \in P_{cl}(X)$, such that the pairs (A,B) and (B,A) satisfy the property UC. Let $T : A \cup B \to P_{prox}(X)$ be a multi-valued operator. Then the following statements hold:

(i) If T is a multi-valued Ćirić type cyclic operator, then T has at least one best proximity point in A and at least one best proximity point in B;

(ii) If T satisfies the following stronger condition:

for any $x \in A$, $y \in B$,

$$\delta(T(x), T(y)) \leq \varphi(M(x,y) - D(A,B)) + D(A,B),$$

then there exist a best proximity $x_A^* \in A$ and a best proximity point $x_B^* \in B$ such that:

$$d(x_A^*, x_B^*) \leq \sup \{t \geq 0 \mid t - \varphi(t) \leq 3D(A,B)\}.$$

Proof. (i) It is a consequence of Theorem 3.2.4.

(ii) $d(x_A^*, x_B^*) \leq D(x_A^*, T(x_A^*)) + \delta(T(x_A^*), T(x_B^*)) + D(x_B^*, T(x_B^*))$

$$\begin{aligned}
&= 2D(A,B) + \delta(T(x_A^*), T(x_B^*) \leq \\
&\leq 2D(A,B) + \varphi(\max\{d(x_A^*, x_B^*), D(x_A^*, T(x_A^*)), D(x_B^*, T(x_B^*)), \\
&\quad \frac{1}{2}[D(x_A^*, T(x_B^*)) + D(x_B^*, T(x_A^*))]\} - D(A,B)) + D(A,B) \\
&\leq 3D(A,B) + \varphi(\max\{d(x_A^*, x_B^*), D(A,B), D(A,B), \\
&\quad \frac{1}{2}[d(x_A^*, x_B^*) + D(A,B) + d(x_B^*, x_A^*) + D(A,B)]\} - D(A,B)) \\
&= 3D(A,B) + \varphi(d(x_A^*, x_B^*))
\end{aligned}$$

Thus, $d(x_A^*, x_B^*) - \varphi(d(x_A^*, x_B^*)) \leq 3D(A,B)$. □

Corollary 3.2.7. (Magdaş [37]) *Let X be a uniformly convex Banach space, $A, B \in P_{cl,cv}(X)$ and $T : A \cup B \to P_{cl,cv}(X)$ be a multi-valued operator. Then the following statements hold:*

(i) If T is a multi-valued Ćirić type cyclic operator, then T has at least one best proximity point in A and at least one best proximity point in B;

(ii) If T satisfies the following stronger condition:

for any $x \in A$, $y \in B$,

$$\delta(T(x), T(y)) \leq \varphi(M(x,y) - D(A,B)) + D(A,B),$$

then there exist a best proximity $x_A^* \in A$ and a best proximity point $x_B^* \in B$ such that:

$$\|x_A^* - x_B^*\| \leq \sup\{t \geq 0 \mid t - \varphi(t) \leq 3D(A,B)\}.$$

Proof. (i) By Remark 1.1.4, any closed and convex set is proximinal.

Since A and B are convex, by Proposition 1.4.4, the pairs (A, B) and (B, A) satisfy the property UC.

Applying Theorem 3.2.4 we get the existence of a best proximity point $x_A^* \in A$ and a best proximity point $x_B^* \in B$.

(ii) It is an immediate consequence of Theorem 3.2.6. □

If, in Theorem 3.2.6, φ is a subadditive strong comparison function, then the condition that the multi-valued operator takes proximinal values can be removed. More precisely, we obtain the second main result, as follows.

Theorem 3.2.8. (Magdaş [37]) *Let (X, d) be a complete metric space, $A, B \in P_{cl}(X)$, such that (A, B) satisfies the property UC. If $T : A \cup B \to P(X)$ is a multi-valued Ćirić type cyclic operator, with a subadditive strong comparison function φ, then the following statements hold:*

(i) T has a best proximity point $x_A^* \in A$;

(ii) there exists a sequence $(x_n)_{n\in\mathbf{N}}$ with $x_{n+1} \in T(x_n)$ starting from an arbitrary $(x_0, x_1) \in Graph(T)$, such that $(x_{2n})_{n\in\mathbf{N}}$ converges to x_A^*.

Proof. (i)+(ii) Let $(x, y) \in Graph(T)$ be arbitrary. We construct a sequence of successive approximations of T starting from (x, y) in the following way:

$x_0 = x \in A$ and $x_1 = y \in T(x) \subseteq T(A) \subseteq B$;

If $d(x_0, x_1) > D(A, B)$ then $\varphi(z_0) < z_0$, where $z_0 := d(x_0, x_1) - D(A, B)$.

For $\varepsilon_1 \in (0, z_0 - \varphi(z_0))$ there exists $x_2 \in T(x_1) \subseteq T(B) \subseteq A$ such that

$$d(x_1, x_2) \leq H(T(x_0), T(x_1)) + \varepsilon_1.$$

If $d(x_1, x_2) > D(A, B)$ then $\varphi(z_1) < z_1$, where $z_1 := d(x_1, x_2) - D(A, B)$.

For $\varepsilon_2 \in (0, \min\{\varepsilon_1, z_1 - \varphi(z_1)\})$, there exists $x_3 \in T(x_2) \subseteq T(A) \subseteq B$ such that

$$d(x_2, x_3) \leq H(T(x_1), T(x_2)) + \varepsilon_2.$$

Following this procedure in the case $z_{n-1} := d(x_{n-1}, x_n) - D(A, B) > 0$, for $n \geq 2$ we choose

$$\varepsilon_n \in (0, \min\{\varepsilon_{n-1}, z_{n-1} - \varphi(z_{n-1})\}). \tag{3.2.3}$$

There exists $x_{n+1} \in T(x_n)$ such that

$$d(x_n, x_{n+1}) \leq H(T(x_{n-1}), T(x_n)) + \varepsilon_n, n \geq 1,$$

the existence of x_{n+1} being assured by Lemma 1.1.1.
Since T is a multi-valued Ćirić type cyclic operator, using the same reasoning as in Theorem 3.2.4, we have

$$z_n \leq \varphi(\max\{z_{n-1}, z_n\}) + \varepsilon_n, \text{ for } n \geq 1. \tag{3.2.4}$$

Using (3.2.3), we obtain

$$z_n < \varphi(\max\{z_{n-1}, z_n\}) + z_{n-1} - \varphi(z_{n-1}), \text{ for } n \geq 1. \tag{3.2.5}$$

We suppose that $z_{n-1} \leq z_n$. Using the subadditivity of φ and Lemma 1.2.2,

$$\varphi(z_n) = \varphi(z_n - z_{n-1} + z_{n-1}) \leq \varphi(z_n - z_{n-1}) + \varphi(z_{n-1}) \leq z_n - z_{n-1} + \varphi(z_{n-1}),$$

so $z_n \geq \varphi(z_n) + z_{n-1} - \varphi(z_{n-1})$ which contradicts (3.2.5).
We have $z_n \leq z_{n-1}$ and (3.2.4) becomes

$$\begin{aligned}
z_n &\leq \varphi(z_{n-1}) + \varepsilon_n \\
&\leq \varphi\left(\varphi\left(z_{n-2}\right) + \varepsilon_{n-1}\right)\right) + \varepsilon_n \\
&\leq \varphi^2(z_{n-2}) + \varphi\left(\varepsilon_{n-1}\right) + \varepsilon_n \\
&\quad \ldots \\
&\leq \varphi^n(z_0) + \sum_{k=0}^{n-1} \varphi^k(\varepsilon_{n-k}) \\
&\leq \varphi^n(z_0) + \sum_{k=0}^{n-1} \varphi^k(\varepsilon_1) \to 0, \text{ when } n \to \infty.
\end{aligned}$$

Then

$$d(x_n, x_{n+1}) \to D\left(A, B\right) \text{ when } n \to \infty.$$

Applying Lemma 3.2.3 for the sequences

$$(x_{2n})_{n\in\mathbf{N}} \subset A, (x_{2n+2})_{n\in\mathbf{N}} \subset A, \text{ and } (x_{2n+1})_{n\in\mathbf{N}} \subset B,$$

results that $(x_{2n})_{n\in\mathbf{N}}$ is a Cauchy sequence. Because the metric space X is complete and A is closed, the sequence $(x_{2n})_{n\in\mathbf{N}} \subset A$ converges to a point $x_A^* \in A$. Using the same reasoning as in Theorem 3.2.4,

$$D(x_{2n}, T(x_A^*)) \to D(A, B), \text{ when } n \to \infty.$$

Then we have

$$D(A, B) \leq D(x_A^*, T(x_A^*)) \leq d(x_A^*, x_{2n}) + D(x_{2n}, T(x_A^*)) \to D(A, B).$$

Therefore

$$D(x_A^*, T(x_A^*)) = D(A, B).$$

If in the above construction, there exists $k \geq 1$ such that $d(x_{k-1}, x_k) = D(A, B)$, then

$$D(A, B) \leq D(x_{k-1}, T(x_{k-1})) \leq d(x_{k-1}, x_k) = D(A, B)$$

so x_{k-1} is a best proximity point of T.
We will show that, in this situation, x_k is also a best proximity point of T.

$$D(x_k, T(x_k)) \leq H(T(x_{k-1}), T(x_k)) \leq \varphi(M(x_{k-1}, x_k) - D(A, B)) + D(A, B).$$

where

$$\begin{aligned} M(x_{k-1}, x_k) &= \max\Big\{d(x_{k-1}, x_k), D(x_{k-1}, T(x_{k-1})), D(x_k, T(x_k)), \\ &\quad \frac{1}{2}[D(x_{k-1}, T(x_k)) + D(x_k, T(x_{k-1}))]\Big\}\Big) \\ &\leq max\Big\{D(A, B), D(x_k, T(x_k)), \\ &\quad \frac{1}{2}[d(x_{k-1}, x_k) + D(x_k, T(x_k))]\Big\}\Big) \\ &\leq D(x_k, T(x_k)). \end{aligned}$$

Thus $D(x_k, T(x_k)) - D(A, B) \leq \varphi(D(x_k, T(x_k)) - D(A, B))$, which means

$$D(x_k, T(x_k)) = D(A, B).$$

There exists $x_{k+1} \in T(x_k)$ such that

$$d(x_k, x_{k+1}) = D(x_k, T(x_k)) = D(A, B),$$

From now on, following this procedure we construct the terms of our sequence $(x_n)_{n \in \mathbf{N}}$ with $x_{n+1} \in T(x_n)$ such that

$$d(x_n, x_{n+1}) = D(x_n, T(x_n)) = D(A, B), \text{ for any } n \geq k.$$

From this point, the proof runs in the same manner as in the case

$$d(x_n, x_{n+1}) > D(A, B), \text{ for any } n \in \mathbb{N}.$$

□

Hereinafter we define and study the generalized Ulam-Hyers stability of the best proximity problem (3.2.1) for a cyclic multi-valued operator.

Definition 3.2.9. (Magdaş [37]) Let (X, d) be a complete metric space and let $A, B \in P(X)$. Let $T : A \cup B \to P(X)$ be a multi-valued operator satisfying the cyclic condition $T(A) \subset B, T(B) \subset A$. The best proximity problem (3.2.1) is called generalized Ulam-Hyers stable if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous at 0, with $\psi(0) = 0$ and there exists $c > 0$ such that for any $\varepsilon > 0$ and $x \in B$ with

$$D(x, T(x)) \leq \varepsilon + D(A, B),$$

there exists a solution $x_A^* \in A$ of (3.2.1) such that

$$d(x, x_A^*) \leq \psi(\varepsilon) + c \cdot D(A, B).$$

Our stability result is the following.

Theorem 3.2.10. (Magdaş [37]) *Let* (X, d) *be a complete metric space,* $A \in P_{cl}(X), B \in P(X)$*, such that* (A, B) *satisfies the property UC and* φ *be a comparison function. Let* $T : A \cup B \to P_{prox}(X)$ *be a multi-valued operator. Assume that:*

(i) $T(A) \subset B, T(B) \subset A$*;*

(ii) for any $x \in A$, $y \in B$,

$$\delta(T(x), T(y)) \leq \varphi(\max\{D(x, T(x)), D(y, T(y))\} - D(A, B)) + D(A, B).$$

Then the best proximity problem (3.2.1) is generalized Ulam-Hyers stable.

Proof. T is a multi-valued Ćirić type cyclic operator, so the best proximity problem has at least one solution $x_A^* \in A$.

$$\begin{aligned} d(x, x_A^*) &\leq D(x, T(x)) + \delta(T(x), T(x_A^*)) + D(x_A^*, T(x_A^*)) \\ &\leq \varepsilon + D(A, B) + \varphi(\max\{D(x, T(x)), D(x_A^*, T(x_A^*))\} - D(A, B)) \\ &\quad +2D(A, B) \\ &\leq \varepsilon + \varphi(\max\{\varepsilon + D(A, B), D(A, B)\} - D(A, B)) + 3D(A, B). \end{aligned}$$

In conclusion,

$$d(x, x_A^*) \leq \varepsilon + \varphi(\varepsilon) + 3D(A, B),$$

proving that the best proximity problem (3.2.1) is generalized Ulam-Hyers stable. □

3.3 Coupled fixed point and coupled best proximity point theorems for multi-valued cyclic contraction type operators

The purpose of this section is to study the coupled fixed point problem and the coupled best proximity problem for multi-valued cyclic contraction type operators. The approach is based on fixed point results and best proximity point results for appropriate operators generated by the initial problems.

Definition 3.3.1. Let (X, d) be a metric space, $A, B \in P(X)$, $Y = A \cup B$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a strong comparison function. A multi-valued operator $F : Y \times Y \to P(Y)$ is called a cyclic coupled φ-contraction of Ćirić type multi-valued operator if the following statements hold:

(i) F is cyclic with respect to A and B, that is

$$F(A \times B) \subseteq B \text{ and } F(B \times A) \subseteq A;$$

(ii)

$$H(F(x,y), F(u,v)) \leq \varphi(\widetilde{M}(x,v,y,u)), \text{ for any } x, v \in A,\ y, u \in B \quad (3.3.1)$$

where

$$\widetilde{M}(x,v,y,u) = \max\Big\{d(x,u), d(v,y), D(x, F(x,y)), D(u, F(u,v)), D(v, F(v,u)), D(y, F(y,x)), \frac{1}{2}[D(x, F(u,v)) + D(u, F(x,y))], \frac{1}{2}[D(y, F(v,u)) + D(v, F(y,x))]\Big\}.$$

The following theorem which is a particular case of Theorem 3.1.4 will be used to prove the first result in this section.

Theorem 3.3.2. *Let (X, d) be a complete metric space, $A, B \in P_{cl}(X)$ and $T : A \cup B \to P_{prox}(A \cup B)$ a multi-valued cyclic φ-contraction of Ćirić type, that is:*

(i) $T(A) \subseteq B$ and $T(B) \subseteq A$;

(ii) there exists a strong comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $x \in A$ and $y \in B$,

$$H(T(x),T(y)) \le \varphi\Big(\max\Big\{d(x,y), D(x,T(x)), D(y,T(y)),$$
$$\frac{1}{2}[D(x,T(y))+D(y,T(x))]\Big\}\Big).$$

Then the following statements hold:

(1) *there exists $x^* \in A\cap B$ such that $x^* \in T(x^*)$;*

(2) *for any $x \in A$ and $y \in T(x)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_0 = x$, $x_1 = y$ and $x_n \in T(x_{n-1})$, $n \ge 1$, that converges to a fixed point $x^* \in A\cap B$ of T.*

The following lemma presents well-known results throughout literature (see for example Mleşniţe, Petruşel [39]).

Lemma 3.3.3. *Let (X,d) be a metric space, d^* the metric defined on $X\times X$ by (2.3.5) and D^* the gap functional, respectively H^* the generalized Pompeiu-Hausdorff functional generated by d^*. Then for any $a,b \in X$ and any $A,B,C,D \in P_{prox}(X)$, the following statements hold:*

(1) $D^*((a,b),C\times D) = \max\{D(a,C), D(b,D)\}$;

(2) $D^*(A\times B, C\times D) = \max\{D(A,C), D(B,D)\}$;

(3) $H^*(A\times B, C\times D) = \max\{H(A,C), H(B,D)\}$;

(4) $D^*(A\times B, B\times A) = D(A,B)$.

Proof. (1)+(2) Since the sets C and D are proximinal then there exist $c_0 \in C, d_0 \in D$ such that $D(a,C) = d(a,c_0)$ and $D(b,D) = d(b,d_0)$.

$$\text{Then } D^*((a,b),C\times D) = \inf\{d^*((a,b),(c,d)) \mid c\in C,\ d\in D\}$$
$$= \inf\{\max\{d(a,c), d(b,d)\} \mid c\in C,\ d\in D\}$$
$$= \max\{d(a,c_0), d(b,d_0)\}.$$

Similarly, we can prove (2).

(3) $H^*(A\times B, C\times D) =$

$\max\{\sup_{(a,b)\in A\times B}\{D^*((a,b),C\times D)\}, \sup_{(c,d)\in C\times D}\{D^*((c,d),A\times B)\}\}.$

Using statement (1), we have

$H^*(A\times B, C\times D) =$

$\max\{\sup_{(a,b)\in A\times B}\{D(a,C), D(b,D)\}, \sup_{(c,d)\in C\times D}\{D(c,A), D(d,B)\}\}$

$= \max\{H(A,C), H(B,D)\}.$

(4) We use statement (2) for $C = A, D = B$. □

Lemma 3.3.4. *Let (X,d) be a metric space, d^* the metric defined on $X \times X$ by (2.3.5). If a multi-valued operator $F : X \times X \to P(X)$ takes proximinal values with respect to d then the multi-valued operator $T : X \times X \to P(X \times X)$, $T(x,y) = (F(x,y), F(y,x))$ takes proximinal values with respect to d^*.*

Proof. For any pair $(a,b) \in X \times X, F(a,b)$ is a proximinal set, which means that for any $x \in X$, there exists $c \in F(a,b)$ such that

$$d(x,c) = D(x, F(a,b)).$$

In a similar way, for any $y \in X$, there exists $d \in F(b,a)$ such that

$$d(y,d) = D(y, F(b,a)).$$

Then for any $(x,y) \in X \times X$, there exists $(c,d) \in T(a,b)$ such that

$$\begin{aligned} d^*((x,y),(c,d)) &= \max\{d(x,c), d(y,d)\} \\ &= \max\{D(x,F(a,b)), D(y,F(b,a))\} \\ &= D^*((x,y),T(a,b)). \end{aligned}$$

□

The first result in this section is the following theorem.

Theorem 3.3.5. (Magdaş [35]) *Let (X,d) be a complete metric space, $A, B \in P_{cl}(X)$, $Y = A \cup B$ and $F : Y \times Y \to P_{prox}(Y)$ be a cyclic coupled φ-contraction of Ćirić type multi-valued operator.*

Then the following statements hold:

(1) *there exist $x^*, y^* \in A \cap B$ such that*

$$x^* \in F(x^*, y^*), \; y^* \in F(y^*, x^*),$$

(that is the pair (x^*, y^*) is a coupled fixed point of F);

(2) *for each $(a,b) \in A \times B$ there exists a sequence $(a_n, b_n)_{n \in \mathbb{N}^*} \in Y \times Y$ with $a_0 = a$, $b_0 = b$ and*

$$a_n \in F(b_{n-1}, a_{n-1}), \; b_n \in F(a_{n-1}, b_{n-1}) \text{ for } n \geq 1$$

that converges to a coupled fixed point $(x^*, y^*) \in A \cap B$ of F.

Proof. It is easy to observe that

$$\widetilde{M}(x,v,y,u)=\widetilde{M}(v,x,u,y), \text{ for any } x,v\in A,\ y,u\in B.$$

If we change the roles between x and v and similarly for y and u, then the inequality (3.3.1) becomes

$$H(F(v,u),F(y,x))\le\varphi(\widetilde{M}(x,v,y,u)). \tag{3.3.2}$$

From (3.3.1) and (3.3.2) we obtain

$$\max\{H(F(x,y),F(u,v)),H(F(y,x),F(v,u))\}\le\varphi(\widetilde{M}(x,v,y,u)).$$

Let $T:Y\times Y\to P(Y\times Y)$, $T(x,y)=(F(x,y),F(y,x))$.

We consider on $Y\times Y$ the metric d^* defined by (2.3.5), using the same functionals D^* and H^* as in lemma 3.3.3.

For $z=(x,y)\in A\times B$, $w=(u,v)\in B\times A$, using Lemma 3.3.3,

$$\begin{aligned}H^*(T(z),T(w))&=H^*((F(x,y),F(y,x)),(F(u,v),F(v,u)))\\&=\max\{H(F(x,y),F(u,v)),H(F(y,x),F(v,u))\}\\&\le\varphi(\widetilde{M}(x,v,y,u)).\end{aligned}\tag{3.3.3}$$

By Lemma 3.3.3,

$$D^*(z,T(z))=\max\{D(x,F(x,y)),D(y,F(y,x))\},$$
$$D^*(w,T(w))=\max\{D(u,F(u,v)),D(v,F(v,u))\},$$

$$\begin{aligned}\frac{1}{2}[D^*(w,T(z))+D^*(z,T(w))]&=\frac{1}{2}[\max\{D(u,F(x,y)),D(v,F(y,x))\}\\&\quad+\max\{D(x,F(u,v)),D(y,F(v,u))\}]\\&\ge\max\left\{\frac{1}{2}[D(u,F(x,y))+D(x,F(u,v))],\right.\\&\left.\frac{1}{2}[D(v,F(y,x))+D(y,F(v,u))]\right\}.\end{aligned}$$

Using the monotonicity of φ, (3.3.3) becomes

$$\begin{aligned}H^*(T(z),T(w))\le&\varphi(\max\{d^*(z,w),D^*(z,T(z)),D^*(w,T(w)),\\&\frac{1}{2}[D^*(w,T(z))+D^*(z,T(w))]\}),\\&\text{for any } z\in A\times B,\ w\in B\times A,\end{aligned}$$

and because T satisfies the cyclic condition

$$T(A \times B) = (F(A \times B), F(B \times A)) \subseteq B \times A, \; T(B \times A) \subseteq A \times B,$$

where $A \times B, B \times A \in P_{cl}(Y \times Y)$, we conclude that T is a multi-valued cyclic φ-contraction of Ćirić type.

By Lemma 3.3.4, the property of the operator F to have proximinal values is transferred to the operator T, so we are in the conditions of Theorem 3.3.2.

Then there exists $(x^*, y^*) \in (A \times B) \cap (B \times A)$ such that $(x^*, y^*) \in (F(x^*, y^*), F(y^*, x^*))$ and for each $(a, b) \in A \times B$ there exists a sequence $(a_n, b_n)_{n \in \mathbb{N}} \in Y \times Y$ with $a_0 = a$, $b_0 = b$ and

$$(a_n, b_n) \in (F(b_{n-1}, a_{n-1}), F(a_{n-1}, b_{n-1})), \; n \geq 1$$

that converges to (x^*, y^*). □

Hereinafter we define and study the generalized Ulam-Hyers stability of the following coupled fixed point problem.

Definition 3.3.6. (Magdaş [35]) Let (X, d) be a metric space, $Y \in P(X)$, $F : Y \times Y \to P(Y)$ be a multi-valued operator. By definition, the coupled fixed point problem

$$\begin{cases} x \in F(x, y) \\ y \in F(y, x) \end{cases}, \; x, y \in Y, \tag{3.3.4}$$

is said to be generalized Ulam-Hyers stable if there exists an increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, continuous at 0, with $\psi(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $(x, y) \in Y \times Y$ of the inequality

$$\max\{D(x, F(x, y)), D(y, F(y, x))\} \leq \varepsilon,$$

there exists a solution $(x^*, y^*) \in Y \times Y$ of the coupled fixed point problem such that

$$\max\{d(x, x^*), d(y, y^*)\} \leq \psi(\varepsilon).$$

Our stability result is a consequence of the Theorem 3.1.8.

Theorem 3.3.7. (Magdaş [35]) *If all the hypotheses of Theorem 3.3.5 take place, then the coupled fixed point problem (3.3.4) is generalized Ulam-Hyers stable.*

Proof. Let any $\varepsilon > 0$ and let $(\overline{x}, \overline{y}) \in Y \times Y$ such that

$$\begin{cases} D(\overline{x}, F(\overline{x}, \overline{y})) \leq \varepsilon \\ D(\overline{y}, F(\overline{y}, \overline{x})) \leq \varepsilon. \end{cases}$$

As before, we consider $T : Y \times Y \to P(Y \times Y)$,

$$T(x, y) = (F(x, y), F(y, x)).$$

For $z = (\overline{x}, \overline{y})$,

$$D^*(z, T(z)) = \max\{D(\overline{x}, F(\overline{x}, \overline{y})), D(\overline{y}, F(\overline{y}, \overline{x}))\} \leq \varepsilon.$$

Applying Theorem 3.1.8, there exists a fixed point $z^* = (x^*, y^*)$ of T such that $d^*(z, z^*) \leq s(\varepsilon)$, that is there exists a solution (x^*, y^*) of the coupled fixed point problem (3.3.4) such that

$$\max\{d(\overline{x}, x^*), d(\overline{y}, y^*)\} \leq s(\varepsilon).$$

□

In the last part of this section we will consider the following best proximity problem for a cyclic coupled multi-valued operator:

If (X, d) is a metric space, $A, B \in P(X)$, $Y = A \cup B$, $F : Y \times Y \to P(Y)$ is a coupled multi-valued operator satisfying the cyclic condition

$$F(A \times B) \subseteq B, F(B \times A) \subseteq A,$$

then we are interested to find $(x^*, y^*) \in A \times B$ such that

$$D(x^*, F(x^*, y^*)) = D(y^*, F(y^*, x^*)) = D(A, B). \tag{3.3.5}$$

(x^*, y^*) is said to be a coupled best proximity point of F.

Notice that, in particular, if $A \cap B \neq \emptyset$ then (x^*, y^*) is a coupled fixed point of F.

Definition 3.3.8. (Magdaş [35]) Let (X, d) be a metric space, $A, B \in P(X)$, $Y = A \cup B$. A multi-valued operator $F : Y \times Y \to P(Y)$ is called a cyclic coupled Ćirić type multi-valued operator if:

(i) $F(A \times B) \subseteq B$ and $F(B \times A) \subseteq A$;

(ii) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$H(F(x, y), F(u, v)) \leq \varphi(\widetilde{M}(x, v, y, u) - D(A, B)) + D(A, B),$$

for any $x, v \in A$, $y, u \in B$.

Lemma 3.3.9. *Let A and B nonempty subsets of a metric space (X, d), and d^* the metric defined on $X \times X$ by (2.3.5). If (A, B) and (B, A) satisfy the property UC with respect to d then $(A \times B, B \times A)$ satisfy the property UC with respect to d^*.*

Proof. We denote $D^*(A \times B, B \times A) = D(A, B) = D$. Let $x_n = (a_n, b_n), z_n = (a'_n, b'_n) \in A \times B, y_n = (\beta_n, \alpha_n) \in B \times A$ such that $d^*(x_n, y_n) \to D$ and $d^*(z_n, y_n) \to D$ as $n \to \infty$.

Then

$$\max\{d(a_n, \beta_n), d(b_n, \alpha_n)\} \to D \text{ and}$$

$$\max\{d(a'_n, \beta_n), d(b'_n, \alpha_n)\} \to D \text{ as } n \to \infty.$$

It is obvious that $d(a_n, \beta_n) \to D, d(a'_n, \beta_n) \to D$ and because (A, B) satisfies the property UC we get $d(a_n, a'_n) \to 0$.

From $d(b_n, \alpha_n) \to D, d(b'_n, \alpha_n) \to D$ as $n \to \infty$ and using (B, A) satisfies the property UC we get $d(b_n, b'_n) \to 0$.

Finally,

$$d^*(x_n, z_n) = \max\{d(a_n, a'_n), d(b_n, b'_n)\} \to 0 \text{ as } n \to \infty.$$

□

The next result is a consequence of the Theorem 3.2.4.

Theorem 3.3.10. (Magdaş [35]) *Let (X, d) be a complete metric space, $A, B \in P_{cl}(X)$ such that (A, B) and (B, A) satisfy the property UC, and $Y = A \cup B$. If $F : Y \times Y \to P_{prox}(Y)$ is a cyclic coupled Ćirić type multi-valued operator, then the following statements hold:*

(i) F has a coupled best proximity point $(x^*, y^*) \in A \times B$;

(ii) there exist two sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ with

$$(x_0, y_0) \in A \times B,\ x_{n+1} \in F(x_n, y_n),\ y_{n+1} \in F(y_n, x_n),$$

such that $((x_{2n}, y_{2n}))_{n\in\mathbb{N}}$ converges to (x^*, y^*).

Proof. Considering again on $Y \times Y$ the metric d^* defined by (2.3.5), in a similar manner as in Theorem 3.3.5, we obtain that the operator

$$T : Y \times Y \to P(Y \times Y), T(x, y) = (F(x, y), F(y, x))$$

is a multi-valued Ćirić type cyclic operator which takes proximinal values.

Using Lemma 3.3.9, the pair $(A\times B, B\times A)$ satisfies the property UC with respect to d^*.

Consequently, we are in the conditions of Theorem 3.2.4, so T has a best proximity point $(x^*, y^*) \in A \times B$ and there exists a sequence $(x_n, y_n)_{n\in\mathbb{N}}$ with $(x_0, y_0) \in A\times B$ and $(x_{n+1}, y_{n+1}) \in T(x_n, y_n)$ such that $(x_{2n}, y_{2n})_{n\in\mathbb{N}}$ converges to (x^*, y^*) with respect to d^*. $\square$

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