Sándor Kajántó

RICCATI PAIRS: A NEW APPROACH FOR HARDY-RELLICH INEQUALITIES

Presa Universitară Clujeană

Sándor Kajántó

RICCATI PAIRS: A NEW APPROACH FOR HARDY-RELLICH INEQUALITIES

Presa Universitară Clujeană

2025

Referenți științifici:

Prof. univ. dr. Csaba Farkas

Prof. univ. dr. Alexandru Kristály

ISBN 978-606-37-2744-3

Universitatea Babeş-Bolyai
Presa Universitară Clujeană
Director: Codruța Săcelean
Str. B.P. Hasdeu nr. 51
400371 Cluj-Napoca, România
Tel.: (+40)-744-687.884
E-mail: editura@ubbcluj.ro
http://www.editura.ubbcluj.ro/
https://libraria.ubbcluj.ro/

Contents

Preface **5**

1 Introduction **7**

2 Preliminaries and notations **15**
- 2.1 Riemannian manifolds 15
  - 2.1.1 Model space forms 17
- 2.2 Finsler manifolds 19
  - 2.2.1 Curvatures and comparison 22
  - 2.2.2 Special Finsler metrics 23
- 2.3 Eigenvalue problems 24
  - 2.3.1 Fixed membrane problem 24
  - 2.3.2 Clamped plate problem 26
  - 2.3.3 Buckling problem 28
- 2.4 The approach of Bessel pairs 28

3 Hardy inequalities on general Riemannian manifolds **33**
- 3.1 General functional inequalities 34
  - 3.1.1 Riccati pairs 36
- 3.2 Applications I: Additive Hardy-type inequalities 41
  - 3.2.1 Caccioppoli inequalities 41
  - 3.2.2 Improved Hardy inequalities 43
  - 3.2.3 Spectral estimates on Riemannian manifolds 50
  - 3.2.4 Interpolation: Hardy inequality versus McKean spectral gap . . 54
  - 3.2.5 Ghoussoub–Moradifam-type inequalities 58
- 3.3 Applications II: Multiplicative Hardy-type inequalities 62
  - 3.3.1 Sharp uncertainty principles 62
  - 3.3.2 Caffarelli–Kohn–Nirenberg inequalities 66

4 Rellich inequalities on general Riemannian manifolds **69**
- 4.1 General functional inequalities 70
- 4.2 Applications I: Spectral gap estimates 74
  - 4.2.1 Clamped plate problem 74
  - 4.2.2 Buckling problem 76
  - 4.2.3 Higher-order estimates 78

4.3 Applications II: Rellich inequalities 80
4.3.1 Classical and weighted Rellich inequalities 80
4.3.2 Higher-order variants of the classical Rellich inequality 81
4.3.3 Further Rellich inequalities 82

5 Rellich inequalities on space forms 85
5.1 General functional inequalities 86
5.1.1 Simplified Riccati pairs and dual Riccati pairs 86
5.2 Applications I: Inequalities on Euclidean spaces 90
5.3 Applications II: Inequalities on hyperbolic spaces 93

6 Hardy inequalities on Finsler manifolds 97
6.1 A Finslerian extension of Riccati pairs 99
6.1.1 Applications 101
6.2 Sharp Hardy inequalities – non-positive curvature 103
6.2.1 Construction 103
6.2.2 Sufficient conditions for sharpness 106
6.2.3 The sharpness of a weighted Hardy inequality 110
6.3 Sharp Hardy inequalities – strong negative curvature 111
6.3.1 Construction 111
6.3.2 Sufficient conditions for sharpness 115
6.3.3 The sharpness of a McKean-type spectral gap estimate 120

Bibliography 123

Preface

This monograph presents alternative approaches to Hardy and Rellich inequalities. Concerning Hardy-type inequalities on Riemannian manifolds, we establish a generic functional inequality, both in *additive* and *multiplicative* forms, which produces well-known and genuinely new inequalities. For the additive version we introduce the notion of *Riccati pairs*, which enables us to give short/elegant proofs for several celebrated functional inequalities on Riemannian manifolds with sectional curvature bounded from above, by simply solving a Riccati-type ODE. The multiplicative version allows us to cover *uncertainty principles* as well.

Concerning the Rellich inequalities on Riemannian manifolds, we establish two generic functional inequalities on Riemannian manifolds of the same type as before. As applications, on the one hand, we prove *sharp spectral gap estimates* for various higher-order eigenvalue problems. On the other hand, we provide extensions for some well-known Rellich inequalities. The latter methods differ from the approach of Riccati pairs, thus, as a next point, we discuss the applicability of Riccati pairs in the context of Rellich inequalities.

As a next step, we revisit Hardy inequalities in view of Finsler geometry, and extend the method of Riccati pairs to this setting. The proof of this extension provides an other explanation of the appearance of the so called reversibility constant in Finslerian Hardy inequalities. We also address the issue of sharpness (covering both cases of non-positive and strong negative curvature; assuming finite reversibility) and we provide a partial answer: both for the (Finslerian versions of the) classical Hardy inequality and the McKean-type spectral gap estimate, we construct some families of Finsler manifolds on which they turn out to be sharp. We also provide sufficient conditions for the sharpness of more involved Hardy-type inequalities on the aforementioned spaces.

The elegance of our approaches lies in their simplicity: the proofs are based on convexity arguments and applications of divergence/comparison theorems; moreover, they are symmetrization-free. Consequently, the validity of the generalized Cartan–Hadamard conjecture is not required, which broadens the range of applicability of our results.

Keywords and phrases

Hardy inequalities, Rellich inequalities, Riemannian manifolds, Finsler manifolds, Riccati pairs, Spectral gap estimates, Symmetrization-free approach

2020 Mathematics Subject Classification

26D10, 35A23, 35P15, 35R01, 46E35, 49R05, 58C40, 58J05, 58J60.

A note on sources and citations

The present monograph serves as an extension of the PhD thesis of the author entitled *New approach for Hardy–Rellich inequalities* defended in 2024 at the Babeș-Bolyai University, Cluj-Napoca, Romania. The monograph consists of six chapters, among which the first two have an introductory role. The next four chapters – containing the original results of the monograph – are grounded in the following individual articles:

- Chapter 3: Kajántó, Kristály, Peter, and Zhao [59];
- Chapter 4: Farkas, Kajántó, and Kristály [39];
- Chapter 5: Kajántó [57];
- Chapter 6: Kajántó [58].

Among these, Chapters 3, 4 & 5 are taken from the PhD thesis of the author, while Chapter 6 is entirely a supplement to the thesis.

The primary results of the monograph are either *generic functional inequalities* that provide Hardy–Rellich inequalities or some special constructions of manifolds on which we investigate sharpness. These results are also presented in one of the above papers. In these cases, we provide citations as follows: **Theorem X.Y (see [REF]).**

The secondary results of the monograph are *applications* of the primary results. These typically involve extensions and alternative proofs of some *formally* well-known Hardy and Rellich inequalities. These results are also present in one of the above papers; we indicate this fact as follows: **Theorem X.Y** (see [REF])**.**

We note that particular versions of the secondary results can also be found in other publications beyond the papers mentioned above; additional references are provided in the text surrounding the corresponding result in the following manner: see e.g., Author1, Author2 [REF].

Citations of auxiliary results taken from other papers than the above-mentioned ones are provided in the text surrounding the corresponding result, using the latter format. Due to their explicatory role, the citations of remarks, which are partially or entirely present in the above papers, are omitted.

Acknowledgements

I would like to express my sincere gratitude to Prof. Dr. Alexandru Kristály for his guidance and support throughout and after my PhD studies. His expertise, encouragement, and unwavering commitment have been invaluable to me and I am incredibly grateful for the opportunity to learn and grow under his mentorship. I would also like to thank Professors Dr. Csaba Farkas, Dr. Ioan Radu Peter, and Dr. Wei Zhao for their indispensable contributions to the research. Finally, I extend my profound thanks to my family, friends, and colleagues, whose reliable support has been a constant source of strength and encouragement throughout this process.

The writing of the book was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

Sincerely,
the author.

Chapter 1

Introduction

More than one hundred years elapsed since the celebrated (one-dimensional) *Hardy inequality* appeared (see Hardy [53]). A multi-dimensional version of the original result can be stated as follows: Given a smooth, compactly supported function u on $\Omega \subseteq \mathbb{R}^n$, the L^2-norm of the singular term $u(x)/|x|$ is controlled by the L^2-norm of $|\nabla u(x)|$. More precisely, one has

$$\int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x \geq \frac{(n-2)^2}{4} \int_\Omega \frac{|u(x)|^2}{|x|^2} \,\mathrm{d}x, \qquad \forall u \in C_0^\infty(\Omega).$$

Moreover, the above inequality is *sharp*, in the sense that the constant $\frac{(n-2)^2}{4}$ cannot be improved.

Several extensions and improvements of this inequality can be found by now in the literature, involving more general weights, additional correction terms, and/or various underlying geometrical settings; sometimes in alternative formulation, often referred to as *uncertainty principles*. In general, they can be written in the following forms:

$$\int_\Omega V|\nabla u|^p \,\mathsf{dm} \geq \int_\Omega W|u|^p \,\mathsf{dm}, \qquad \forall u \in C_0^\infty(\Omega), \tag{$\mathbf{H}$}$$

$$\left(\int_\Omega V|\nabla u|^p \,\mathsf{dm}\right)^{\frac{1}{p}} \left(\int_\Omega \overline{W}|u|^p \,\mathsf{dm}\right)^{\frac{1}{p'}} \geq \int_\Omega \widetilde{W}|u|^p \,\mathsf{dm}, \qquad \forall u \in C_0^\infty(\Omega), \tag{$\mathbf{UP}$}$$

respectively, where $p > 1$ and $p' \stackrel{\text{def}}{=} \frac{p}{p-1}$ is the *conjugate* of p; Ω is an open subset of an ambient space M, which could be the Euclidean space $\mathbb{R}^n$, any Riemannian/Finsler manifold, or a stratified group; m is a measure on M, while $V, W, \overline{W}, \widetilde{W} \colon \Omega \to (0, \infty)$ are certain potentials, possibly containing singular terms. We notice that the analysis of the sharpness of the above inequalities and the determination of the best constants has now developed into its own distinct body of literature.

These inequalities became indispensable from the point of view of applications. On the one hand, solutions of a large class of elliptic problems involving singular terms are based on the validity of corresponding Hardy-type inequalities; hence, they appear in several fields of mathematics. In particular, they play a crucial role in the spectral theory of the *fixed membrane problem*, describing the fundamental tones.

On the other hand, uncertainty principles have an especially important role in quantum mechanics, formulating one of its most fundamental and yet most surprising facts: Given an arbitrary particle of the Universe, one cannot determine precisely both its position and momentum, i.e., the more precisely we know its position, the less precisely we know its momentum, and vice versa. In this study we restrict ourselves to the mathematical point of view of such uncertainty principles.

Key observations related to Hardy-type inequalities have been made (mainly for $p = 2$ and the Euclidean setting) e.g., by Adimurthi, Chaudhuri, and Ramaswamy [1], Brezis and Marcus [17], Brezis and Vázquez [18], Devyver, Fraas, and Pinchover [34], Fefferman [41], Filippas, Maz'ya, and Tertikas [42, 43], Filippas and Tertikas [44], Muckenhoupt [85], Ruzhansky and Suragan [98] and Tertikas and Zographopoulos [102].

The higher-order variants of the inequality (**H**) are the following two Rellich-type inequalities:

$$\int_\Omega U|\Delta u|^p \,\mathsf{dm} \geq \int_\Omega W|u|^p \,\mathsf{dm}, \qquad \forall u \in C_0^\infty(\Omega), \tag{$\mathbf{R}_1$}$$

$$\int_\Omega U|\Delta u|^p \,\mathsf{dm} \geq \int_\Omega V|\nabla u|^p \,\mathsf{dm}, \qquad \forall u \in C_0^\infty(\Omega), \tag{$\mathbf{R}_2$}$$

where $p > 1$ and U, V, W are given positive potentials on Ω. Both inequalities of types ($\mathbf{R}_1$) and ($\mathbf{R}_2$) have numerous applications; here we highlight the spectral theory of the *clamped plate problem* and the *buckling problem*, respectively. The first problem describes the vibrations within a thin elastic plate with a clamped boundary. The second problem investigates a similar plate that is subjected to a compressive load. We also notice that inequality ($\mathbf{R}_1$) can be obtained by combining either ($\mathbf{R}_2$) and (**H**), or their alternative versions, where the classical gradient ∇u is replaced by the radial directional derivative $\nabla^{\mathrm{rad}} u = \langle \nabla u, \nabla d_{x_0} \rangle$ and d_{x_0} is the distance from a point $x_0 \in \Omega$.

For both problems, it is increasingly true that they are mainly considered in the Euclidean setting for $p = 2$. The classical version of ($\mathbf{R}_1$) dates back to the 1950s and is due to Rellich [97]. Surprisingly, as claimed by the authors, the corresponding version of ($\mathbf{R}_2$) only appeared relatively recently (in the 2000s) in the paper by Tertikas and Zographopoulos [102]. These facts suggest inequality ($\mathbf{R}_2$) being more problematic. Further pioneering results in the Euclidean setting can be found e.g., in the papers by Davies and Hinz [32] and Mitidieri [82]. For results on more general structures, see e.g., Kombe and Özaydin [64, 65] and Kristály and Repovš [73]. Comprehensive discussions about Hardy and Rellich inequalities can also be found in the monographs by Balinsky, Evans, and Lewis [7], Ghoussoub and Moradifam [51], and Ruzhansky and Suragan [99].

A milestone result – concerning the problems (**H**) and ($\mathbf{R}_2$) – has been provided by Ghoussoub and Moradifam [49, 50], again for $p = 2$ and the Euclidean setting. On the one hand, the authors showed that the inequality (**H**) holds if and only if (V, W) is a *Bessel pair*. The latter notion is based on the solvability of a second-order linear Bessel-type ordinary differential equation (ODE for short) containing the potentials V and W. On the other hand, they showed that under certain conditions, inequality ($\mathbf{R}_2$) holds if and only if $(U, \widetilde{V})$ is a Bessel pair, where $\widetilde{V}$ is a potential involving V and an additional correction term.

The proof of the results concerning Bessel-pairs are heavily based on the technique of *spherical harmonics decomposition*, which works well on the model space forms (Euclidean space, hyperbolic space, and the sphere), but it cannot be applied to general Riemann manifolds. The concept was extended to general $p > 1$ (see Duy, Lam, and Lu [36]), and has applications on non-positively curved Riemannian manifolds (see Flynn, Lam, Lu, and Mazumdar [46] and Berchio, Ganguly, and Grillo [11]), where still the usual notion of Bessel pairs and fine comparison arguments are used.

In this work, we present an alternative approach to Hardy–Rellich inequalities, which completely avoids the use of spherical harmonics decomposition. As we shall see, our forthcoming proofs are built upon simple convexity arguments, multiple uses of divergence theorems, and comparison theorems that encode curvature information of the ambient manifold. In the sequel, we present our results simultaneously with the structure of this study, as follows.

In Chapter 2 we recall some definitions and preliminary results that are relevant for our presentation. In Section 2.1 we present several concepts and their corresponding notations concerning general Riemannian manifolds. In Section 2.1.1 we present various notions regarding model space forms and relevant comparison results (see Theorem 2.1). In Section 2.2 we display some relevant elements of Finsler geometry. In Section 2.2.1 we recall several curvature notions and relevant comparison result (see Theorem 2.3). In Section 2.2.2 some special classes of Finsler manifolds are discussed. In Section 2.3 we address various eigenvalue problems on Riemannian manifolds with their corresponding spectral gap estimates. Namely, we discuss the *fixed membrane problem* (Section 2.3.1), the *clamped plate problem* (Section 2.3.2) and the *buckling problem* (Section 2.3.3). Finally, in Section 2.4 we summarize the method of Bessel pairs mentioned earlier; for the central result of the approach, see Theorem 2.14. Here, we also discuss the relation between Bessel pairs and the classical method of *supersolutions*, also known as the *Allegretto–Moss–Piepenbrink approach*, which emerges from the early works of Allegretto [4] and Moss and Piepenbrink [84] (see Remark 2.16).

In Chapter 3 we restrict our attention to Hardy inequalities; our approach for them is presented in Section 3.1: First, in Theorem 3.1, we provide a general functional inequality on Riemannian manifolds in both *additive* and *multiplicative* forms that turn out to be *equivalent* to each other. Both forms involve several parameters besides the unknown function u; substituting concrete parameters yields inequalities of types (**H**) and (**UP**), respectively. Next, in Section 3.1.1, we make a key observation: Both forms contain the Laplacian of a given potential (implicitly encoding curvature information about the manifold), which suggests the application of an appropriate comparison argument. This comparison furnishes – in the additive form – a Riccati-type ordinary differential inequality (ODI), which leads to the notion of *Riccati pairs* for certain potentials (see Definition 3.3). Incorporating this notion into the additive form yields Theorem 3.5, which turns out to be extremely efficient in proving inequalities of type (**H**). Indeed, to prove an inequality of this type, it is enough to solve a Riccati-type ODI, and to apply Theorem 3.5. Finally, in Proposition 3.7 and Remark 3.8 we show that Riccati pairs extend Bessel pairs (slightly in the Euclidean case); however, the difference is mainly in the underlying technique, not in the ODE/ODI.

In Section 3.2 we present simple alternative proofs for various functional inequalities of type (**H**) using Theorem 3.1/(i) and Theorem 3.5. We highlight that a part of these results is formally well-known in the Euclidean setting. Our method, however, extends them to *Cartan–Hadamard manifolds*, which are complete, simply connected Riemannian manifolds, with non-positive sectional curvature. This demonstrates the efficiency of our main results, mostly based on Riccati pairs. We shall consider the following inequalities:

- In Section 3.2.1 we present two *L^p-Caccioppoli-type inequalities* on Riemannian manifolds, providing alternative proofs for the results obtained by D'Ambrosio and Dipierro [30] (see Theorems 3.9 & 3.12). Some new improvements are also established in the case $p \in (1, 2]$.

- In Section 3.2.2 we discuss a number of *improved Hardy-type inequalities* on Cartan–Hadamard manifolds, including results by Carron [20] and Kombe and Özaydin [64, 65] (see Theorem 3.13), Edmunds and Triebel [37] (see Theorem 3.17), Adimurthi, Chaudhuri, and Ramaswamy [1] (see Theorem 3.18), and Brezis and Vázquez [18] (see Theorem 3.19).

- In Section 3.2.3 we present *spectral estimates* on Riemannian manifolds. First, we establish a simple alternative proof of the celebrated Cheng's comparison result on Riemannian manifolds with sectional curvature bounded from above (see Theorem 3.21). Next, we consider the well-known Faber–Krahn inequality and the McKean spectral gap estimate (see Theorems 3.23 & 3.24). In addition, we give a short proof of the main spectral result of Carvalho and Cavalcante [21] (see Theorem 3.26).

- In Section 3.2.4 we establish an *interpolation inequality* connecting the Hardy inequality and McKean's spectral gap estimate on Cartan–Hadamard manifolds, in the spirit of Berchio, Ganguly, Grillo, and Pinchover [12] (see Theorem 3.27). A simple modification of the latter argument also provides a short alternative proof of the inequality by Akutagawa and Kumura [3] (see Theorem 3.30).

- In Section 3.2.5 we consider two parameter-dependent *Ghoussoub–Moradifam-type weighted inequalities* in the Euclidean case (cf. [50]), where the weights are of non-singular type (see Theorem 3.31). For a certain parameter range, we also extend these inequalities to Cartan–Hadamard manifolds (see Theorem 3.33).

In Section 3.3 we provide alternative proofs for various multiplicative Hardy-type inequalities, which are simple consequences of Theorem 3.1/(ii). We proceed as follows:

- In Section 3.3.1 we establish a sharp parameter-dependent *uncertainty principle* on Cartan–Hadamard manifolds, which implies the Heisenberg–Pauli–Weyl and the Hydrogen uncertainty principles (see Theorem 3.35). We also establish a rigidity result: If the quantitative uncertainty principle holds on an n-dimensional Cartan–Hadamard manifolds with Ricci curvature bounded from below, then the manifold is isometric to the corresponding model space form (see Theorem 3.37).

- In Section 3.3.2 we present two sharp *Caffarelli–Kohn–Nirenberg inequalities* on Cartan–Hadamard manifolds (see Theorems 3.40 & 3.41).

In Chapter 4 we focus on Rellich inequalities. In Section 4.1 we present two general functional inequalities on Riemannian manifolds, built upon convexity arguments and divergence/comparison theorems. The first inequality involves a second-order ODI, and provides inequalities of type ($\mathbf{R}_1$), for general $p > 1$ (see Theorem 4.1). The second inequality involves a second-order *partial* differential inequality (PDI) and produces inequalities of type ($\mathbf{R}_2$), for $p = 2$ (see Theorem 4.2). We notice that for special choices of parameters, the latter PDI reduces to an ODI, becoming easier to deal with (see Remark 4.3/(a)).

In the rest of the chapter, several applications are presented on Cartan–Hadamard manifolds. Here we highlight that our proofs are *symmetrization-free*, hence they do not require the validity of the *generalized Cartan–Hadamard conjecture*, i.e., the validity of the *sharp isoperimetric inequality* in this geometrical setting. We note that the conjecture is valid for general Cartan–Hadamard manifolds in dimension $n \in \{2, 3\}$ (see Bol [15] and Kleiner [62]), and for space forms in any dimension (see Dinghas [35]).

In Section 4.2 we establish fourth-order variants of the celebrated spectral gap result by McKean [79]. The latter result states that a strong negative curvature (when it is less than or equal to a negative number) produces a universal, domain-independent spectral gap for the first/principal eigenvalue of the fixed membrane problem, which is in radical contrast to the Euclidean case.

- In Section 4.2.1 we prove a spectral gap estimate and its *sharpness* for the clamped plate problem on Cartan–Hadamard manifolds with strong negative curvature, for $p > 1$ (see Theorem 4.4) We notice that the same estimate was also established by Kristály [68] (on Cartan–Hadamard manifolds satisfying the generalized Cartan–Hadamard conjecture) and Ngô and Nguyen [87] (on space forms) using symmetrization. In this case, our result completes the picture.

- In Section 4.2.2 we provide a *sharp* spectral gap estimate for the buckling problem on Cartan–Hadamard manifolds with strong negative curvature, for $p = 2$ (see Theorem 4.5). The same estimate is known on space forms due to Ngô and Nguyen [87]. In this case, our result adds a new piece of puzzle to the picture; however, the pieces corresponding to the general case when $p > 1$ are still missing.

- In Section 4.2.3 we establish higher-order *sharp* spectral gap estimates by simply combining our previous inequalities. The results are valid on Cartan–Hadamard manifolds with strong negative curvature. The clamped plate-type results hold for $p > 1$, while the buckling-type results hold for $p = 2$ (see Theorems 4.6 & 4.7).

In Section 4.3 we consider additional Rellich-type inequalities on Cartan–Hadamard manifolds, namely:

- In Section 4.3.1 we extend the classical and weighted Rellich inequalities to Cartan–Hadamard manifolds (see Theorem 4.8 and Corollary 4.9). These results are well-known in the Euclidean settings (see e.g., Mitidieri [83]).

- In Section 4.3.2 we provide higher-order variants of the classical Rellich inequality on Cartan–Hadamard manifolds (see Theorem 4.10).

- In Section 4.3.3 we provide further Rellich-type inequalities on general Cartan–Hadamard manifolds: Theorems 4.11 & 4.12 are improvements of type ($\mathbf{R}_1$), the second result is valid for strong negative curvature. In Theorem 4.14 we extend the classical Rellich inequality of type ($\mathbf{R}_2$) to Cartan–Hadamard manifolds, but only in dimensions $n \geq 8$. This constraint arises from technical conditions; for a similar phenomenon, see Kristály and Repovš [73].

In Chapter 5 we present a second alternative approach for establishing Rellich inequalities on space forms, using Riccati pairs. Parallel to presenting the structure of this chapter, which contains the final versions of this approach and its applications, let us briefly describe the development process and the intermediate observations. This additional information is intended to motivate the choice of presentation of our results.

Our initial goal was to develop a general functional inequality built upon Riccati pairs, which provides Rellich inequalities of type ($\mathbf{R}_2$). Having such an inequality and combining it with the results from Chapter 3 providing inequalities of type ($\mathbf{H}$), would automatically yield inequalities of type ($\mathbf{R}_1$). Clearly, we intended to formulate our inequality on the most general manifolds possible. Keeping in mind the requirement of the Laplace comparison in our argument, manifolds with sectional curvature bounded from above seemed to be good candidates in this case as well.

During the study, it turned out that the suitable convexity argument requires both a Laplace and a Hessian comparison, having 'opposite' direction to each other. Thus, one either imposes a lower bound on the sectional curvature as well, or simply considers space forms. We decided on the second option because several applications are also formulated in this setting. Additionally, in this particular case, the results can be presented in a formally simpler and more accessible manner.

In Section 5.1 we present our approach for Rellich inequalities on space forms. In Section 5.1.1 we start with two definitions. First, motivated by the simplicity of the ambient geometrical setting, in Definition 5.1 we introduce *simplified Riccati pairs* including the same ODI, but more straightforward conditions on parameter functions. Next, in Definition 5.2 we introduce *dual Riccati pairs* including an ODI, which is the true driving force behind the Rellich inequalities. We note that the latter ODI differs from the former by a change of function, the introduction of the second concept is a personal decision that hopefully enhances the presentation of the results.

Using these simplified concepts, we established a general functional inequality that furnishes inequalities of type ($\mathbf{R}_2$) (see Theorem 5.3/(ii)). Unfortunately, it turned out that the technical conditions resulting from the convexity argument typically imply a dimension constraint, similarly to Theorem 4.14. At this point, the idea of radial derivatives seemed useful. First, in Theorem 5.3/(i) we developed a general functional inequality, which provides radial versions of inequality ($\mathbf{R}_2$), where the gradient is replaced with the radial derivative. Here, the technical conditions are less restrictive, at least they do not require unnecessary dimensional constraints. Next, in Theorem 5.5 we established a general functional inequality providing radial versions of inequality ($\mathbf{H}$). We note that due to the Cauchy–Schwarz inequality (2.10), this inequality is stronger than its non-radial variant, i.e., the reformulation of Theorem 3.5 in these geometrical settings. We also notice that this inequality does not require any additional conditions.

In the rest of the chapter, we present applications of type ($\mathbf{R}_1$) exploiting the idea of radial derivatives. We note that each of the obtained intermediate radial inequalities admits a non-radial counterpart, whose technical condition is typically more restrictive; for the simplicity of presentation, these inequalities are omitted.

In Section 5.2 we provide applications to our method in Euclidean spaces; here we intend to present both the strengths and limitations of our approach. To do this, we test our result against two recent inequalities proved by Adimurthi, Grossi, and Santra [2]. First, we combine Theorem 5.3/(i) and Theorem 5.5 to provide a general functional inequality (see Theorem 5.10) formulated in terms of Bessel potentials (a notion related to Bessel pairs cf. Proposition 2.12), which allows us to have a simpler presentation. Next, we show that the first inequality meets the conditions of the latter result (see Corollary 5.11), while the second does not (see Remark 5.12).

In Section 5.3 we apply our approach to hyperbolic spaces. In Theorem 5.13 we provide a radial inequality as a consequence of Theorem 5.3/(i), interpolating between Rellich-type and spectral gap estimate-type inequalities. This result can be seen as a higher-order radial variant of the interpolation inequality from Theorem 3.27. For the extremal values of the parameter, we obtain formally well-known inequalities (see Corollaries 5.14 & 5.15). In Theorem 5.16 lower-order counterparts of Theorem 5.13 are presented, which are simple consequences of Theorem 5.5. Finally, in Theorem 5.18 we combine the previous two results and provide a sophisticated inequality of type ($\mathbf{R}_1$) on hyperbolic spaces.

In Chapter 6, we revisit Riccati pairs and Hardy inequalities in the Finsler setting. Here, the distance measurement is direction-dependent; in particular, the distance from a point in one direction can differ from the distance measured in the opposite direction. The *reversibility* of the manifold is precisely the supremum of the ratio of the aforementioned quantities; see relation (2.15). We focus on the following two questions: On the one hand, is it possible to the extend the previously obtained results to the Finsler context, in particular, how does the reversibility of the Finlser metric comes into the play? On the other hand, what can we say about the sharpness of the obtained inequalities? Note that all the inequalities achieved through Riccati pairs in the Riemannian setting turn out to be sharp (see Chapter 3).

In Section 6.1 we answer the first question: In form of a general functional inequality we provide a Finslerian extension of Riccati pairs (see Theorem 6.1). The proof of this extension also motivates the typical presence of the reversibility constant in Hardy inequalities formulated on Finsler manifolds. In Section 6.1.1 we present a number of applications. In their spirit, our extension naturally yield Finslerian counterparts of the Riemannian inequalities previously obtained by Riccati pairs. A similar functional inequality can be derived to cover uncertainty principles as well.

In the rest of the chapter, we address the second question: the issue of sharpness. On the one hand, it is well known that if the manifold is reversible (i.e., the reversibility equals one) and some suitable curvature bounds hold, then the best constants of Hardy inequalities are the same as the best constants on Riemannian manifolds satisfying a corresponding curvature bound. On the other hand, if the reversibility is infinite, the best constants may vanish, we refer to Huang, Kristály, and Zhao [56] and Kristály, Li, and Zhao [75] for details.

Concerning the remaining case, we provide a partial answer: we present some special families of Finsler spaces on which the obtained functional inequalities (involving the Riemannian optimal constants and the reversibility constants) turn out to be sharp. We note, however, that these result are one step away from the complete answer, which is to provide the best constants for arbitrary manifolds satisfying some suitable curvature bounds.

In Section 6.2 we consider the case of non-positive curvature. In Section 6.2.1 we construct a family of Finsler spaces on which our sharpness results are established (see Theorem 6.5). The main idea of the construction stems from the proof of Theorem 6.1. We observe that the key inequalities of the proof can be controlled by geometrical restrictions of ambient space: The ratio appearing in the reversibility (2.15) should be constant along the geodesics starting from a point. This observation led us to a candidate metric, which is a projectively flat perturbation of the Euclidean metric; see (6.13). Unfortunately, this candidate turns out to be discontinuous at the origin, the aforementioned family is in fact a smooth approximation of this candidate. In Section 6.2.2, we provide some sufficient conditions ensuring the sharpness of Hardy inequalities obtained by Riccati pairs on the constructed spaces (see Theorem 6.7). Finally, in Section 6.2.3 we present an application: In Theorem 6.8 we prove the sharpness of the Finslerian weighted Hardy inequality. As a consequence we recover the classical version (see Corollary 6.9).

In Section 6.3 we proceed similarly concerning the case of strong negative curvature. In Section 6.3.1 we present a second family of spaces (see Theorem 6.10), which is again a smooth approximation of a candidate metric. In this case, the candidate is obtained by a perturbation of the *Klein metric*, see (6.31). The underlying geometrical considerations are similar to the previous case. In Section 6.3.2 we provide sufficient conditions for sharpness of Hardy-type inequalities on the aforementioned family (see Theorem 6.12). Finally, in Section 6.3.3 we present an application: In Theorem 6.13 we prove a sharpness result concerning a McKean-type spectral gap estimate on the Finsler setting.

Chapter 2

Preliminaries and notations

In this chapter, we recall those definitions and preliminary results that are used in the subsequent chapters for presenting our approaches. Simultaneously, we also specify our notations. First, in Section 2.1 we recall the relevant concepts and notations concerning general Riemannian manifolds. In Section 2.1.1 we focus on Riemannian manifolds having constant sectional curvature, also called model space forms. Here, we provide exact formulas for several operators; in addition, relevant comparison results are recalled. Next, in Section 2.2 we recall various elements from Finsler geometry. Within this, in Section 2.2.1 we present various concepts of curvature and a relevant comparison theorem, while in Section 2.2.2 we discuss some special classes of Finsler manifolds. Next, in Section 2.3 we list those eigenvalue problems that are related to Hardy–Rellich inequalities, namely the fixed membrane problem, the clamped plate problem, and the buckling problem. Finally, in Section 2.4 we present the recent notion of Bessel pairs and the related approach to Hardy–Rellich inequalities, developed by Ghoussoub and Moradifam [50]. This approach is compared later to ours.

2.1 Riemannian manifolds

In this section we recall various elements of Riemannian geometry. We mainly follow Gallot, Hulin, and Lafontaine [48] and Hebey [54].

Let (M, g) be a complete n-dimensional Riemannian manifold, with $n \geq 2$. At each point $x \in M$, the *tangent space* T_xM consists of all tangent vectors at x. The disjoint union of tangent spaces is the *tangent bundle* TM, which consists of all tangent vector fields. In both cases, the canonical norm and the scalar product induced by the metric g are denoted by $|\cdot|$ and $\langle \cdot, \cdot \rangle$, respectively.

Let (x^i) be a local coordinate system in a coordinate neighborhood of $x \in M$. Denote the local components of the metric g and g^{-1} by (g^{ij}) and (g_{ij}), respectively. The *gradient* of a smooth function u on M is $\nabla_g u \in TM$, its local components are

$$u^i = g^{ij} \frac{\partial u}{\partial x^j}.$$

Hereafter, by convention, repeated upper and lower indexes are automatically summed.

The *divergence* of a smooth vector field V with components (V^i) is a scalar function and is given by

$$\operatorname{div}_g V = \frac{1}{\sqrt{\det g}} \cdot \frac{\partial}{\partial x^i} \left(\sqrt{\det g} \cdot V^i \right).$$

If $p > 1$ and u is a smooth function, the *p-Laplace–Beltrami operator* is given by

$$\Delta_{g,p} u = \operatorname{div}_g \left(|\nabla_g u|^{p-2} \nabla_g u \right).$$

Note that $\Delta_{g,2}$ is in fact Δ_g, the usual *Laplace–Beltrami operator* on (M, g).

Denote by $d_g \colon M \times M \to [0, \infty)$ the *distance function* induced by the metric g on the manifold M. For fixed $x_0 \in M$ define $d_{g,x_0} \colon M \to [0, \infty)$ by

$$d_{g,x_0}(x) = d_g(x_0, x), \qquad \forall x \in M,$$

the *distance from* x_0. The *eikonal equation* states that $\mathrm{d}v_g$-a.e. on M one has

$$|\nabla_g d_{g,x_0}| = 1. \tag{2.1}$$

For every $p > 1$, the above relations yield $\mathrm{d}v_g$-a.e. on M that

$$\Delta_{g,p} d_{g,x_0} = \operatorname{div}_g \left(\nabla_g d_{g,x_0} \right) = \Delta_g d_{g,x_0}. \tag{2.2}$$

For a fixed set $S \subseteq M$, define $d_{g,S} \colon M \to [0, \infty)$, the *distance from* S, by

$$d_{g,S}(x) = \inf_{y \in S} d_g(x, y) = d_g(x, S), \qquad \forall x \in M.$$

The *open metric ball* with center $x_0 \in M$ and radius $R > 0$ is defined by

$$B_{g,x_0}(R) = \{ y \in M : d_{g,x_0}(y) < R \}.$$

A generic ball with radius R (and arbitrary center) is simply denoted by $B_g(R)$.

The *volume* of a bounded open set $S \subseteq M$ is given by

$$\operatorname{Vol}_g(S) = \int_S \mathrm{d}v_g = \mathcal{H}_g^n(S),$$

where $\mathrm{d}v_g$ is the *canonical volume form* and $\mathcal{H}_g^n$ stands for *Hausdorff measure.*

If u_1 and u_2 are smooth functions, both vanishing at the boundary, the *integration by parts* formula reads as:

$$\int_M u_1 \Delta_{g,p} u_2 \,\mathrm{d}v_g = -\int_M |\nabla_g u_2|_g^{p-2} \langle \nabla_g u_2, \nabla_g u_1 \rangle \,\mathrm{d}v_g. \tag{2.3}$$

Applying twice the latter relation for $p = 2$ yields *Green's second identity*:

$$\int_M u_1 \Delta_g u_2 \,\mathrm{d}v_g = \int_M \Delta_g u_1 u_2 \,\mathrm{d}v_g. \tag{2.4}$$

2.1.1 Model space forms

The model space form $(\mathbf{M}^n_\kappa, g_\kappa)$ is an n-dimensional complete Riemannian manifold with constant sectional curvature $\mathbf{K} \equiv \kappa$, where $\kappa \in \mathbb{R}$. In particular,

$$\mathbf{M}^n_\kappa = \begin{cases} \mathbb{S}^n_\kappa \text{ – the sphere,} & \text{if } \kappa > 0, \\ \mathbb{R}^n \text{ – the Euclidean space,} & \text{if } \kappa = 0, \\ \mathbb{H}^n_\kappa \text{ – the hyperbolic space,} & \text{if } \kappa < 0. \end{cases}$$

For a more precise definition, let us construct the auxiliary function

$$p_\kappa(x) = \begin{cases} \frac{2}{\sqrt{\kappa}(1+|x|^2)}, & \text{if } \kappa > 0 \text{ and } x \in \mathbb{R}^n, \\ 1, & \text{if } \kappa = 0 \text{ and } x \in \mathbb{R}^n, \\ \frac{2}{\sqrt{-\kappa}(1-|x|^2)}, & \text{if } \kappa < 0 \text{ and } x \in \mathbb{R}^n \text{ with } |x| < 1, \end{cases}$$

where $|\cdot|$ stands for the Euclidean norm. Thus, the model space form can be modelled on the Euclidean space (if $\kappa \geq 0$) or on the unit ball of the Euclidean space (if $\kappa < 0$), endowed with the metric

$$(g_\kappa)_{ij}(x) = p^2_\kappa(x)\delta_{ij},$$

where δ_{ij} is the *Kronecker symbol*. Note that in the case when $\kappa < 0$ we use the standard Poincaré ball model, while in the case when $\kappa > 0$ we use the *stereographic projection* of the standard sphere model.

For better readability, hereafter, we simply use the subscript κ instead of g_κ. In case of Euclidean quantities, the subscript κ is also omitted. By this convention, the canonical volume form, the Riemannian gradient, and the Laplace–Beltrami operator are written as

$$\mathrm{d}x_\kappa = p^n_\kappa(x)\,\mathrm{d}x, \quad \nabla_\kappa u = \frac{\nabla u}{p^2_\kappa}, \quad \text{and} \quad \Delta_\kappa u = p^{-n}_\kappa \operatorname{div}(p^{n-2}_\kappa \nabla u),$$

respectively, where u is a smooth function. Note that div and ∇ denote the Euclidean divergence and gradient, respectively.

In the sequel, we define a number of functions that are frequently used in the subsequent chapters. First, for every $\kappa \in \mathbb{R}$, define

$$\mathbf{s}_\kappa(t) = \begin{cases} \frac{\sin(\sqrt{\kappa}t)}{\sqrt{\kappa}}, & \text{if } \kappa > 0, \\ t, & \text{if } \kappa = 0, \\ \frac{\sinh(\sqrt{-\kappa}t)}{\sqrt{-\kappa}}, & \text{if } \kappa < 0, \end{cases} \quad \text{and} \quad \mathbf{c}_\kappa(t) = \begin{cases} \cos(\sqrt{\kappa}t), & \text{if } \kappa > 0, \\ 1, & \text{if } \kappa = 0, \\ \cosh(\sqrt{-\kappa}t), & \text{if } \kappa < 0, \end{cases}$$

for all $t \in (0, \pi/\sqrt{\kappa})$, using the *convention*

$$\pi/\sqrt{\kappa} \stackrel{\text{def}}{=} \infty, \quad \text{when} \quad \kappa \leq 0. \tag{2.5}$$

Next, using the same convention, let us construct

$$\mathbf{ct}_\kappa(t) = \frac{\mathbf{c}_\kappa(t)}{\mathbf{s}_\kappa(t)} = \begin{cases} \sqrt{\kappa}\cot(\sqrt{\kappa}t), & \text{if } \kappa > 0, \\ \frac{1}{t}, & \text{if } \kappa = 0, \\ \sqrt{-\kappa}\coth(\sqrt{-\kappa}t), & \text{if } \kappa < 0, \end{cases} \quad \forall t \in \left(0, \pi/\sqrt{\kappa}\right). \tag{2.6}$$

Next, let

$$L_\kappa(t) = (n-1)\mathbf{ct}_\kappa(t), \qquad \forall t \in \left(0, \pi/\sqrt{\kappa}\right). \tag{2.7}$$

Finally, for every $\kappa \le 0$, define $\mathbf{D}_\kappa \colon [0,\infty) \to [0,\infty)$ by

$$\mathbf{D}_\kappa(t) = \begin{cases} 0, & \text{if } t = 0, \\ t\mathbf{ct}_\kappa(t) - 1, & \text{if } t > 0. \end{cases} \tag{2.8}$$

The volume of an arbitrary ball $B_\kappa(R)$ in $\mathbf{M}^n_\kappa$ with radius $R > 0$ is given by

$$\mathrm{VB}_\kappa(R) = n\omega_n \int_0^R \mathbf{s}_\kappa^{n-1}(t)\,\mathrm{d}t, \tag{2.9}$$

where $\omega_n = \pi^{\frac{n}{2}}/\Gamma(\frac{n}{2}+1)$ is the volume of the unit ball in $\mathbb{R}^n$.

The *radial derivative* of a smooth function u is defined by

$$\nabla_\kappa^{\mathrm{rad}} u = \langle \nabla_\kappa u, \nabla_\kappa d_{\kappa,x_0} \rangle,$$

and it satisfies the following Cauchy–Schwarz inequality:

$$|\nabla_\kappa^{\mathrm{rad}} u|^2 = \langle \nabla_\kappa u, \nabla_\kappa d_{\kappa,x_0} \rangle^2 \le |\nabla_\kappa u|^2 \cdot |\nabla_\kappa d_{\kappa,x_0}|^2 = |\nabla_\kappa u|^2. \tag{2.10}$$

In case of the model space form $\mathbf{M}^n_\kappa$ the eikonal equation

$$|\nabla_\kappa d_{\kappa,x_0}| = 1 \tag{2.11}$$

holds everywhere except the point x_0 and its *cut locus* (which is empty for $\kappa \le 0$, and is the antipodal point of x_0 on the sphere $\mathbb{S}^n_\kappa$ for $\kappa > 0$). Excepting those points, the Laplacian of the distance function can be computed as follows:

$$\Delta_\kappa d_{\kappa,x_0} = (n-1)\mathbf{ct}_\kappa(d_{\kappa,x_0}) = L_\kappa(d_{\kappa,x_0}). \tag{2.12}$$

For any vector field $V \in T\mathbf{M}^n_\kappa$ the Hessian of the distance function verifies the following relation:

$$\mathrm{Hess}_\kappa(d_{\kappa,x_0})(V,V) = \mathbf{ct}_\kappa(d_{\kappa,x_0})(|V|^2 - \langle V, \nabla_\kappa d_{\kappa,x_0} \rangle^2). \tag{2.13}$$

Denote by inj_{x_0} the *injectivity radius* of $x_0 \in M$. We have the following Laplace comparison and Bishop–Gromov volume comparison principles (see e.g., Gallot, Hulin, and Lafontaine [48, Theorem 3.101]).

Theorem 2.1. *Let (M,g) be an n-dimensional complete Riemannian manifold with $n \ge 2$, and let $x_0 \in M$ be a fixed point. The following statements hold with the usual convention (2.5).*

(I) *If the sectional curvature satisfies $\mathbf{K} \le \kappa$ for some $\kappa \in \mathbb{R}$, then:*

  (i) (Laplace comparison) $\Delta_g d_{g,x_0} \ge (n-1)\mathbf{ct}_\kappa(d_{g,x_0})$ *with* $d_{g,x_0} < \pi/\sqrt{\kappa}$*;*

  (ii) (Volume comparison) $\mathrm{Vol}_g(B_{g,x_0}(R)) \ge \mathrm{VB}_\kappa(R)$, $\forall R \in (0, \min(\mathrm{inj}_{x_0}, \pi/\sqrt{\kappa}))$.

(II) *If the Ricci curvature satisfies $\mathbf{Ric} \ge \kappa(n-1)g$ for some $\kappa \in \mathbb{R}$, then:*

  (i) (Laplace comparison) $\Delta_g d_{g,x_0} \le (n-1)\mathbf{ct}_\kappa(d_{g,x_0})$ *with* $d_{g,x_0} < \pi/\sqrt{\kappa}$*;*

  (ii) (Volume comparison) $\mathrm{Vol}_g(B_{g,x_0}(R)) \le \mathrm{VB}_\kappa(R)$, $\forall R > 0$.

If equality holds in any of the above statements, then (M,g) is isometric to the model space form $\mathbf{M}^n_\kappa$.

2.2 Finsler manifolds

In this section we recall various elements of Riemannian geometry. For further details, we refer to the monographs by Bao, Chern, and Shen [8] and Shen [101].

Let M be a smooth manifold with dimension $n \geq 2$. At each point $x \in M$, the *tangent space* T_xM consists of tangent vectors (or simply vectors) at x, while the *cotangent space* T_x^*M consists of cotangent vectors (or covectors) at x, sometimes also referred to as one forms. These spaces are *dual* to each other, the *canonical pairing* between them is a bilinear form defined by

$$\langle \xi, y \rangle = \xi(y), \qquad \forall x \in M,\ y \in T_xM,\ \xi \in T_x^*M.$$

The *tangent bundle* TM is the collection of all tangent spaces, consisting of *bound vectors*, the *cotangent bundle* T^*M is the collection of all cotangent spaces, consisting of *bound covectors*. More precisely, we have

$$TM = \{(x, y) : x \in M, y \in T_xM\} \quad \text{and} \quad T^*M = \{(x, \xi) : x \in M, \xi \in T_x^*M\}.$$

A *vector field* on M is a map that assigns to each point $x \in M$ a bound vector $(x, y) \in TM$, while a *covector field* assigns a bound covector $(x, \xi) \in T^*M$.

Remark 2.2. We avoid the introduction of additional notation or letters for bound vectors and vector fields. If a statement involves a bound vector $(x, y) \in TM$ or a vector field $x \mapsto (x, y)$, but the positional coordinate x is not particularly important, we simply omit x and formulate the statement using the bound vector $y \in TM$ or the vector field y, respectively. Similarly, ξ may denote a covector, a bound covector, or a covector field. The specific underlying objects are always revealed by the context.

The couple (M, F) is said to be a *Finsler manifold* if the function $F\colon TM \to [0, \infty)$ is a *Finsler metric* satisfying the following conditions:

(i) *Regularity*: $F(y)$ is smooth on $TM \setminus \{0\}$, which consists of non-zero bound vectors;

(ii) *Positive homogeneity*: $F(\lambda y) = \lambda F(y)$, for every $y \in TM$ and $\lambda \geq 0$;

(iii) *Strong convexity*: For every $y \in TM \setminus \{0\}$, the following quadratic form is positive definite:

$$g_y(v, w) \stackrel{\text{def}}{=} \frac{\partial^2}{\partial s \partial t} \left(\frac{1}{2} F(y + sv + tw)^2 \right) \bigg|_{s=t=0}, \qquad \forall v, w \in TM. \tag{2.14}$$

According to Remark 2.2, we use both $F(y)$ and $F(x, y)$ depending on the context.

The *reversibility* of a Finsler manifold (M, F), introduced by Rademacher [94], is defined by

$$\lambda_F(M) \stackrel{\text{def}}{=} \sup_{x \in M} \sup_{y \in T_xM} \mathbf{rev}(x, y), \quad \text{where} \quad \mathbf{rev}(x, y) = \frac{F(x, -y)}{F(x, y)}. \tag{2.15}$$

Clearly $1 \leq \lambda_F(M) \leq +\infty$. The manifold is said to be *reversible*, if $\lambda_F(M) = 1$.

The *dual Finsler metric* of F is the function $F^*\colon T^*M \to [0,\infty)$, defined by

$$F^*(\xi) \stackrel{\text{def}}{=} \sup_{y\in TM\setminus\{0\}} \frac{\langle \xi, y\rangle}{F(y)}, \qquad \forall \xi \in T^*M, \tag{2.16}$$

which is also a Finsler metric on M. The reversibility of (M, F^*) can be defined by

$$\lambda_{F^*}(M) \stackrel{\text{def}}{=} \sup_{x\in M} \sup_{\xi\in T_x^*M} \mathbf{rev}^*(x,y), \quad \text{where} \quad \mathbf{rev}^*(x,y) = \frac{F^*(x,-\xi)}{F^*(x,\xi)}.$$

In particular, according to e.g., Huang, Kristály, and Zhao [56, Lemma 2.1], one has

$$\lambda_F(M) = \lambda_{F^*}(M). \tag{2.17}$$

Using a Finsler metric F, one can define the *length* of a Lipschitz continuous curve $c\colon [a,b] \to M$ by

$$L(c) \stackrel{\text{def}}{=} \int_a^b F(\dot{c}(t))\,\mathrm{d}t,$$

where the dot denotes the derivative with respect to t. Given two points $x_0, x_1 \in M$, the *distance* from x_0 to x_1 is defined by

$$d(x_0, x_1) = \inf_c L(c),$$

where the infimum is taken over all Lipschitz continuous curves $c\colon [0,1] \to M$ with $c(0) = x_0$ and $c(1) = x_1$. In general, $d(x_0,x_1) \neq d(x_1,x_0)$, unless F is reversible.

If $\rho = d(x_0,\cdot)$ is the distance from a point $x_0 \in M$, then the *eikonal equations* hold dm-a.e. in M:

$$F(\nabla\rho) = F^*(D\rho) = 1. \tag{2.18}$$

For fixed $x \in M$, the *Legendre transform* $\ell^*\colon T_x^*M \to T_xM$ assigns to each covector ξ the unique vector y, which minimizes the map

$$E^*(y) \stackrel{\text{def}}{=} \langle \xi, y\rangle - \frac{1}{2}F(y)^2.$$

The minimizer $y = \ell^*(\xi)$ can also be interpreted as the unique bound vector with

$$F(\ell^*(\xi)) = F^*(\xi) \quad \text{and} \quad \langle \xi, \ell^*(\xi)\rangle = F(\ell^*(\xi))F^*(\xi). \tag{2.19}$$

Combining the properties from relation (2.19) yields

$$\langle \xi, \ell^*(\xi)\rangle = F^*(\xi)^2, \qquad \forall \xi \in T_x^*M. \tag{2.20}$$

Similarly, the Legendre transform $\ell\colon T_xM \to T_x^*M$ assigns to each vector y the unique covector ξ that minimizes the map

$$E(\xi) \stackrel{\text{def}}{=} \langle \xi, y\rangle - \frac{1}{2}F^*(\xi)^2.$$

For the minimizer $\xi = \ell(y)$ similar relations hold as in (2.19) & (2.20). One also has $\ell^{-1} = \ell^*$, hence $y = \ell^*(\xi)$. Using the fact that $\frac{\partial E}{\partial \xi} = 0$, we obtain

$$\ell^*(\xi) = F^*(\xi)\frac{\partial F^*(\xi)}{\partial \xi}, \qquad \forall \xi \in T_x^*M, \tag{2.21}$$

thus, ℓ^* is positive homogeneous. Note that the definitions and properties of the Legendre transforms ℓ and ℓ^* naturally extend to bound vectors and vector fields. Moreover, using the conventions of Remark 2.2, all of the above formulas are formally valid. We only need to make sure that the positional coordinates of y and ξ coincide, otherwise the canonical pairing does not make sense.

For every $x \in M$, there exists a local coordinate system $\{x^i\}$ defined on a coordinate neighborhood of x. Denote by $\{\frac{\partial}{\partial x^i}\}$ and $\{\mathrm{d}x^i\}$ the induced basis on T_xM and T_x^*M, respectively.

A *Finsler metric measure manifold* (M, F, m) (abbreviated by FMMM) is a Finsler manifold equipped with a smooth positive measure m. Unlike the Riemannian setting, there is no canonical measure in Finsler geometry. However, one often considers the Busemann–Hausdorff measure induced by F, which is given by

$$\mathrm{d}\mathsf{m}_F \stackrel{\text{def}}{=} \sigma_F(x)\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n, \quad \text{where} \quad \sigma_F(x) = \frac{\mathrm{vol}(\mathbb{B}^n)}{\mathrm{vol}(B_x(1))}, \tag{2.22}$$

$\mathrm{vol}(\cdot)$ denotes the Euclidean volume, $\mathbb{B}^n \subset \mathbb{R}^n$ is the Euclidean unit ball, while

$$B_x(1) = \{y \in T_xM : F(y) < 1\}.$$

The *derivative* of a smooth function $u\colon M \to \mathbb{R}$ is a covector field defined by

$$Du \stackrel{\text{def}}{=} \frac{\partial u}{\partial x^i}\,\mathrm{d}x^i. \tag{2.23}$$

The *gradient* of a smooth function $u\colon M \to \mathbb{R}$ is a vector field that can be computed as

$$\nabla u \stackrel{\text{def}}{=} \ell^*(Du). \tag{2.24}$$

The *divergence* of a smooth vector field v with components (v^i) is given by

$$\mathrm{div}(v) \stackrel{\text{def}}{=} \sum_i \frac{1}{\sigma}\frac{\partial \sigma v^i}{\partial x^i}.$$

If $p > 1$, then the *p-Laplacian* of a smooth function u is

$$\Delta_{F,p}(u) \stackrel{\text{def}}{=} \mathrm{div}(F(\nabla u)^{p-2}\nabla u).$$

We note that for $p = 2$, the p-Laplacian is precisely the Finsler Laplacian

$$\Delta_F(u) = \mathrm{div}(\nabla u).$$

We also notice that, if ρ is a distance function, then $\Delta_{F,p}(\rho) = \Delta_F(\rho)$ holds $\mathrm{d}\mathsf{m}$-a.e. in M.

Finally, if u_1 and u_2 are smooth functions, both vanishing at the boundary, then the following *integration by parts formula* holds:

$$\int_M u_1\Delta_F(u_2)\,\mathrm{d}\mathsf{m} = -\int_M \langle Du_1, \nabla u_2\rangle\,\mathrm{d}\mathsf{m}, \tag{2.25}$$

see e.g., Ohta and Sturm [90].

2.2.1 Curvatures and comparison

In the sequel, we recall the notions of the flag and S-curvature of a FMMM (M, F, m), which can be interpreted more easily using tensors. According to the notation of the previous section, the fundamental tensor from relation (2.14) can be rewritten as

$$g_{(x,y)} = g_{ij}(x,y)\,\mathrm{d}x^i\,\mathrm{d}x^j, \quad \text{where} \quad g_{ij}(x,y) \stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial^2 F(x,y)^2}{\partial y^i \partial y^j}, \qquad \forall (x,y) \in TM.$$

Note that, by convention, repeated indexes are automatically summed. The *geodesic spray coefficients* are defined as follows:

$$G^i(x,y) \stackrel{\text{def}}{=} \frac{1}{4} g^{il}(x,y)\left(2\frac{\partial g_{jl}(x,y)}{\partial x^k} - \frac{\partial g_{jk}(x,y)}{\partial x^l}\right) y^j y^k, \qquad \forall (x,y) \in TM.$$

Recall that a smooth curve $t \mapsto \gamma(t)$ is *geodesic* if it satisfies the equation

$$\ddot{\gamma}^i(t) + 2G^i(\gamma(t), \dot{\gamma}(t)) = 0.$$

The *Riemannian curvature* is defined by

$$R_{(x,y)} \stackrel{\text{def}}{=} R^i_k(x,y)\frac{\partial}{\partial x^i} \otimes dx^k, \qquad \forall (x,y) \in TM,$$

where

$$R^i_k(x,y) \stackrel{\text{def}}{=} 2\frac{\partial G^i(x,y)}{\partial x^k} - y^i \frac{\partial^2 G^i(x,y)}{\partial x^j \partial y^k} + 2G^j(x,y)\frac{\partial^2 G^i(x,y)}{\partial y^j \partial y^k} - \frac{\partial G^i(x,y)}{\partial y^j}\frac{\partial G^j(x,y)}{\partial y^k}.$$

The definition of *flag curvature* is as follows:

$$\mathbf{K}(x,y,v) = \frac{g_{(x,y)}(R_{(x,y)}(v), v)}{g_{(x,y)}(y,y) g_{(x,y)}(v,v) - g_{(x,y)}(y,v)^2}, \qquad \forall x \in M,\ y, v \in T_x M. \tag{2.26}$$

The name “flag” refers to the plane spanned by the vectors y and v; the vector y is sometimes referred to as the *flagpole*.

The *distortion* is defined by

$$\tau(x,y) \stackrel{\text{def}}{=} \log \frac{\sqrt{\det g_{ij}(x,y)}}{\sigma(x)}, \qquad \forall (x,y) \in TM.$$

The S*-curvature* measures the rate of change of the distortion along geodesics, more precisely

$$\mathbf{S}(x,y) = \frac{\mathrm{d}}{\mathrm{d}t}\tau(\gamma(t), \dot{\gamma}(t))\bigg|_{t=0}, \qquad \forall (x,y) \in TM, \tag{2.27}$$

where γ denotes the unique geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = y$. For computational convenience (due to the homogeneity of degree zero), we prefer to use the *reduced* S*-curvature* defined by

$$\overline{\mathbf{S}}(x,y) = \frac{\mathbf{S}(x,y)}{F(x,y)}, \quad \forall (x,y) \in TM,\ y \neq 0.$$

We conclude this subsection, with the following Laplace comparison theorem, which is a slight modification of the result by Wu and Xin [105, Theorem 5.1]. Here we only assume curvature conditions alongside the geodesics starting from some $x_0 \in M$, which does not affect the original proof. For the Riemannian counterpart, see Theorem 2.1. See also relation (2.6) for the definition of $\mathbf{ct}_\kappa$.

Theorem 2.3. *Let (M, F) be a Finsler manifold and $\rho = d(x_0, \cdot)$ be the distance from a point $x_0 \in M$. Suppose that $\mathbf{K} \leq \kappa$ and $\overline{\mathbf{S}} \leq (n-1)h$ along $\nabla\rho$, for some $\kappa \leq 0$ and $h \in \mathbb{R}$. Then one has*

$$\Delta_F(\rho) \geq (n-1)\mathbf{ct}_\kappa(\rho) - (n-1)h.$$

Moreover, equality holds if and only if $\mathbf{K} = \kappa$ and $\overline{\mathbf{S}} = (n-1)h$, along $\nabla\rho$.

2.2.2 Special Finsler metrics

A Finsler metric on $\Omega \subseteq \mathbb{R}^n$ is said to be *projectively flat* if the geodesics are straight (Euclidean) lines as point sets. The following characterization is due to Rapcsák [95].

Theorem 2.4. *Let $F\colon T\Omega \to [0, \infty)$ be a Finlser metric on $\Omega \subseteq \mathbb{R}^n$. The following conditions are equivalent:*

(i) *The Finsler space (Ω, F) is projectively flat.*

(ii) *The geodesic coefficients of F are given by*

$$G^i(x, y) = P(x, y)y^i, \quad \textit{where} \quad P(x, y) = \frac{1}{2F(x, y)} \cdot \frac{\partial F(x, y)}{\partial x^k} y^k. \tag{2.28}$$

(iii) *The metric F satisfies*

$$\frac{\partial^2 F(x, y)}{\partial x^k \partial y^i} y^k = \frac{\partial F(x, y)}{\partial x^i}, \quad \forall i \in \{1, \ldots, n\}. \tag{2.29}$$

The following proposition is useful for computing the flag and S-curvature of a projectively flat metric. For the proof, we refer to Shen [101, Sections 6.3 and 7.3]

Proposition 2.5. *Suppose that (M, F, m) is a FMMM equipped with a projectively flat Finsler metric F and a smooth positive measure m having density σ. The flag and S-curvature can be computed as follows:*

$$\begin{aligned}
\mathbf{K}(x, y) &= \frac{1}{F(x, y)^2}\left(P(x, y)^2 - \frac{\partial P(x, y)}{\partial x^i} y^i\right), \\
\mathbf{S}(x, y) &= \frac{\partial G^i(x, y)}{\partial y^i} - \frac{y^i}{\sigma(x)} \frac{\partial \sigma(x)}{\partial x^i},
\end{aligned}$$

where P and G^i are given by relation (2.28). In this case $\mathbf{K}$ only depends on x and y.

In the sequel, we consider a special class of Finsler spaces, called *Randers spaces*, which are important examples of Finsler manifolds: In many cases, they help us to observe and investigate several conceptual differences between Riemannian and Finsler geometries. A Randers space is a differentiable manifold M endowed with a special Finsler metric $F\colon TM \to [0,\infty)$ called *Randers metric*, which is defined by

$$F(x,y) = \sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i, \tag{2.30}$$

where $a_{ij}(x)\,\mathrm{d}x^i\,\mathrm{d}x^j$ is a Riemannian metric, $b_i(x)\,\mathrm{d}x^i$ is a a *one-form*, such that

$$\|b(x)\|_a \stackrel{\text{def}}{=} \sqrt{a^{ij}(x)b_i(x)b_j(x)} < 1, \quad \forall x \in M,$$

and $\{a^{ij}(x)\}$ denotes the inverse of $\{a_{ij}(x)\}$. The density of the Busemann–Hausdorff measure induced by a Raders metric F is given by

$$\sigma_F(x) = (1 - \|b(x)\|_a^2)^{\frac{n+1}{2}} \sqrt{\det a_{ij}(x)}. \tag{2.31}$$

For later use, define the *Kronecker delta* by

$$\delta_{ij} \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{2.32}$$

Remark 2.6. Let F be a Randers metric given by (2.30). It the Riemannian metric $\overline{F}(x,y) = \sqrt{a_{ij}(x)y^i y^j}$ satisfies (2.29), and $b_i\,\mathrm{d}x^i = D\beta(x)$ for some smooth function β, then F satisfies (2.29) as well. Thus, according to Theorem 2.4 the metric F is projectively flat. The claim easily follows from the symmetry of second order partial derivatives of the smooth function β.

2.3 Eigenvalue problems

In his celebrated book entitled *The Theory of Sound*, Lord Rayleigh [96] formulated various questions concerning the qualitative behavior of the first eigenvalue for the fixed membrane, the clamped plate, and the buckling problems. Initially, these problems were posed in the Euclidean setting, but later, the mathematical community started to study them in curved spaces as well. In this section, we recall these eigenvalue problems and briefly discuss the relevant result regarding their first eigenvalue. For a clearer presentation, we formulate the problems on Riemannian manifolds, although they can naturally also be addressed in the Finsler setting.

2.3.1 Fixed membrane problem

Due to its second-order character, the *fixed membrane problem* turned out to be the most accessible among the aforementioned eigenvalue problems that can be formulated as follows.

Let (M, g) be an n-dimensional Riemannian manifold, with $n \geq 2$, let $\Omega \subseteq M$ is a bounded, open domain, and consider the problem

$$\begin{cases} \Delta_g u = -\lambda_{\mathrm{m}} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.33}$$

In the Euclidean setting, Faber [38] and Krahn [66] independently proved that the first eigenvalue of (2.33) satisfies

$$\Lambda_{\mathrm{m}}(\Omega) \stackrel{\text{def}}{=} \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla_g u|^2 \, \mathrm{d}v_g}{\displaystyle\int_\Omega u^2 \, \mathrm{d}v_g} \geq j^2_{n/2-1,1}(\omega_n/\mathrm{Vol}(\Omega))^{2/n}, \tag{2.34}$$

where $j_{\mu,1}$ stands for the first positive zero of the Bessel function J_μ of the first kind with order μ. In addition, equality is achieved in (2.34) whenever Ω is a ball; moreover, the eigenvalues converge to zero when the radius of the balls tend to infinity. The proof relies crucially on the Pólya–Szegő inequality, which is based on Schwarz symmetrization and the sharp isoperimetric inequality in $\mathbb{R}^n$. For that reason, it can be naturally extended to Cartan–Hadamard manifolds that satisfy the Cartan–Hadamard conjecture; formally being the same as its Euclidean counterpart. We note, however, that this conjecture is confirmed only in low dimensions when $n \in \{2, 3, 4\}$ (see Bol [15], Kleiner [62], and Croke [29], respectively). For a detailed discussion on this conjecture and its generalization, see Kloeckner and Kuperberg [63].

One of the most surprising facts in spectral theory on Riemannian manifolds is due to McKean [79], which roughly states that strong negative curvature produces a universal, domain-independent, sharp spectral gap for the first eigenvalue of (2.33). This fact is in radical contrast to the Euclidean case. The precise formulation of this result is as follows. Let (M, g) be a Cartan–Hadamard manifold with sectional curvature $\mathbf{K} \leq \kappa$ for some $\kappa < 0$, and $\Omega \subseteq M$ be an open bounded domain. Then the first eigenvalue of (2.33) satisfies

$$\Lambda_{\mathrm{m}}(\Omega) = \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla_g u|^2 \, \mathrm{d}v_g}{\displaystyle\int_\Omega u^2 \, \mathrm{d}v_g} \geq \left(\frac{(n-1)\sqrt{-\kappa}}{2} \right)^2. \tag{2.35}$$

Moreover, the bound in (2.35) is *sharp*. Indeed, if we consider the model space form $M = \mathbf{M}^n_\kappa$ of constant sectional curvature $\mathbf{K} \equiv \kappa$, and the ball $\Omega = B_\kappa(R) \subseteq \mathbf{M}^n_\kappa$ with radius R, then the first eigenvalue of (2.33) has the limiting property

$$\lim_{R\to\infty} \Lambda_{\mathrm{m}}(B_\kappa(R)) = \left(\frac{(n-1)\sqrt{-\kappa}}{2} \right)^2; \tag{2.36}$$

see e.g., Chavel [24]. For further asymptotically improved versions, see e.g., Borisov and Freitas [16], Cheng and Yang [26], Kristály [70] and Savo [100].

A natural extension of the problem (2.33) is the *p-fixed membrane problem*, which can be obtained by replacing the PDE with

$$\Delta_{g,p} u = -\lambda_{\mathrm{m},p} |u|^{p-2} u.$$

In this case, the sharp estimate for the first eigenvalue reads as

$$\Lambda_{\mathrm{m},p}(\Omega) \stackrel{\mathrm{def}}{=} \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla_g u|^p \, \mathrm{d}v_g}{\displaystyle\int_\Omega u^p \, \mathrm{d}v_g} \geq \left(\frac{(n-1)\sqrt{-\kappa}}{p} \right)^p, \tag{2.37}$$

see He and Yin [108]. We refer to Theorem 3.24 for a simple alternative proof. For a nonlocal version of the above problem, see Kajántó and Kristály [61]. For a Kirchhoff-type perturbation, we refer to Farkas, Kajántó, and Varga [40].

2.3.2 Clamped plate problem

The *clamped plate problem* is definitely more sophisticated than the fixed membrane problem due to its fourth-order character. It can be formulated as follows:

$$\begin{cases} \Delta_g^2 u = \lambda_{\mathrm{c}} u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \mathbf{n}} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.38}$$

where $\Omega \subseteq M$ is a bounded, open domain and (M,g) is an n-dimensional Riemannian manifold, with $n \geq 2$, while Δ_g^2 denotes the *biharmonic operator* defined by

$$\Delta_g^2 u = \Delta_g(\Delta_g u)$$

for every smooth function u, and $\frac{\partial}{\partial \mathbf{n}}$ stands for the outward pointing *normal derivative*.

In the Euclidean setting, Ashbaugh and Benguria [6] and Nadirashvili [86] proved in dimensions $n \in \{2,3\}$ that the first eigenvalue of (2.38) satisfies

$$\Lambda_{\mathrm{c}}(\Omega) \stackrel{\mathrm{def}}{=} \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\Delta_g u|^2 \, \mathrm{d}v_g}{\displaystyle\int_\Omega u^2 \, \mathrm{d}v_g} \geq \mathfrak{h}_{n/2-1}^4 (\omega_n / \mathrm{Vol}(\Omega))^{4/n}, \tag{2.39}$$

where $\mathfrak{h}_\mu$ is the first positive critical point of J_μ / I_μ, while J_μ and I_μ represent the Bessel and modified Bessel functions of the first kind with order μ, respectively. Similarly to the fixed membrane problem, equality is achieved in (2.39) whenever Ω is a ball. Additionally, larger and larger domains produce smaller and smaller first eigenvalues, which tend to zero. For a quantitative form, see Antunes, Buoso, and Freitas [5].

Clamped plate problems have been recently studied on Riemannian manifolds, in both the positively and negatively curved cases; see Kristály [68, 69]. The author proved that if (M,g) is a Cartan–Hadamard manifold with sectional curvature $\mathbf{K} \leq \kappa$

for some $\kappa < 0$ and verifying the generalized Cartan–Hadamard conjecture, then the first eigenvalue of (2.38) satisfies the *sharp* estimate

$$\Lambda_c(\Omega) = \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\Delta_g u|^2 \, dv_g}{\displaystyle\int_\Omega u^2 \, dv_g} \geq \frac{(n-1)^4 \kappa^4}{16}. \tag{2.40}$$

The proof of the inequality (2.40) deeply relies on Schwarz-type symmetrization and the validity of the aforementioned generalized Cartan–Hadamard conjecture.

The natural extension of the problem (2.38) is the *p-clamped plate problem*, which can be obtained by replacing the PDE with

$$\Delta_{g,p}^2 u = \lambda_{c,p} |u|^{p-2} u,$$

where $\Delta_{g,p}^2$ denotes the *p-biharmonic operator* defined by

$$\Delta_{g,p}^2 u = \Delta_g(|\Delta_g u|^{p-2} \Delta_g u),$$

for every smooth function u. The latter problem was discussed in the context of hyperbolic spaces by Ngô and Nguyen [87]. More precisely, the authors proved that if $\Omega \subseteq \mathbb{H}_\kappa^n$ is a bounded open domain for some $\kappa < 0$, then the *sharp* estimate for the first eigenvalue reads as

$$\Lambda_{c,p}(\Omega) \stackrel{\text{def}}{=} \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\Delta_g u|^p \, dv_g}{\displaystyle\int_\Omega u^p \, dv_g} \geq \left(\frac{(n-1)^2 |\kappa| (p-1)}{p^2} \right)^p. \tag{2.41}$$

The proof of the inequality (2.41) deeply explores symmetrization techniques combined with the validity of the generalized Cartan–Hadamard conjecture, where the model structure of $M = \mathbf{M}_\kappa^n$ plays a crucial role.

Remark 2.7. Later in Theorem 4.4 we prove the estimate (2.41) and its *sharpness* for *general* Cartan–Hadamard manifolds, with sectional curvature $\mathbf{K} \leq \kappa$ for some $\kappa < 0$, using a symmetrization-free approach. We note that Theorem 4.4 extends not only the validity of (2.40) to any dimension (this estimate being proved only in dimensions 2 and 3, cf. Kristály [68]), but also solves the claim raised in Cheng and Yang [27] and Li, Mao, and Zeng [78]. In the latter two works the authors proved that *if* the first eigenvalue of the clamped plate problem satisfies

$$\lim_{R \to \infty} \Lambda_c(B_{-1}(R)) = \frac{(n-1)^4}{16}, \tag{2.42}$$

where $B_{-1}(R)$ is the geodesic ball in the hyperbolic space $\mathbf{M}_{-1}^n$, then the same limit should be valid also for the l^{th} eigenvalues of (2.38), for every $l \geq 2$. In view of Theorem 4.4, the assumption (2.42) in papers [27, 78] turns out to be superfluous.

2.3.3 Buckling problem

The *buckling problem* is a fourth-order problem, which can be stated as follows:

$$\begin{cases} \Delta_g^2 u = -\lambda_{\mathrm{b}} \Delta_g u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \mathbf{n}} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.43}$$

where $\Omega \subseteq M$ is a bounded, open domain and (M, g) is an n-dimensional Riemannian manifold, with $n \geq 2$.

To the best of our knowledge, only a few pieces of qualitative information are known concerning the first eigenvalue of the problem (2.43). A milestone result is due to Ngô and Nguyen [87], where the authors provided a spectral gap estimate for the p-buckling problem in the hyperbolic setting, for general $p > 1$. In particular, when $p = 2$, the estimate can be stated as follows: If $\Omega \subseteq H_\kappa^n$ is a bounded, open domain for some $\kappa < 0$, then the first eigenvalue of (2.43) satisfies the *sharp* estimate

$$\Lambda_{\mathrm{b}}(\Omega) \stackrel{\text{def}}{=} \inf_{u \in C_0^\infty(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega |\Delta_g u|^2 \, \mathrm{d}v_g}{\displaystyle\int_\Omega |\nabla_g u|^2 \, \mathrm{d}v_g} \geq \frac{(n-1)^2 |\kappa|}{4}. \tag{2.44}$$

The proof is again based on symmetrization and on the validity of the generalized Cartan–Hadamard conjecture.

Remark 2.8. Later in Theorem 4.5 we prove the estimate (2.44) and its *sharpness* for *general* Cartan–Hadamard manifolds, with sectional curvature $\mathbf{K} \leq \kappa$ for some $\kappa < 0$, using a symmetrization-free approach. We note, however, that certain technical difficulties prevent the extension of our approach to the general case when $p > 1$.

2.4 The approach of Bessel pairs

In this section, we recall the key ingredients of a recent approach to establish Hardy–Rellich inequalities on the Euclidean space $\mathbb{R}^n$. This method is based on the notion of *Bessel pairs* introduced by Ghoussoub and Moradifam [50].

Let us start with an auxiliary definition that is relevant for applications.

Definition 2.9. A positive function Z is said to be a *Bessel potential* on $(0, R)$ if there exist a constant $c > 0$ and a function $z > 0$ such that following ordinary differential equation holds:

$$z''(t) + \frac{z'(t)}{t} + c \cdot Z(t) \cdot z(t) = 0, \qquad \forall t \in (0, R). \tag{2.45}$$

The largest c, for which there exists $z > 0$ verifying (2.45) is called the *best constant*.

In the sequel, we present some relevant examples of Bessel potentials. Hereafter, for every function h let us denote

$$h_{[i]}(t) = \begin{cases} t, & \text{if } i = 0, \\ h(t), & \text{if } i = 1, \\ h(h_{[i-1]}(t)), & \text{if } i \geq 2. \end{cases} \tag{2.46}$$

Example 2.10. (a) For every $R > 0$, the functions

$$Z \equiv 1 \quad \text{and} \quad z(t) = J_0\left(\frac{t \cdot j_{0,1}}{R}\right)$$

satisfy (2.45), for all $t \in (0, R)$, where $j_{0,1}$ denotes the first positive root of the Bessel function J_0 of the first kind and order zero. The best constant is $c = \frac{j_{0,1}^2}{R^2}$.

(b) Let $k \geq 1$, $R > 0$, and $r = R \cdot \exp_{[k-1]}(e)$. The functions

$$Z_{k,r}(t) = \sum_{j=1}^{k} \frac{1}{t^2} \left(\prod_{i=1}^{j} \log_{[i]}\left(\frac{r}{t}\right) \right)^{-2} \quad \text{and} \quad z_{k,r}(t) = \left(\prod_{i=1}^{k} \log_{[i]}\left(\frac{r}{t}\right) \right)^{\frac{1}{2}}$$

satisfy (2.45), for all $t \in (0, R)$. The best constant is $c = \frac{1}{4}$.

(c) Let $k \geq 1$, $R > 0$, and define

$$\ell(t) = \frac{1}{1 - \log(t)}, \qquad \forall t \in (0, 1).$$

The functions

$$\widetilde{Z}_{k,R}(t) = \sum_{j=1}^{k} \frac{1}{t^2} \prod_{i=1}^{j} \ell_{[i]}^2\left(\frac{t}{R}\right) \quad \text{and} \quad \widetilde{z}_{k,R}(t) = \left(\prod_{i=1}^{k} \ell_{[i]}\left(\frac{t}{R}\right) \right)^{-\frac{1}{2}}$$

satisfy (2.45), for all $t \in (0, R)$. The best constant is $c = \frac{1}{4}$.

The central concept of this approach is the notion of Bessel pairs. The definition is as follows.

Definition 2.11. The couple (X, Y) of positive functions is said to be a *Bessel pair* on $(0, R)$ if there exist a constant $C > 0$ and a function $y > 0$ such that the following ordinary differential equation holds:

$$y''(t) + \left(\frac{n-1}{t} + \frac{X'(t)}{X(t)}\right) y'(t) + \frac{C \cdot Y(t)}{X(t)} y(t) = 0, \qquad \forall t \in (0, R). \tag{2.47}$$

The largest C, for which there exists $y > 0$ verifying (2.47) is called the *best constant.*

The relations between the two concepts are given by the following proposition.

Proposition 2.12. *If equation* (2.45) *holds on* $(0, R)$*, then equation* (2.47) *holds as well for the choices:*

$$y(t) = z(t) \cdot t^{\frac{2-n}{2}}, \quad C = 1, \quad X(t) \equiv 1, \quad \textit{and} \quad Y(t) = \frac{(n-2)^2}{4t^2} + c \cdot Z(t).$$

The next result sheds more light on the connection between the two notions, being particularly useful when dealing with Rellich inequalities. It is a simple consequence of [50, Theorem 2.6], the proof is based on the comparison of solutions of (2.47).

Proposition 2.13. *Suppose that Z is a Bessel potential on $(0,R)$ with best constant c, such that*

$(\mathbf{Z})\ :\ \frac{Z'(t)}{Z(t)} = -\frac{\lambda}{t} + f(t)$, *where* $\lambda < n-2$, $f(t) \geq 0$ *and* $\lim_{t\to 0} t f(t) = 0$.

Then the couples $\left(\frac{1}{t^2}, \frac{(n-4)^2}{4t^4} + \frac{c\cdot Z}{t^2}\right)$ *and* $\left(Z, \frac{(n-\lambda-2)^2\cdot Z}{4t^2}\right)$ *are also Bessel pairs on* $(0,R)$*, with best constant* $C = 1$.

Finally, we present the main result of this approach (see [50, Theorem 1.1]).

Theorem 2.14. *Let $\Omega = B_O(R)$ denote the Euclidean ball centered at the origin O with radius $R > 0$. Suppose that X and Y are positive radial C^1 functions on $\Omega \setminus \{O\}$, with*

$$\int_0^R \frac{1}{t^{n-1}X(t)}\,\mathrm{d}t = +\infty \quad \textit{and} \quad \int_0^R t^{n-1}X(t)\,\mathrm{d}t < +\infty.$$

The following statements are equivalent:

(1) *The couple (X,Y) is a Bessel pair on $(0,R)$ with the best constant $C \geq 1$.*

(2) *One has*

$$\int_\Omega V(x)|\nabla u|^2\,\mathrm{d}x \geq \int_\Omega W(x)u^2\,\mathrm{d}x, \tag{2.48}$$

for all $u \in C_0^\infty(B)$.

(3) *If* $\lim_{t\to 0} t^\alpha V(t) = 0$ *for some* $\alpha < n-2$*, then one has*

$$\int_\Omega V(x)|\Delta u|^2\,\mathrm{d}x \geq \int_\Omega W(x)|\nabla u|^2\,\mathrm{d}x + (n-1)\int_\Omega \left(\frac{V(x)}{|x|^2} - \frac{\nabla^{\mathrm{rad}}V(|x|)}{|x|}|\nabla u|^2\right)\mathrm{d}x, \tag{2.49}$$

for all radial *functions* $u \in C_0^\infty(B)$.

(4) *If, in addition,*

$$W(t) - \frac{2V(t)}{t^2} + \frac{2\nabla^{\mathrm{rad}}V(t)}{t} - \nabla^{\mathrm{rad}}\nabla^{\mathrm{rad}}V(t) \geq 0, \qquad \forall t \in (0,R),$$

then the inequality (2.49) holds for all $u \in C_0^\infty(B)$.

The notion of classical Bessel pairs was extended to general $p > 1$ as follows (see Duy, Lam, and Lu [36, Definition 1.1]).

Definition 2.15. If $p > 1$ and $A, B\colon (0,R) \to \mathbb{R}$ are functions with A being of class C^1, the couple (A,B) is a *p-Bessel pair in* $(0,R)$ if the ODE

$$\left(t^{n-1}A(t)|y'(t)|^{p-2}y'(t)\right)' + t^{n-1}B(t)|y(t)|^{p-2}y(t) = 0 \tag{2.50}$$

has a positive solution in $(0,R)$.

We note that there exists a connection between the ODE of Bessel pairs, and the classical method of supersolutions, sometimes also called the Allegretto–Moss–Piepenbrink approach. We have the following

Remark 2.16. By the theory of supersolutions, if a second-order elliptic operator admits a positive supersolution, then the operator is positive in the sense of quadratic forms (see Davies [31, Theorem 4.2.1]). Accordingly, if $\Omega \subseteq \mathbb{R}^n$ is a domain that contains the origin, $W \in L^1_{\text{loc}}(\Omega) \cap C(\Omega \setminus \{0\})$ and $\varphi \in C^2(\Omega \setminus \{0\})$ is positive such that

$$(-\Delta - W)\varphi(x) \geq 0, \qquad \forall x \in \Omega \setminus \{0\}, \tag{2.51}$$

then the operator $(-\Delta - W) \geq 0$ is *positive*; more precisely, one has

$$\int_\Omega |\nabla u|^2 \, \mathrm{d}x \geq \int_\Omega W u^2 \, \mathrm{d}x, \qquad \forall u \in C_0^\infty(\Omega).$$

Observe that if $\varphi(x) = y(|x|)$ for some function y, then (2.51) translates to

$$-y''(t) - \frac{n-1}{t} y'(t) - W(t)y(t) \geq 0, \qquad \forall t \in (0, \sup\{|x| : x \in \Omega\}).$$

The equality case of the above relation perfectly agrees with equation (2.50) for $A \equiv 1$, $B = W$ and $p = 2$. We note that similar arguments apply in the weighted case, as well as in the case when $p > 1$.

We conclude this section with the following additional comments.

Remark 2.17. (a) We notice that equation (2.50) for $p = 2$ is equivalent to (2.47).

(b) The notion of Bessel pairs and p-Bessel pairs was used to establish Hardy–Rellich inequalities on both Euclidean and hyperbolic spaces, in spite of the fact that their definition is purely Euclidean, i.e., they do not encode curvature information. Later, in Remark 3.8 we suggest a more natural extension.

(c) By using a suitable Laplace comparison, the arguments from Remark 2.16 can be extended to Riemannian manifolds in the view of Remark 3.8. These relations, together with Proposition 3.7 – linking Bessel pairs with our approach to Hardy inequalities – allow us to incorporate in our method various elements of the *criticality theory* of Hardy inequalities, developed originally in context of supersolutions; see Remark 3.29/(b) for an illustration. Concerning criticality theory, we refer to the foundational works by Pinchover [92, 93] and Devyver, Fraas, and Pinchover [34].

(d) In the subsequent chapters, we present alternative methods for proving Hardy and Rellich inequalities. These methods do not rely on spherical harmonics decomposition and are also symmetrization-free.

Chapter 3

Hardy inequalities on general Riemannian manifolds

In this chapter, we present a simple, elegant approach for proving well-known and genuinely new Hardy-type inequalities on Riemannian manifolds.

In Section 3.1 we present our abstract approach. First, in Theorem 3.1 we provide a general functional inequality both in additive and multiplicative forms, producing Hardy inequalities of the corresponding forms, i.e., inequalities of types (**H**) and (**UP**), respectively. We also show that the two forms are equivalent. Next, in Section 3.1.1 we introduce the notion of Riccati pairs for the additive form. This concept enables us to give short/elegant proofs for a number of celebrated functional inequalities on Riemannian manifolds with sectional curvature bounded from above, by simply solving a Riccati-type ODI; see Theorem 3.5. Finally, in Proposition 3.7 and Remark 3.8 we show that Riccati pairs extend the notion of Bessel pairs developed by Ghoussoub and Moradifam [50]. We refer to Section 2.4 for a short summary of Bessel pairs.

In Section 3.2 we present various applications of the additive inequality; specifically, we shall consider the following inequalities.

- In Section 3.2.1 we give alternative proofs to L^p-Caccioppoli-type inequalities on Riemannian manifolds (see Theorems 3.9 & 3.12).
- In Section 3.2.2 we prove various improved Hardy-type inequalities on Cartan–Hadamard manifolds (see Theorems 3.13 & 3.17 & 3.18 & 3.19).
- In Section 3.2.3 we establish several spectral estimates on Riemannian manifolds, including the Cheng-type comparison result, the Faber–Krahn inequality, the McKean-type spectral gap estimate, and the Carvalho–Cavalcante-type spectral result (see Theorems 3.21 & 3.23 & 3.24 & 3.26).
- In Section 3.2.4 we prove interpolation inequalities connecting the classical Hardy inequality and McKean's spectral gap estimate on Cartan–Hadamard manifolds (see Theorems 3.27 & 3.30).
- In Section 3.2.5 we provide Ghoussoub–Moradifam-type weighted inequalities in the Euclidean setting; for a certain parameter range these results extend to Cartan–Hadamard manifolds (see Theorems 3.31 & 3.33).

In Section 3.3 we present applications to the multiplicative inequality; specifically, we shall consider the following inequalities.

- In Section 3.3.1 we provide a sharp uncertainty principle on Cartan–Hadamard manifolds (see Theorem 3.35), which implies both the Heisenberg–Pauli–Weyl and Hydrogen uncertainty principles; a rigidity result is proved in Theorem 3.37.
- In Section 3.3.2 we establish sharp Caffarelli–Kohn–Nirenberg inequalities on Cartan–Hadamard manifolds (see Theorems 3.40 & 3.41).

3.1 General functional inequalities

In this section, we present our abstract approach for proving Hardy inequalities. We put ourselves into the realm of Riemannian manifolds. The utilized concepts, preliminary results, and the corresponding notations are summarized in Sections 2.1 & 2.1.1.

Our first result is a general functional inequality formulated in two equivalent forms, which provide inequalities of types (**H**) and (**UP**). It can be stated as follows.

Theorem 3.1 (see [59]). *Let (M,g) be a complete, non-compact, $n\geq 2$-dimensional Riemannian manifold. Let $\Omega\subseteq M$ be a domain, $p>1$, and let $\rho\in W^{1,p}_{\mathrm{loc}}(\Omega)$ be non-constant and positive with $\mathcal{H}^n_g(\rho^{-1}(\sup_\Omega\rho))=0$. Suppose that $w\colon(0,\sup_\Omega\rho)\to(0,\infty)$, $G\colon(0,\sup_\Omega\rho)\to\mathbb{R}$, and $H\colon\mathbb{R}\to\mathbb{R}$ are C^1 functions such that*

$$(\mathbf{G})_{\rho,w}\ :\ G(\rho)w(\rho)^{\frac{1}{p'}}|\nabla_g\rho|^{p-1}\in L^{p'}_{\mathrm{loc}}(\Omega)\ \text{and}\ (G(\rho)w(\rho))'|\nabla_g\rho|^p, w(\rho)\in L^1_{\mathrm{loc}}(\Omega),$$

and $H(0)=H'(0)=0$. The following inequalities hold:

(i) (Additive form) *For every $u\in C_0^\infty(\Omega)$ one has*

$$\int_\Omega w(\rho)|\nabla_g u|^p\,\mathrm{d}v_g\geq p\int_\Omega\left[(G(\rho)w(\rho))'|\nabla_g\rho|^p+G(\rho)w(\rho)\Delta_{g,p}\rho\right]H(u)\,\mathrm{d}v_g$$
$$+(1-p)\int_\Omega|G(\rho)|^{p'}|\nabla_g\rho|^p w(\rho)|H'(u)|^{p'}\,\mathrm{d}v_g. \tag{3.1}$$

(ii) (Multiplicative form) *For every $u\in C_0^\infty(\Omega)$ one has*

$$\int_\Omega w(\rho)|\nabla_g u|^p\,\mathrm{d}v_g\geq\frac{\left|\displaystyle\int_\Omega\left[(G(\rho)w(\rho))'|\nabla_g\rho|^p+G(\rho)w(\rho)\Delta_{g,p}\rho\right]H(u)\,\mathrm{d}v_g\right|^p}{\left(\displaystyle\int_\Omega|G(\rho)|^{p'}|\nabla_g\rho|^p w(\rho)|H'(u)|^{p'}\,\mathrm{d}v_g\right)^{p-1}}, \tag{3.2}$$

provided that there exists a neighborhood $\mathcal{V}\subseteq\mathbb{R}$ of zero satisfying

$$H'(s)\neq 0,\forall s\in\mathcal{V}\setminus\{0\},\ \ G(t)\neq 0,\forall t\in(0,\sup_\Omega\rho),\ \text{and}\ \mathcal{H}^n_g(|\nabla_g\rho|^{-1}(0))=0. \tag{3.3}$$

Proof. (i) By the convexity of $\xi \mapsto |\xi|^p$, it turns out that

$$|\xi|^p \geq |\eta|^p + p|\eta|^{p-2}(\xi - \eta)\eta = p|\eta|^{p-2}\xi\eta + (1-p)|\eta|^p, \qquad \forall \xi, \eta \in T\Omega.$$

Fix $u \in C_0^\infty(\Omega)$ arbitrarily and choose

$$\xi = \nabla_g u \quad \text{and} \quad \eta = -|G(\rho)|^{\frac{2-p}{p-1}}|H'(u)|^{\frac{2-p}{p-1}} G(\rho)H'(u)\nabla_g \rho.$$

On the one hand, we have

$$|\eta|^p = |G(\rho)|^{\frac{p}{p-1}}|\nabla_g \rho|^p |H'(u)|^{\frac{p}{p-1}} = |G(\rho)|^{p'}|\nabla_g \rho|^p |H'(u)|^{p'}.$$

On the other hand, the chain rule

$$\nabla_g H(u) = H'(u)\nabla_g u$$

implies that

$$|\eta|^{p-2}\xi\eta = -G(\rho)|\nabla_g \rho|^{p-2}\nabla_g \rho \nabla_g H(u).$$

Therefore, one has

$$|\nabla_g u|^p \geq -pG(\rho)|\nabla_g \rho|^{p-2}\nabla_g \rho \nabla_g H(u) + (1-p)|G(\rho)|^{p'}|\nabla_g \rho|^p |H'(u)|^{p'}.$$

Multiplying by $w(\rho) > 0$, integrating over Ω, and using $(\mathbf{G})_{\rho,w}$ lead to

$$\begin{aligned} \int_\Omega w(\rho)|\nabla_g u|^p \,\mathrm{d}v_g \geq -p \int_\Omega G(\rho)w(\rho)|\nabla_g \rho|^{p-2}\nabla_g \rho \nabla_g H(u) \,\mathrm{d}v_g \\ + (1-p)\int_\Omega |G(\rho)|^{p'}|\nabla_g \rho|^p w(\rho)|H'(u)|^{p'} \,\mathrm{d}v_g \\ \stackrel{\text{def}}{=} pI_H + (1-p)J_H. \end{aligned} \tag{3.4}$$

Integration by parts (2.3) and the boundary conditions on H yield

$$\begin{aligned} I_H &= \int_\Omega \mathrm{div}_g \left(G(\rho)w(\rho)|\nabla_g \rho|^{p-2}\nabla_g \rho\right) H(u) \,\mathrm{d}v_g \\ &= \int_\Omega \left[(G(\rho)w(\rho))'|\nabla_g \rho|^p + G(\rho)w(\rho)\Delta_{g,p}\rho\right] H(u) \,\mathrm{d}v_g. \end{aligned}$$

The above arguments imply inequality (3.1).

(ii) Recall the definitions of I_H and J_H from (3.4). On the one hand, assumption (3.3) implies that $J_H > 0$. On the other hand, observe that for every $c \in \mathbb{R}$, inequality (3.1) is valid for cH instead of H, thus,

$$\int_\Omega w(\rho)|\nabla_g u|^p \,\mathrm{d}v_g \geq pI_{cH} + (1-p)J_{cH} = pcI_H + (1-p)|c|^{p'}J_H.$$

Note that $1 - p < 0$; thus, we can maximize the right-hand side of the latter expression with respect to c. The maximum is achieved for $c = J_H^{1-p}|I_H|^{p-2}I_H$, obtaining that

$$\int_\Omega w(\rho)|\nabla_g u|^p \,\mathrm{d}v_g \geq \frac{|I_H|^p}{J_H^{p-1}},$$

which is precisely inequality (3.2).

□

Remark 3.2. We notice that the additive and multiplicative forms are equivalent in the sense that they can be deduced from each other. On the one hand, the relation (3.2) follows from (3.1) (see Theorem 3.1/(ii)). On the other hand, we have that

$$J_H^{1-p}|I_H|^p \geq pI_H + (1-p)J_H;$$

indeed, for $I_H \leq 0$, the latter inequality is trivial, while for $I_H > 0$, Young's inequality applies. Accordingly, we may roughly say that inequalities of types (**H**) and (**UP**) have the same 'algebraic' origin.

As one shall see, both inequalities in Theorem 3.1 are *generic*, that is, for suitable choices of parameters p, ρ, w, G, and H, they produce various functional inequalities. On the one hand, the additive form (i) is tailored to produce Hardy-type inequalities of type (**H**) (see Section 3.2). On the other hand, the multiplicative form (ii) provides various uncertainty principles of type (**UP**) (see Section 3.3).

In the sequel we demonstrate that the aforementioned procedure can be reversed: for a number of Hardy-type inequalities one can find suitable choices of parameters providing short/elegant proofs for them. The central element of this idea is the concept of Riccati pairs, which is presented in the next section.

3.1.1 Riccati pairs

Let us focus on the additive form (i) of Theorem 3.1. Let $p > 1$, and observe that if

$$H(s) = \frac{|s|^p}{p}, \ \forall s \in \mathbb{R}, \quad \text{then} \quad pH(s) = |H'(s)|^{p'} = |s|^p.$$

The above observation, together with the Laplace comparison (see Theorem 2.1/(I)/(i)) suggest the following notion.

Definition 3.3 (see [59]). Let (M, g) be a complete, non-compact $n \geq 2$-dimensional Riemannian manifold. Let $\Omega \subseteq M$ be a domain, $p > 1$, and $\rho \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a positive function with $|\nabla_g \rho| = 1$ dv_g-a.e. in Ω. Fix the continuous functions $L, W\colon (0, \sup_\Omega \rho) \to (0, \infty)$ and the function $w\colon (0, \sup_\Omega \rho) \to (0, \infty)$ of class C^1. We say that the couple (L, W) is a (p, ρ, w)-*Riccati pair in* $(0, \sup_\Omega \rho)$ if there exists a function $G\colon (0, \sup_\Omega \rho) \to \mathbb{R}$ such that

(c1) $(\mathbf{G})_{\rho,w}$ holds (from Theorem 3.1);

(c2) $\Delta_g \rho \geq L(\rho)$ in the distributional sense in Ω, and

$$G \geq 0, \quad \text{if} \quad \mathcal{H}^n_g(\{x \in \Omega : \Delta_g \rho(x) > L(\rho(x))\}) \neq 0;$$

(c3) for every $t \in (0, \sup_\Omega \rho)$ one has

$$(G(t)w(t))' + G(t)w(t)L(t) + (1-p)|G(t)|^{p'}w(t) \geq W(t)w(t). \tag{3.5}$$

A function G satisfying the above conditions is said to be *admissible* for (L, W).

Remark 3.4. For later use, we provide the following particular forms of (3.5).

(i) If $p = 2$, then inequality (3.5) reduces to

$$(G(t)w(t))' + G(t)w(t)L(t) - |G(t)|^2 w(t) \geq W(t)w(t), \qquad \forall t \in (0, \sup_\Omega \rho). \tag{3.6}$$

(ii) If $w \equiv 1$, then inequality (3.5) reduces to

$$G'(t) + G(t)L(t) + (1-p)|G(t)|^{p'} \geq W(t), \qquad \forall t \in (0, \sup_\Omega \rho). \tag{3.7}$$

(iii) If both $p = 2$ and $w \equiv 1$, then inequality (3.5) reduces to

$$G'(t) + G(t)L(t) - |G(t)|^2 \geq W(t), \qquad \forall t \in (0, \sup_\Omega \rho). \tag{3.8}$$

An efficient application of Theorem 3.1/(i) based on the concept of Riccati pairs can be stated as follows.

Theorem 3.5 (see [59]). *Let (M, g) be a complete, non-compact $n \geq 2$-dimensional Riemannian manifold. Let $\Omega \subseteq M$ be a domain, $p > 1$, and $\rho \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a positive function with $|\nabla_g \rho| = 1$ $\mathrm{d}v_g$-a.e. in Ω. Suppose that $L, W\colon (0, \sup_\Omega \rho) \to (0, \infty)$ are continuous functions and $w\colon (0, \sup_\Omega \rho) \to (0, \infty)$ is of class C^1 such that (L, W) is a (p, ρ, w)-Riccati pair in $(0, \sup_\Omega \rho)$. Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega w(\rho)|\nabla_g u|^p \,\mathrm{d}v_g \geq \int_\Omega W(\rho)w(\rho)|u|^p \,\mathrm{d}v_g. \tag{3.9}$$

Proof. Since (L, W) is a (p, ρ, w)-Riccati pair in $(0, \sup_\Omega \rho)$, there exists a function $G\colon (0, \sup_\Omega \rho) \to \mathbb{R}$, such that Conditions **(c1)**–**(c3)** are satisfied. We shall apply Theorem 3.1/(i) for the parameters L, W, p, ρ, w, G, as well as

$$H(s) = \frac{|s|^p}{p}, \qquad \forall s \in \mathbb{R}.$$

Condition **(c1)** yields $(\mathbf{G})_{\rho,w}$; moreover, the boundary conditions $H(0) = H'(0) = 0$ hold as well. Accordingly, we obtain that

$$\int_\Omega w(\rho)|\nabla_g u|^p \,\mathrm{d}v_g \geq p \int_\Omega \left[(G(\rho)w(\rho))'|\nabla_g \rho|^p + G(\rho)w(\rho)\Delta_{g,p}\rho\right] \frac{|u|^p}{p} \,\mathrm{d}v_g$$
$$+ (1-p) \int_\Omega |G(\rho)|^{p'} |\nabla_g \rho|^p w(\rho)|u|^p \,\mathrm{d}v_g.$$

Now we can use relations (2.1) & (2.2), namely, since $|\nabla_g \rho| = 1$ $\mathrm{d}v_g$-a.e. in Ω, one has $\Delta_{g,p}\rho = \Delta_g \rho$ $\mathrm{d}v_g$-a.e. in Ω. Therefore, we have

$$\int_\Omega w(\rho)|\nabla_g u|^p \,\mathrm{d}v_g \geq \int_\Omega \left[(G(\rho)w(\rho))' + G(\rho)w(\rho)\Delta_g \rho\right] |u|^p \,\mathrm{d}v_g$$
$$+ (1-p) \int_\Omega |G(\rho)|^{p'} w(\rho)|u|^p \,\mathrm{d}v_g. \tag{3.10}$$

Observe that Condition (c2) implies

$$G(\rho)\Delta_g\rho \geq G(\rho)L(\rho) \tag{3.11}$$

in the distributional sense in Ω. Indeed, on the one hand, if $\Delta_g\rho = L(\rho)$ $\mathrm{d}v_g$-a.e. in Ω, then the inequality (3.11) is trivial. On the other hand, if

$$\mathcal{H}^n_g(\{x \in \Omega : \Delta_g\rho(x) > L(\rho(x))\}) \neq 0,$$

then G is non-negative by (c2), hence the inequality (3.11) holds in this case as well.

Finally, combining inequalities (3.10) & (3.11) with Condition (c3) yields

$$\begin{aligned}
\int_\Omega w(\rho)|\nabla_g u|^p \,\mathrm{d}v_g &\geq p\int_\Omega \left[(G(\rho)w(\rho))' + G(\rho)w(\rho)\Delta_g\rho\right]\frac{|u|^p}{p}\,\mathrm{d}v_g \\
&\quad + (1-p)\int_\Omega |G(\rho)|^{p'}w(\rho)|u|^p \,\mathrm{d}v_g \\
&\geq \int_\Omega \left[(G(\rho)w(\rho))' + G(\rho)w(\rho)L(\rho)\right]|u|^p \,\mathrm{d}v_g \\
&\quad + (1-p)\int_\Omega |G(\rho)|^{p'}w(\rho)|u|^p \,\mathrm{d}v_g \\
&\geq \int_\Omega W(\rho)w(\rho)|u|^p \,\mathrm{d}v_g,
\end{aligned}$$

which is precisely relation (3.9). □

To present the efficiency of Theorem 3.5, we sketch a short proof of the celebrated *McKean's sharp spectral gap estimate*: If (M, g) is an n-dimensional Cartan–Hadamard manifold with $n \geq 2$ and sectional curvature $\mathbf{K} \leq \kappa$ for some $\kappa < 0$, then the *essential spectrum* of the Laplace–Beltrami operator on (M, g) is $[K_{\kappa,n}, \infty)$, where

$$K_{\kappa,n} \stackrel{\text{def}}{=} \left(\frac{(n-1)\sqrt{-\kappa}}{2}\right)^2.$$

Indeed, fix $x_0 \in M$ and choose $\rho = d_{g,x_0}$, i.e., the distance from x_0, as well as

$$w \equiv 1, \quad L \equiv (n-1)\sqrt{-\kappa}, \quad W \equiv (n-1)c\sqrt{-\kappa} - c^2, \quad \text{and} \quad G \equiv c,$$

for some $c > 0$. For these choices, the ODI of (3.8) is verified with equality, thus, Conditions (c1) & (c3) clearly hold. The Laplace comparison (see Theorem 2.1/(I)/(i)) implies $\Delta_g\rho \geq L$, hence (c2) holds as well. We obtained that (L, W) is a (p, ρ, w)-Riccati pair in $(0, \infty)$, and G is admissible for (L, W). A simple computation yields

$$\max_{c>0}\left((n-1)\sqrt{-\kappa}c - c^2\right) = K_{\kappa,n},$$

thus, Theorem 3.5 immediately implies the required spectral estimate:

$$\int_M |\nabla_g u|^2 \,\mathrm{d}v_g \geq K_{\kappa,n}\int_M |u|^2 \,\mathrm{d}v_g, \qquad \forall u \in C_0^\infty(M).$$

For a similar proof of the general case when $p > 1$, see Theorem 3.24. We also note that the eigenvalues of the Laplace–Beltrami operator are precisely the eigenvalues of the fixed membrane problem (see Section 2.3.1).

Remark 3.6. (a) In the majority of applications, as in the previous one, ρ is a distance function either from a fixed point x_0 or from the boundary of the domain; additionally, the other parameter functions are typically smooth. However, we have presented our results in such a general way that we can also include delicate applications, such as Caccioppoli-type inequalities (see Theorem 3.9).

(b) If Ω is a domain in the model space form $\mathbf{M}^n_\kappa$ and $\rho = d_{\kappa,x_0}$ is the distance from some fixed point $x_0 \in \mathbf{M}^n_\kappa$, then by the relation (2.12) we have

$$\Delta_\kappa d_{\kappa,x_0} = (n-1)\mathbf{ct}_\kappa(d_{\kappa,x_0}) = L_\kappa(d_{\kappa,x_0})$$

on $\mathbf{M}^n_\kappa$, except the point x_0 and its cut locus (which is empty for $\kappa \leq 0$, and is the antipodal point of x_0 on the sphere $\mathbb{S}^n_\kappa$ for $\kappa > 0$). In such cases we may choose

$$L(t) = (n-1)\mathbf{ct}_\kappa(t) = L_\kappa(t),$$

with the usual interval restriction when $\kappa > 0$, and then *no sign restriction* should be imposed on the function G (see **(c2)** and the proof of Theorem 3.5). However, on generic Riemannian manifolds, the non-negativeness of G is indispensable in our arguments.

(c) Based on these observations, in Chapter 5 – where we discuss Rellich inequalities on space forms – we will use a simplified definition of Riccati pairs (see Definition 5.1).

In the sequel, we establish the connection between Riccati and Bessel pairs. Let us recall Definition 2.15: Let $p > 1$, $R > 0$, and $A, B\colon (0,R) \to \mathbb{R}$ are functions with A being of class C^1. The couple (A, B) is a *p-Bessel pair* in $(0,R)$, if the ODE

$$\left(t^{n-1}A(t)|y'(t)|^{p-2}y'(t)\right)' + t^{n-1}B(t)|y(t)|^{p-2}y(t) = 0 \tag{3.12}$$

has a positive solution in $(0,R)$. This concept is related to Riccati pairs as follows.

Proposition 3.7 (see [59]). *Let $R > 0$ and $w, W\colon (0,R) \to (0,\infty)$ be two potentials with w of class C^1. The function $y > 0$ is a solution of (3.12) on $(0,R)$ for the couple*

$$(A,B) = (w, wW)$$

if and only if

$$G(t) = -\frac{|y'(t)|^{p-2}y'(t)}{y(t)^{p-1}} \tag{3.13}$$

is a solution of

$$(G(t)w(t))' + G(t)w(t)L_0(t) + (1-p)|G(t)|^{p'}w(t) = W(t)w(t), \tag{3.14}$$

on $(0,R)$, that is precisely (3.5) with equality and

$$L(t) = \frac{n-1}{t} = L_0(t).$$

Proof. Insert G of (3.13) into (3.14); a simple computation yields the equation (3.12). Since all steps can be reversed, the two equations are equivalent. $\square$

By Proposition 3.7, we obtain that Riccati pairs extend Bessel pairs. For a better analysis of the relationship between the two concepts, we make a few observations.

First, we note that the concept of Bessel pairs is purely Euclidean, since it does not implicitly encode any curvature information. Nevertheless, several authors managed to apply it in the hyperbolic context with additional comparisons of Jacobi fields; see e.g., Flynn, Lam, Lu, and Mazumdar [46] and Berchio, Ganguly, and Grillo [11]. In contrary, Riccati pairs 'inherit' the geometry of the ambient Riemannian manifold, which is reflected in the L term of the ODI (3.5) of Riccati pairs.

Next, we observe that in the Euclidean setting, the two notions are 'essentially' equivalent to each other: The equality case of the ODI (3.5) of Riccati pairs is equivalent to the ODE (3.12) of Bessel pairs, in the sense that performing a change of functions makes the transition between them. Based on this observation, we suggest a more general definition of p-Bessel pairs, which has similar properties to the original, but can also be used on general manifolds. Our suggestion is the following.

Remark 3.8. In a Riemannian manifold (M, g) with sectional curvature $\mathbf{K} \leq \kappa$ for some $\kappa \in \mathbb{R}$, a more appropriate ODE in the definition of a p-Bessel pair (A, B) instead of (3.12), is as follows:

$$\left(\mathbf{s}_\kappa^{n-1}(t)A(t)|y'(t)|^{p-2}y'(t)\right)' + \mathbf{s}_\kappa^{n-1}(t)B(t)|y(t)|^{p-2}y(t) = 0, \qquad \forall t \in (0, R). \tag{3.15}$$

Indeed, when $\kappa = 0$, equation (3.15) reduces to (3.12), while for $\kappa \neq 0$, the density $\mathbf{s}_\kappa$ encodes the curvature and explains the choice of

$$L(t) = (n-1)\mathbf{ct}_\kappa(t) = L_\kappa(t), \qquad \forall t \in (0, R).$$

This observation will be crucial in some functional inequalities in the forthcoming sections, which will be obtained by means of Riccati pairs; see e.g., Cheng's comparison principle for the first eigenvalue in Theorem 3.21.

Finally, in view of Remark 3.8, we motivate the decision of presenting our results using Riccati pairs over the recently suggested version of Bessel pairs. On the one hand, Riccati pairs *naturally appear* from our generic inequality of Theorem 3.1, and the corresponding convexity arguments, thus, no redundant change of variables is required. On the other hand, Riccati pairs are *flexible* from technical point of view, as both L and W can be easily adjusted to meet our needs. Such an example is provided by the proof of McKean's sharp spectral gap estimate, where the choice of $L \equiv (n-1)\sqrt{-\kappa}$ significantly reduces the computations compared to the expected form

$$L(t) = (n-1)\sqrt{-\kappa}\coth(\sqrt{-\kappa}t) = (n-1)\mathbf{ct}_\kappa(t) = L_\kappa(t).$$

We note that, despite the similarities between the Riccati ODI (3.5) and the Bessel ODE (3.12), the corresponding approaches and arguments are mutually different. The former is based on convexity and comparison arguments, and hence it can be used on general manifolds with sectional curvature bounded from above. The latter is based on spherical harmonics decomposition; thus, it is limited to model space forms.

We also notice that our arguments can be easily extended to reversible Finsler manifolds; in the irreversible case, there are certain technical issues, which require the presence of the reversibility constant. For such a phenomenon, see Kajántó and Kristály [60].

3.2 Applications I: Additive Hardy-type inequalities

3.2.1 Caccioppoli inequalities

The first simple consequence of Theorem 3.1 is a Caccioppoli-type inequality, proved by D'Ambrosio and Dipierro [30, Theorems 2.1 & 3.1; Corollary 2.3].

Theorem 3.9 (see [59]). *Let (M, g) be a complete, non-compact $n \geq 2$-dimensional Riemannian manifold. Let $\Omega \subseteq M$ be a domain, $p > 1$, and $\rho \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a non-negative function. If $\alpha \in \mathbb{R}$ such that $-(p-1-\alpha)\Delta_{g,p}\rho \geq 0$ in the distributional sense in Ω and $|\nabla_g \rho|^p \rho^{\alpha-p}, \rho^\alpha \in L^1_{\mathrm{loc}}(\Omega)$, then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega \rho^\alpha |\nabla_g u|^p \,\mathrm{d}v_g \geq \left(\frac{|p-1-\alpha|}{p}\right)^p \int_\Omega \rho^\alpha \frac{|u|^p}{\rho^p} |\nabla_g \rho|^p \,\mathrm{d}v_g. \tag{3.16}$$

Proof. Without loss of generality, based on the assumptions $|\nabla_g \rho|^p \rho^{\alpha-p}, \rho^\alpha \in L^1_{\mathrm{loc}}(\Omega)$, we may assume that ρ is positive. Indeed, eventually we may replace ρ by $\rho+\epsilon$ for some $\epsilon > 0$, and then take the limit with respect to $\epsilon > 0$, by using the above integrability assumptions. A similar reason is developed in [30].

Let us choose in Theorem 3.1/(i) the functions

$$w(t) = t^\alpha, \quad G(t) = -(p-1-\alpha)\frac{|p-1-\alpha|^{p-2}}{p^{p-1}} t^{1-p}, \qquad \forall t > 0,$$

as well as

$$H(s) = \frac{|s|^p}{p}, \qquad \forall s \in \mathbb{R}.$$

Observe that the Condition $(\mathbf{G})_{\rho,w}$ is equivalent to $|\nabla_g \rho|^p \rho^{\alpha-p}, \rho^\alpha \in L^1_{\mathrm{loc}}(\Omega)$. Hence, using assumption$-(p-1-\alpha)\Delta_{g,p}\rho \geq 0$ and Theorem 3.1/(i), we directly obtain inequality (3.16). □

Remark 3.10. (a) The above argument provides a simple, alternative proof of the L^p-Caccioppoli inequality discussed by D'Ambrosio and Dipierro [30, Theorems 2.1 & 3.1; Corollary 2.3]. The sharpness (and eventually the existence of nonzero extremizers) of (3.16) is a very delicate issue, even in the Euclidean setting; see e.g., Brezis and Marcus [17] and Filippas, Maz'ya, and Tertikas [43]. In [30, Theorem 4.1] a simple criterion is formulated for $\alpha = 0$ under the assumptions of Theorem 3.9; namely, the constant $((p-1)/p)^p$ is sharp; moreover,

(i) if $\rho^{1/p'} \in D^{1,p}(\Omega)$, then the function $\rho^{1/p'}$ is an extremizer;

(ii) if $\rho^{1/p'} \notin D^{1,p}(\Omega)$ and $p \geq 2$, then there is no extremal function.

Here, $D^{1,p}(\Omega)$ stands for the completion of $C_0^\infty(\Omega)$ with respect to the Dirichlet norm

$$u \mapsto \left(\int_\Omega |\nabla_g u|^p \,\mathrm{d}v_g\right)^{1/p}.$$

(b) A typical example of ρ verifying the assumptions of Theorem 3.9 is provided by

$$\rho(x) = d_{g,\partial\Omega}(x) = \inf_{y\in\partial\Omega} d_g(x,y), \qquad \forall x \in \Omega,$$

where $\Omega \subseteq M$ is a bounded domain. Note that $|\nabla_g \rho| = 1$ $\mathrm{d}v_g$-a.e. in Ω and $(\mathbf{G})_{\rho,w}$ is verified for the choices of w and G in Theorem 3.9. Indeed, for any constant $\beta \in \mathbb{R}$ and compact set $K \subseteq \Omega$, it follows that

$$\rho_1 \stackrel{\text{def}}{=} \max_{x\in K} \rho(x) \geq \min_{x\in K} \rho(x) \stackrel{\text{def}}{=} \rho_0 > 0,$$

thus,

$$\int_K \rho^\beta \, \mathrm{d}v_g \leq \max\left\{\rho_0^\beta, \rho_1^\beta\right\} \mathrm{Vol}_g(K) < \infty.$$

In particular, we have $\rho^{\alpha-p}, \rho^\alpha \in L^1_{\mathrm{loc}}(\Omega)$; moreover, relations (2.1) & (2.2) imply that $\Delta_{g,p}\rho = \Delta_g \rho$ $\mathrm{d}v_g$-a.e. in Ω, for every $p > 1$.

(c) Let (M, g) be an n-dimensional complete Riemannian manifold with *non-negative sectional curvature* and N be a minimal closed submanifold of M. If

$$\rho(x) = d_{g,N}(x) = \inf_{y\in N} d_g(x,y), \qquad \forall x \in M,$$

then ρ is superharmonic, that is $-\Delta_g \rho \geq 0$ $\mathrm{d}v_g$-a.e. in M (see Chen, Leung, and Zhao [25, Theorem 2.8]). Alternatively, if (M, g) is a complete Riemannian manifold with *non-negative Ricci curvature* and $\Omega \subseteq M$ is a domain with non-empty piecewise smooth *weakly mean convex* boundary, then $\rho(x) = d_{g,\partial\Omega}(x)$ is superharmonic a.e. on Ω (see Chen, Leung, and Zhao [25, Corollary 2.9]). In particular, in the Euclidean setting, the above statement can be reversed as well (see Lewis, Li, and Li [77]). Related studies on Finsler-type Minkowski spaces can be found in Della Pietra, di Blasio, and Gavitone [33].

An immediate consequence of Theorem 3.9 is the estimate of the first Dirichlet eigenvalue of the p-Laplace–Beltrami operator, i.e., of the p-fixed membrane problem (see Section 2.3.1). For simplicity, we consider the unweighted case ($\alpha = 0$):

Corollary 3.11 (see [59]). *Let (M, g) be a complete, non-compact $n \geq 2$-dimensional Riemannian manifold, $\Omega \subseteq M$ be a bounded domain, $p > 1$, and $\rho(x) = d_{g,\partial\Omega}(x)$ for every $x \in \Omega$. If $-\Delta_g \rho \geq 0$ in the distributional sense in Ω, then the first Dirichlet eigenvalue of the Riemannian p-Laplacian can be estimated as*

$$\Lambda_{\mathrm{m},p}(\Omega) = \inf_{u\in C_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla_g u|^p \, \mathrm{d}v_g}{\int_\Omega |u|^p \, \mathrm{d}v_g} \geq \left(\frac{p-1}{p}\right)^p \frac{1}{R_\Omega^p},$$

where $R_\Omega = \sup_{x\in\Omega} \rho(x)$ is the Riemannian-inradius *of the domain $\Omega \subseteq M$.*

Using the notation $R_\Omega = \sup_{x\in\Omega} \rho(x)$ from Corollary 3.11, in the spirit of Brezis and Marcus [17] and Barbatis, Filippas, and Tertikas [9], we provide an improvement of Theorem 3.9 with a suitable reminder term, whenever $1 < p \leq 2$, as follows.

Theorem 3.12 (see [59]). *Under the assumptions of Corollary 3.11, if $1 < p \leq 2$, then one has*

$$\int_\Omega |\nabla_g u|^p \, \mathrm{d}v_g \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{\rho^p}(1 + \mathcal{R}_p(\rho)) \, \mathrm{d}v_g, \tag{3.17}$$

for every $u \in C_0^\infty(\Omega)$, where

$$\mathcal{R}_p(t) = \left(1 + \log^{-1}\left(\frac{t}{eR_\Omega}\right)\right)^{p-2}\left(1 + (2-p)\log^{-1}\left(\frac{t}{eR_\Omega}\right) + \log^{-2}\left(\frac{t}{eR_\Omega}\right)\right) - 1 \geq 0,$$

for every $t \in (0, R_\Omega)$. In particular, if $p = 2$, then we have

$$\int_\Omega |\nabla_g u|^2 \, \mathrm{d}v_g \geq \frac{1}{4}\int_\Omega \frac{u^2}{\rho^2} \, \mathrm{d}v_g + \frac{1}{4}\int_\Omega \frac{u^2}{\rho^2} \log^{-2}\left(\frac{\rho}{eR_\Omega}\right) \mathrm{d}v_g. \tag{3.18}$$

Proof. In Theorem 3.1/(i) we choose the functions $w \equiv 1$, as well as

$$G(t) = -\left(\frac{p-1}{p}\right)^{p-1} t^{1-p}\left(1 + \log^{-1}\left(\frac{t}{eR_\Omega}\right)\right)^{p-1}, \qquad \forall t \in (0, R_\Omega),$$

and

$$H(s) = \frac{|s|^p}{p}, \qquad \forall s \in \mathbb{R}.$$

Note that $\sup_\Omega \rho = R_\Omega$ and observe that $(\mathbf{G})_{\rho,w}$ clearly holds (see Remark 3.10/(b)).

An elementary computation shows the validity of (3.17). Observe that for every $t \in (0, R_\Omega)$, the reminder term satisfies $\mathcal{R}_p(t) \geq 0$, since

$$z \stackrel{\text{def}}{=} \log^{-1}\left(\frac{t}{eR_\Omega}\right) \in (-1, 0),$$

and due to the fact that if $p \in (1, 2]$, then the function

$$h(z) = (1+z)^{p-2}\left(1 + (2-p)z + z^2\right) - 1$$

is decreasing on $(-1, 0)$, with $\lim_{z\to 0} h(z) = 0$. □

3.2.2 Improved Hardy inequalities on Cartan–Hadamard manifolds

In the sequel, we demonstrate that our approach works perfectly on Cartan–Hadamard manifolds. Before doing this, the following L^p-Hardy inequality is stated on general Riemannian manifolds with certain constraints, which has been first established by Kombe and Özaydin [64, Theorem 2.1] for generic $p > 1$; the initial version for $p = 2$ is the celebrated result of Carron [20, Theorem 1.4].

Theorem 3.13 (see [59]). *Let (M, g) be a complete, non-compact $n \geq 2$-dimensional Riemannian manifold. Let $\alpha \in \mathbb{R}$, $p > 1$, and $\rho\colon M \to [0, \infty)$ be a function such that $\rho^{-1}(0) \subseteq M$ is compact, $|\nabla_g \rho| = 1$ and $\Delta_g \rho \geq \frac{C}{\rho}$ in the distributional sense for some $C > 0$ with the property that $C + 1 + \alpha > p > 1$. Then for every $u \in C_0^\infty(M \setminus \rho^{-1}(0))$ one has*

$$\int_M \rho^\alpha |\nabla_g u|^p \, \mathrm{d}v_g \geq \left(\frac{C + 1 + \alpha - p}{p}\right)^p \int_M \rho^\alpha \frac{|u|^p}{\rho^p} \, \mathrm{d}v_g. \tag{3.19}$$

Proof. Denote
$$q = C + 1 + \alpha - p > 0.$$
In Theorem 3.5 let us choose $\Omega = M \setminus \rho^{-1}(0)$, as well as
$$w(t) = t^\alpha, \quad L(t) = \frac{C}{t}, \quad W(t) = \left(\frac{q}{pt}\right)^p, \quad \text{and} \quad G(t) = \left(\frac{q}{pt}\right)^{p-1}, \qquad \forall t > 0.$$
We obtain that (L, W) is a (p, ρ, w)-Riccati pair in $(0, \infty) \supset (0, \sup_\Omega \rho)$ and G is admissible for (L, W). Indeed, the validity of **(c1)** reduces to $\rho^{\alpha-p}, \rho^\alpha \in L^1_{\text{loc}}(\Omega)$, which is true since $\Omega = M \setminus \rho^{-1}(0)$, while Condition **(c2)** holds by assumption. Moreover, since $G > 0$ satisfies (3.5) with equality, i.e., one has
$$(G(t)w(t))' + G(t)w(t)L(t) + (1-p)|G(t)|^{p'}w(t) = W(t)w(t), \qquad \forall t > 0,$$
thus, **(c3)** holds as well. Therefore, applying Theorem 3.5 concludes the proof. □

Remark 3.14. We notice that if the p-capacity of the compact set $\rho^{-1}(0) \subseteq M$ is zero, then inequality (3.19) is valid not only in $C_0^\infty(M \setminus \rho^{-1}(0))$ but also in $C_0^\infty(M)$; see e.g., Carron [20] and D'Ambrosio and Dipierro [30]. In particular, if $n \geq p$ and $\mathcal{H}_g^{n-p}(\rho^{-1}(0)) < \infty$, then the p-capacity of $\rho^{-1}(0) \subseteq M$ is zero (see Heinonen, Kilpeläinen, and Martio [55]).

A simple consequence of Theorem 3.13 is the following weighted Hardy inequality.

Corollary 3.15 (see [59])**.** *Let (M, g) be an $n \geq 2$-dimensional Cartan–Hadamard manifold. Let $x_0 \in M$ be fixed and $p, \alpha \in \mathbb{R}$ such that $1 < p < n + \alpha$. Then for every $u \in C_0^\infty(M \setminus \{x_0\})$ one has*
$$\int_M d_{g,x_0}^\alpha |\nabla_g u|^p \, dv_g \geq \left(\frac{n+\alpha-p}{p}\right)^p \int_M d_{g,x_0}^\alpha \frac{|u|^p}{d_{g,x_0}^p} \, dv_g. \tag{3.20}$$
Moreover, the constant $\left(\frac{n+\alpha-p}{p}\right)^p$ is sharp.

Proof. We apply Theorem 3.13 with the following choices:
$$\rho = d_{g,x_0}, \quad \Omega = M \setminus \{x_0\}, \quad \text{and} \quad C = n - 1.$$
On the one hand, the eikonal equation (2.1) yields that dv_g-a.e. on Ω one has
$$|\nabla_g d_{g,x_0}| = 1.$$
On the other hand, by the Laplace comparison (see Theorem 2.1/(I)/(i)) we obtain
$$\Delta_g \rho \geq \frac{n-1}{\rho},$$
which concludes the proof.

The sharpness of the inequality and the lack of extremal functions can be shown in the usual way; see e.g., Yang, Su, and Kong [107, Theorem 3.1], Zhao [109, Theorem 1.2], and Kristály [67, Theorem 4.1]. □

Remark 3.16. (a) An alternative way to prove inequality (3.20) is to choose

$$\rho = d_{g,x_0}, \quad w(t) = t^\alpha, \quad G(t) = t^{1-p}, \quad \text{and} \quad H(s) = \frac{|s|^p}{p}, \qquad \forall t > 0,\ \forall s \in \mathbb{R},$$

in Theorem 3.1/(ii). The Laplace comparison (see Theorem 2.1/(I)/(i)) yields

$$\Delta_g \rho \geq \frac{n-1}{\rho}$$

in the distributional sense, and hence the following inequality holds:

$$\int_M \left[(G(\rho)w(\rho))'|\nabla_g \rho|^p + G(\rho)w(\rho)\Delta_{g,p}\rho\right] H(u)\, \mathrm{d}v_g \geq \frac{n+\alpha-p}{p} \int_M \rho^\alpha \frac{|u|^p}{\rho^p}\, \mathrm{d}v_g.$$

Using the relation

$$(G(\rho))^{p'} w(\rho)|H'(u)|^{p'} = \rho^\alpha \frac{|u|^p}{\rho^p},$$

and Theorem 3.1/(ii) concludes the proof.

(b) The inequality (3.20) has also been established for $\alpha = 0$ by D'Ambrosio and Dipierro [30, Theorem 6.5]. It was also discussed on special Finsler manifolds by Zhao [109, Theorem 1.2]. Further results on Finsler manifolds can also be found in Mester, Peter, and Varga [81].

(c) The proof of Theorem 3.1 can also be adapted to the compact case, where certain Hardy-type inequalities can be produced; such a result will be provided in the sequel. To see this, let (M, g) be an n-dimensional compact Riemannian manifold, $\Omega \subseteq M$ be a domain, and $x_0 \in \Omega$. Let $w\colon (0, \sup_\Omega \rho) \to (0, \infty)$ and $G\colon (0, \sup_\Omega \rho) \to \mathbb{R}$ be C^1 functions such that

$$(\mathbf{G})^{x_0}_{\rho,w} \ :\ G(\rho)w(\rho)^{\frac{1}{p'}} \in L^{p'}_{\mathrm{loc}}(\Omega\backslash\{x_0\}) \quad \text{and} \quad (G(\rho)w(\rho))', w(\rho) \in L^1_{\mathrm{loc}}(\Omega\backslash\{x_0\}).$$

In particular, Theorem 3.1 holds for every $u \in C_0^\infty(\Omega\backslash\{x_0\})$. Moreover, if $\mathbf{Ric} \geq 0$ and the parameters p and β verify

$$p \in (1, n) \cup (n, \infty), \quad \beta < -n, \quad \text{and} \quad p + \beta > -n,$$

then for every $u \in C^\infty(M, x_0) \overset{\text{def}}{=} \{u \in C^\infty(M) : u(x_0) = 0\}$ one has

$$\int_M |\nabla_g u|^p d_{g,x_0}^{p+\beta}\, \mathrm{d}v_g \geq \left(\frac{|n+\beta|}{p}\right)^p \int_M |u|^p d_{g,x_0}^\beta\, \mathrm{d}v_g. \tag{3.21}$$

The proof goes as follows: Let $\Omega = M$, $\rho = d_{g,x_0}$, as well as

$$w(t) = t^{p+\beta}, \quad G(t) = -t^{1-p} \quad \text{and} \quad H_c(s) = c|s|^p, \qquad \forall t > 0,\ \forall s \in \mathbb{R},$$

and for $c > 0$ chosen later. Clearly, $(\mathbf{G})^{x_0}_{\rho,w}$ holds. Thus, for every $u \in C_0^\infty(M\backslash\{x_0\})$, by Theorem 3.1/(i) and Theorem 2.1/(II)/(i) we have

$$\int_M |\nabla_g u|^p d_{g,x_0}^{p+\beta}\, \mathrm{d}v_g \geq \left(c|\beta+n| + (1-p)c^{p'}\right) \int_M |u|^p d_{g,x_0}^\beta\, \mathrm{d}v_g.$$

It is easy to check that

$$\max_{c>0}\left(c|\beta+n|+(1-p)c^{p'}\right)=\left(\frac{|n+\beta|}{p}\right)^p,$$

which implies that (3.21) holds for every $u\in C_0^\infty(M\backslash\{x_0\})$. By the theory of capacity (see e.g., Meng, Wang, and Zhao [80, Lemma 3.2]), every $u\in C^\infty(M,x_0)$ can be approximated by a sequence of functions belonging to $C_0^\infty(M\backslash\{x_0\})$ and hence (3.21) follows. The sharpness of $(|n+\beta|/p)^p$ also follows from [80].

When $\alpha=0$ in Corollary 3.15, the limit case $p=n$ does not provide any reasonable inequality similar to (3.20). In the next result we prove a parameter-dependent Hardy inequality with logarithmic weights, which is valid also in the limit case $p=n$; similar results were established by Edmunds and Triebel [37] in the Euclidean case, as well as by D'Ambrosio and Dipierro [30, Theorem 6.5], Nguyen [88], and Zhao [109, Theorem 1.3] on Riemannian/Finsler manifolds.

Theorem 3.17 (see [59]). *Let (M,g) be an $n\geq 2$-dimensional Cartan–Hadamard manifold, $x_0\in M$ be a fixed point, $\Omega=B_{g,x_0}(1)$, and $\alpha,p\in\mathbb{R}$ such that $1<p\leq n$ and $\alpha+1<p$. Then for every $u\in C_0^\infty(\Omega\setminus\{x_0\})$ one has*

$$\int_\Omega \log^\alpha(1/d_{g,x_0})|\nabla_g u|^p\,\mathrm{d}v_g\geq\left(\frac{p-\alpha-1}{p}\right)^p\int_\Omega\log^{\alpha-p}(1/d_{g,x_0})\frac{|u|^p}{d_{g,x_0}^p}\,\mathrm{d}v_g. \tag{3.22}$$

Moreover, the constant $\left(\frac{p-\alpha-1}{p}\right)^p$ is sharp.

Proof. Define

$$f(t)=t\log(1/t)>0,\qquad\forall t\in(0,1).$$

In Theorem 3.5, let us choose $\rho=d_{g,x_0}$ on the punctured ball $\Omega\setminus\{x_0\}$, as well as

$$w(t)=\log^\alpha(1/t),\quad L(t)=\frac{p-1}{t},\quad W_C(t)=Cf(t)^{-p},\quad\text{and}\quad G_c(t)=cf(t)^{1-p},$$

for all $t\in(0,1)$ and for some $C,c>0$, which will be determined later. Observe that $f>0$ implies $W_C,G_c>0$. On the one hand, Condition **(c1)** clearly holds on $\Omega\setminus\{x_0\}$. Since $p\leq n$, the Laplace comparison (see Theorem 2.1/(I)/(i)) implies $\Delta_{g,p}\rho\geq L(\rho)$, thus **(c2)** holds as well. On the other hand, a simple computation yields

$$G_c'(t)+\left(\frac{w'(t)}{w(t)}+L(t)\right)G_c(t)+(1-p)G_c(t)^{p'}=\left(c(p-\alpha-1)+(1-p)c^{p'}\right)f(t)^{-p},$$

for every $t\in(0,1)$. Hence, choosing

$$C=c(p-\alpha-1)+(1-p)c^{p'}$$

verifies the ODI (3.5) from Condition **(c3)** with equality. We obtain that (L,W_C) is a (p,ρ,w)-Riccati pair in $(0,1)$ and Theorem 3.5 implies for every $u\in C_0^\infty(\Omega)$ that

$$\int_\Omega\log^\alpha(1/d_{g,x_0})|\nabla_g u|^p\,\mathrm{d}v_g\geq C\int_\Omega\log^{\alpha-p}(1/d_{g,x_0})\frac{|u|^p}{d_{g,x_0}^p}\,\mathrm{d}v_g.$$

The best constant C is

$$\max_{c>0}\left(c(p-\alpha-1)+(1-p)c^{p'}\right)=\left(\frac{p-\alpha-1}{p}\right)^p,$$

and it is achieved for $c=((p-\alpha-1)/p)^{p-1}$.

The sharpness of the inequality is proved by Zhao [109, Theorem 1.3]. $\square$

Since the inequality in Theorem 3.15 is sharp, but no nonzero extremals exist, one can expect improvements even on Riemannian manifolds; see e.g., Berchio, D'Ambrosio, Ganguly, and Grillo [10], Berchio, Ganguly, and Roychowdhury [13], Flynn, Lam, and Lu [46], and Kristály [67].

In the sequel, we provide an alternative approach to establish improved Hardy inequalities on Cartan–Hadamard manifolds. For simplicity of presentation, we shall consider the case when $p=2$ and $\alpha=0$. The first such result 'interpolates' between Corollary 3.15 and Theorem 3.17 (see Adimurthi, Chaudhuri, and Ramaswamy [1]).

Theorem 3.18 (see [59])**.** *Let (M,g) be an $n\geq 3$-dimensional Cartan–Hadamard manifold, and $\Omega\subseteq M$ be a bounded domain. Let $x_0\in\Omega$ and $D_\Omega=\sup_{x\in\Omega}d_g(x_0,x)$. Then for every $u\in C_0^\infty(\Omega)$ one has*

$$\int_\Omega|\nabla_g u|^2\,\mathrm{d}v_g\geq\frac{(n-2)^2}{4}\int_\Omega\frac{u^2}{d_{g,x_0}^2}\,\mathrm{d}v_g+\frac{1}{4}\int_\Omega\log^{-2}\left(\frac{d_{g,x_0}}{eD_\Omega}\right)\frac{u^2}{d_{g,x_0}^2}\,\mathrm{d}v_g. \tag{3.23}$$

Proof. We apply Theorem 3.5 for the set $\Omega\setminus\{x_0\}$, and the positive functions $w\equiv 1$ and $\rho=d_{g,x_0}$ on $\Omega\setminus\{x_0\}$ with $\operatorname{Im}\rho\subseteq(0,D_\Omega)$, as well as

$$L(t)=\frac{n-1}{t}\quad\text{and}\quad W_C(t)=\frac{(n-2)^2}{4t^2}+\frac{C}{t^2}\log^{-2}\left(\frac{t}{eD_\Omega}\right),\qquad\forall t\in(0,D_\Omega),$$

and for some $C>0$, which will be defined later. First, by the Laplace comparison (see Theorem 2.1/(I)/(i)), one has that $\Delta_g\rho\geq L(\rho)$ on $\Omega\setminus\{x_0\}$, thus **(c2)** holds. Next, we intend to guarantee that (L,W_C) is a $(2,\rho,1)$-Riccati pair in $(0,D_\Omega)$. To do this, we are looking for a positive function G_C, which solves the Riccati-type ODI (3.8) with equality, i.e., for which one has

$$G_C'(t)+L(t)G_C(t)-G_C(t)^2=W_C(t),\qquad\forall t\in(0,D_\Omega).$$

The fundamental solution of the latter equation is given by

$$G_C(t)=\frac{n-2}{2t}+\frac{1-\sqrt{1-4C}}{2t\log(\frac{eD_\Omega}{t})},\qquad\forall t\in(0,D_\Omega).$$

The function G_C is well-defined and positive whenever $C\leq\frac{1}{4}$. Clearly, we choose the largest possible value for C, i.e., $C=\frac{1}{4}$, which yields **(c3)**. Finally, with this choice of C, it turns out that $(\mathbf{G})_{\rho,w}$ is verified on $\Omega\setminus\{x_0\}$, thus **(c1)** holds as well. Therefore, $(L,W_{1/4})$ is a $(2,\rho,1)$-Riccati pair in $(0,D_\Omega)$, and Theorem 3.5 provides the proof of (3.23) for functions belonging to $C_0^\infty(\Omega\setminus\{x_0\})$. Since $\rho^{-1}(0)=\{x_0\}$, by Remark 3.14 we have the validity of (3.23) for every function in $C_0^\infty(\Omega)$, which concludes the proof. $\square$

Another relevant improvement of the Hardy inequality in $\mathbb{R}^n$ is due to Brezis and Vázquez [18, Theorem 4.1]; more precisely, if $\Omega \subseteq \mathbb{R}^n$ is a bounded domain ($n \geq 2$), then for every $u \in C_0^\infty(\Omega)$ one has

$$\int_\Omega |\nabla u|^2 \, dx \geq \frac{(n-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} \, dx + j_{0,1}^2 \left(\frac{\omega_n}{\mathrm{Vol}(\Omega)} \right)^{\frac{2}{n}} \int_\Omega u^2 \, dx, \tag{3.24}$$

where $j_{0,1} \approx 2.4048$ is the first positive root of the Bessel function J_0, and ω_n is the volume of the unit Euclidean ball. Inequality (3.24) has been obtained by Schwarz symmetrization and an ingenious 1-dimensional analysis.

In the sequel, by using our approach, we provide a Riemannian version of the result by Brezis and Vázquez [18], which sheds new light on the appearance of $j_{0,1}$ in inequality (3.24). In fact, we prove another kind of interpolation inequality; see also Remark 3.20. As before, let $j_{\nu,k}$ be the k^{th} positive root of the Bessel function J_ν of the first kind and order $\nu \in \mathbb{R}$.

Theorem 3.19 (see [59]). *Let (M, g) be an $n \geq 2$-dimensional Cartan–Hadamard manifold and $\Omega \subseteq M$ be a bounded domain. Let $x_0 \in \Omega$ and $D_\Omega = \sup_{x \in \Omega} d_{g,x_0}(x)$. Then for every $\nu \in \left[0, \frac{n-2}{2}\right]$ and $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega |\nabla_g u|^2 \, dv_g \geq \left(\frac{(n-2)^2}{4} - \nu^2 \right) \int_\Omega \frac{u^2}{d_{g,x_0}^2} \, dv_g + \frac{j_{\nu,1}^2}{D_\Omega^2} \int_\Omega u^2 \, dv_g. \tag{3.25}$$

Proof. We intend to apply Theorem 3.5 for the set $\Omega \setminus \{x_0\}$ and for the functions $w \equiv 1$, $\rho = d_{g,x_0} > 0$, as well as

$$L(t) = \frac{n-1}{t} \quad \text{and} \quad W_C(t) = \frac{1}{t^2} \left(\frac{(n-2)^2}{4} - \nu^2 \right) + C > 0, \qquad \forall t \in (0, D_\Omega),$$

and for some $C > 0$, which will be determined later.

We clearly have $\Delta_g \rho \geq L(\rho)$ on $\Omega \setminus \{x_0\}$, which implies (**c2**). The fundamental solution of the ODE

$$G_C'(t) + L(t) G_C(t) - G_C(t)^2 = W_C(t), \qquad \forall t \in (0, D_\Omega),$$

which is precisely the equality case of (3.8), is given by the function

$$G_C(t) = \frac{n-2-2\nu}{2t} + \sqrt{C} \frac{J_{\nu+1}(\sqrt{C} t)}{J_\nu(\sqrt{C} t)}, \qquad \forall t \in (0, D_\Omega).$$

Accordingly, G_C is well-defined whenever $J_\nu(\sqrt{C} t) \neq 0$ for every $t \in (0, D_\Omega)$; thus, we need $\sqrt{C} D_\Omega \leq j_{\nu,1}$. Clearly, we choose the largest possible value $C = j_{\nu,1}^2 / D_\Omega^2$.

We show that G_C is positive on $(0, D_\Omega)$. Indeed, since $n - 2 - 2\nu > 0$, it is enough to show that $J_{\nu+1}(t)/J_\nu(t) \geq 0$, for every $t \in (0, j_{\nu,1})$, which directly follows from the Mittag–Leffler representation (see Watson [104, p. 498]):

$$\frac{J_{\nu+1}(t)}{J_\nu(t)} = \sum_{k=1}^\infty \frac{2t}{j_{\nu,k}^2 - t^2} > 0, \qquad \forall t \in (0, j_{0,1}). \tag{3.26}$$

Simple reasoning shows that $(\mathbf{G})_{\rho,w}$ holds on $\Omega \setminus \{x_0\}$ for the above choice of G_C, thus **(c1)** & **(c3)** are verified and (L, W_C) is a $(2, \rho, 1)$-Riccati pair in $(0, D_\Omega)$.

Now, we are in the position to apply Theorem 3.5 obtaining

$$\int_\Omega |\nabla_g u|^2 \, \mathrm{d}v_g \geq \int_\Omega u^2 W_C(\rho) \, \mathrm{d}v_g = \left(\frac{(n-2)^2}{4} - \nu^2 \right) \int_\Omega \frac{u^2}{d_{g,x_0}^2} \, \mathrm{d}v_g + \frac{j_{\nu,1}^2}{D_\Omega^2} \int_\Omega u^2 \, \mathrm{d}v_g,$$

for all $u \in C_0^\infty(\Omega \setminus \{x_0\})$. Since $\rho^{-1}(0) = \{x_0\}$, using an argument similar to that in Remark 3.14 implies the validity of (3.25) for every function in $C_0^\infty(\Omega)$. □

Several comments are in order.

Remark 3.20. (a) The Brezis–Vázquez inequality (3.24) in the Euclidean space $\mathbb{R}^n$ contains the constant

$$\left(\frac{\omega_n}{\mathrm{Vol}(\Omega)} \right)^{2/n}$$

instead of D_Ω^{-2}. This fact is explained by *Schwarz symmetrization*, rearranging the sets and functions into radially symmetric objects. Indeed, under symmetrization, the Dirichlet energy functional $u \mapsto \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x$ non-increases (Pólya–Szegő inequality), the L^2-norm $u \mapsto \int_\Omega u(x)^2 \, \mathrm{d}x$ is preserved (Cavalieri principle), while the Hardy term $u \mapsto \int_\Omega u(x)^2/|x|^2 \, \mathrm{d}x$ non-decreases (Hardy–Littlewood–Pólya inequality). In this way, the problem reduces to a *ball* with volume $\mathrm{Vol}(\Omega)$ and radially symmetric functions; thus a 1-dimensional analysis suffices.

These arguments can be partially repeated in Cartan–Hadamard manifolds. In this case, the Pólya–Szegő inequality is valid whenever the *Cartan–Hadamard conjecture* holds (in dimensions 2, 3, and 4; see Hebey [54]), and the L^2-norm is preserved; however, nothing is known about the behavior of the Hardy term $u \mapsto \int_\Omega u^2/d_{g,x_0}^2 \, \mathrm{d}v_g$ under symmetrization.

(b) We note that in the 'complementary' geometric setting, i.e., on Riemannian or Finsler manifolds with non-negative Ricci curvature, the Hardy term non-decreases under symmetrization, that is, a Hardy–Littlewood–Pólya-type inequality holds; see Kristály, Mester, and Mezei [71] and Kristály and Szakál [74]. In this way, Euclidean spaces can be viewed as *threshold* geometric structures of the symmetric rearrangement of Hardy terms.

(c) Note that if in Theorem 3.19 we consider the ball $\Omega = B_{g,x_0}(R)$ for some $R > 0$, then by the Bishop–Gromov volume comparison principle (see Theorem 2.1/(I)/(ii)), we have that

$$D_\Omega^{-2} = R^{-2} \geq \left(\frac{\omega_n}{\mathrm{Vol}_g(\Omega)} \right)^{2/n},$$

which is in a perfect concordance with the original inequality (3.24) containing the volume of the set.

(d) Inequality (3.25) trivially interpolates between the inequality of Brezis and Vázquez ($\nu = 0$) and the celebrated Faber–Krahn inequality ($\nu = \frac{n-2}{2}$). In the latter case, we obtain a spectral estimate for the first Dirichlet eigenvalue on a domain Ω. More details are provided in the next section.

3.2.3 Spectral estimates on Riemannian manifolds

Let (M, g) be an n-dimensional Riemannian manifold with $n \geq 2$. Let $\Omega \subseteq M$ be a domain and $p > 1$. The *first Dirichlet eigenvalue* of Ω for the p-Laplace–Beltrami operator $-\Delta_{g,p}$ on (M, g) is given by

$$\Lambda_{\mathrm{m},p}(\Omega) = \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla_g u|^p \,\mathrm{d}v_g}{\int_\Omega |u|^p \,\mathrm{d}v_g}.$$

The above quantity can also be referred to as the first eigenvalue of the fixed membrane problem, see Section 2.3.1.

In this section, we present the applicability of our method to obtain spectral gap estimates on Riemannian manifolds. First, we prove *Cheng's comparison principle* (see Cheng [28]), whose original proof is based on Barta's argument. Next, we prove the *Faber–Krahn inequality* on Cartan–Hadamard manifolds (see Chavel [24]), as well as *McKean's spectral gap estimate* (see McKean [79]). Finally, we provide a short proof of the main result of Carvalho and Cavalcante [21, Theorem 1.1] concerning the lower bound of $\Lambda_{\mathrm{m},p}$ for the p-Laplacian on general Riemannian manifolds.

Theorem 3.21 (see [59]). *Let (M, g) be an n-dimensional Riemannian manifold with $n \geq 2$ and sectional curvature $\mathbf{K} \leq \kappa$ for some $\kappa \in \mathbb{R}$. Fix $x_0 \in M$ and suppose that $0 < R < \min(\mathrm{inj}_{x_0}, \pi/\sqrt{\kappa})$ with the usual convention (2.5). Then one has*

$$\Lambda_{\mathrm{m},2}(B_{g,x_0}(R)) \geq \Lambda_{\mathrm{m},2}(B_\kappa(R)), \tag{3.27}$$

where $B_\kappa(R)$ is an arbitrary ball of radius R in the model space form $\mathbf{M}_\kappa^n$.

Proof. Using standard compactness and symmetrization arguments, we know that $\Lambda_{\mathrm{m},2}(B_\kappa(R))$ is achieved by a positive, radially symmetric, non-increasing function $v \in W_0^{1,2}(B_\kappa(R))$, such that $v_\kappa = v(d_\kappa(\overline{x}_0, \cdot))$ is a non-increasing profile function of v, where $\overline{x}_0$ is the center of $B_\kappa(R)$. By the Euler–Lagrange equation, it turns out that

$$v_\kappa''(t) + (n-1)\mathbf{ct}_\kappa(t)v_\kappa'(t) + \Lambda_{\mathrm{m},2}(B_\kappa(R))v_\kappa(t) = 0, \qquad \forall t \in (0, R), \tag{3.28}$$

which can be written equivalently in the form

$$G_\kappa'(t) + L(t)G_\kappa(t) - G_\kappa^2(t) = W(t), \qquad \forall t \in (0, R), \tag{3.29}$$

where

$$L(t) = (n-1)\mathbf{ct}_\kappa(t), \quad W \equiv \Lambda_{\mathrm{m},2}(B_\kappa(R)), \quad \text{and} \quad G_\kappa(t) = -\frac{v_\kappa'(t)}{v_\kappa(t)}, \qquad \forall t \in (0, R).$$

Note that $G_\kappa \geq 0$ on $(0, R)$, since v_κ is non-increasing and positive in $(0, R)$. Define $\Omega = B_{g,x_0}(R)$ and $\rho = d_{g,x_0}$. The Laplace comparison (see Theorem 2.1/(I)/(i)) implies that $\Delta_g \rho \geq L(\rho)$ on $\Omega \setminus \{x_0\}$. Since $(\mathbf{G})_{\rho,w}$ also holds, we obtain that (L, W) is a $(2, \rho, 1)$-Riccati pair in $(0, R)$ and G_κ is admissible for (L, W). By Theorem 3.5 one has for every $u \in C_0^\infty(\Omega \setminus \{x_0\})$ that

$$\int_\Omega |\nabla_g u|^2 \,\mathrm{d}v_g \geq \Lambda_{\mathrm{m},2}(B_\kappa(R)) \int_\Omega u^2 \,\mathrm{d}v_g.$$

On account of Remark 3.14, this inequality is valid for every function in $C_0^\infty(\Omega)$, which concludes the proof of (3.27). $\square$

Remark 3.22. We notice that (3.28) is also equivalent to

$$\left(\mathbf{s}_\kappa^{n-1}(t)v_0'(t)\right)' + \Lambda_{\mathrm{m},2}(B_\kappa(R))\mathbf{s}_\kappa^{n-1}(t)v_0(t) = 0, \qquad \forall t \in (0,R),$$

which is in perfect concordance with equation (3.15) from Remark 3.8.

In the case when the domain is not a ball, as in Cheng's result, a more powerful argument is needed. We shall consider only the case when $\kappa = 0$, which corresponds to the famous Faber–Krahn inequality on Cartan–Hadamard manifolds:

Theorem 3.23 (see [59]). *Let (M,g) be an $n \geq 2$-dimensional Cartan–Hadamard manifold, which satisfies the Cartan–Hadamard conjecture, and let $\Omega \subseteq M$ be a bounded domain. Then we have*

$$\Lambda_{\mathrm{m},2}(\Omega) \geq \Lambda_{\mathrm{m},2}(\Omega^*) = j^2_{\frac{n}{2}-1,1}\left(\frac{\omega_n}{\mathrm{Vol}_g(\Omega)}\right)^{2/n}, \tag{3.30}$$

where $\Omega^* \subseteq \mathbb{R}^n$ is a ball with $\mathrm{Vol}(\Omega^*) = \mathrm{Vol}_g(\Omega)$.

Proof. Let $u \in C_0^\infty(\Omega) \setminus \{0\}$ be arbitrarily fixed; without loss of generality, we may assume that $u \geq 0$. Let $u^* \colon \Omega^* \to [0,\infty)$ be the symmetric rearrangement of u on the Euclidean ball $\Omega^* \subseteq \mathbb{R}^n$. Without loss of generality, we may assume that its center is the origin O (see Hebey [54]). By the layer cake representation and the validity of the Cartan–Hadamard conjecture on (M,g), both the Cavalieri principle and Pólya–Szegő inequality hold, thus

$$\frac{\int_\Omega |\nabla_g u|^2 \,\mathrm{d}v_g}{\int_\Omega u^2 \,\mathrm{d}v_g} \geq \frac{\int_{\Omega^*} |\nabla u^*|^2 \,\mathrm{d}x}{\int_{\Omega^*} (u^*)^2 \,\mathrm{d}x}.$$

Observe that the right-hand side falls into the setting of Theorem 3.21 for $\kappa = 0$. Therefore, if we choose

$$R = (\mathrm{Vol}_g(\Omega)/\omega_n)^{1/n}, \quad \Omega^* = B_O(R), \quad \text{and} \quad W_C \equiv C = \Lambda_{\mathrm{m},2}(B_O(R)),$$

in equation (3.29) for $\kappa = 0$, the fundamental solution is

$$G_0(t) = \sqrt{C}\frac{J_{\frac{n}{2}}(\sqrt{C}t)}{J_{\frac{n}{2}-1}(\sqrt{C}t)}, \qquad \forall t \in (0,R).$$

A similar argument as in the proof of Theorem 3.19 shows that the best choice is

$$C = \Lambda_{\mathrm{m},2}(B_O(R)) = \frac{j^2_{\frac{n}{2}-1,1}}{R^2}.$$

Clearly, G_0 is positive on $(0,R)$ and $(\mathbf{G})_{\rho,w}$ holds; therefore, (L, W_C) is a $(2,\rho,1)$-Riccati pair in $(0,R)$. The rest follows similarly as in the proof of Theorem 3.21. □

The following result is McKean's spectral gap estimate, established by McKean [79] for $p = 2$ by using fine properties of Jacobi fields; our argument is based on Riccati pairs.

Theorem 3.24 (see [59]). *Let (M, g) be an $n \geq 2$-dimensional Cartan–Hadamard manifold, with sectional curvature $\mathbf{K} \leq \kappa < 0$. If $p > 1$, then*

$$\Lambda_{\mathrm{m},p}(M) \geq \left(\frac{n-1}{p}\sqrt{-\kappa}\right)^p. \tag{3.31}$$

Proof. Let $x_0 \in M$ be arbitrarily fixed, and choose $\rho = d_{g,x_0} > 0$ on $\Omega = M \setminus \{x_0\}$, as well as

$$w \equiv 1, \quad L \equiv (n-1)\sqrt{-\kappa}, \quad W_C \equiv C, \quad \text{and} \quad G_c \equiv c,$$

for some $C, c > 0$ that will be determined later.

Clearly, (c1) holds. The Laplace comparison (see Theorem 2.1/(I)/(i)) yields

$$\Delta_g \rho \geq (n-1)\mathbf{ct}_\kappa(\rho) = (n-1)\sqrt{-\kappa}\coth(\sqrt{-\kappa}\rho) \geq (n-1)\sqrt{-\kappa} = L(\rho),$$

thus, (c3) is verified by the choice

$$C = (n-1)c\sqrt{-\kappa} + (1-p)c^{p'}.$$

Indeed, in this case, the ODI (3.7) is verified with equality, that is

$$G_c'(t) + L(t)G_c(t) + (1-p)G_c(t)^{p'} \geq (n-1)c\sqrt{-\kappa} + (1-p)c^{p'} = C = W_C(t),$$

for all $t > 0$. Thus, (L, W_C) is a $(p, \rho, 1)$-Riccati pair in $(0, \infty)$. The best constant C is

$$\max_{c>0}\left((n-1)c\sqrt{-\kappa} + (1-p)c^{p'}\right) = \left(\frac{n-1}{p}\sqrt{-\kappa}\right)^p,$$

and it is achieved for

$$c = \left(\frac{n-1}{p}\sqrt{-\kappa}\right)^{p-1}.$$

It remains to apply Theorem 3.5 to conclude the proof of (3.31). □

Remark 3.25. (a) Another proof of (3.31) can be given using the multiplicative form from Theorem 3.1/(ii). Indeed, fix $x_0 \in M$ and choose

$$w \equiv 1, \quad \rho = d_{g,x_0}, \quad G \equiv 1, \quad \text{and} \quad H(s) = \frac{|s|^p}{p}, \qquad \forall s \in \mathbb{R}.$$

The Laplace comparison (see Theorem 2.1/(I)/(i)) implies that

$$\Delta_g \rho \geq (n-1)\mathbf{ct}_\kappa(\rho) \geq (n-1)\sqrt{-\kappa}.$$

Thus, the inequality (3.31) follows at once.

(b) The constant $(\frac{n-1}{p}\sqrt{-\kappa})^p$ in (3.31) is sharp for the hyperbolic space $\mathbb{H}^n_\kappa$, more precisely (similarly to relation (2.36)) one has

$$\Lambda_{\mathrm{m},p}(\mathbb{H}^n_\kappa) = \lim_{R\to\infty}\Lambda_{\mathrm{m},p}(B_\kappa(R)) = \left(\frac{n-1}{p}\sqrt{-\kappa}\right)^p.$$

(c) Instead of (3.31), we can prove an improved version of McKean's spectral gap estimate. Indeed, if we fix $x_0 \in M$ and choose $w \equiv 1$ as well as

$$L(t) = (n-1)\mathbf{ct}_\kappa(t)$$
$$W(t) = \left(\frac{n-1}{p}\sqrt{-\kappa}\right)^p + \frac{(n-1)^p}{p^{p-1}}(\sqrt{-\kappa})^p(\coth(\sqrt{-\kappa}t)-1),$$

for all $t > 0$, it turns out that (L, W) is a $(p, \rho, 1)$-Riccati pair in $(0, \infty)$, and

$$G \equiv \left(\frac{n-1}{p}\sqrt{-\kappa}\right)^{p-1}$$

is admissible for (L, W). By Theorem 3.5 we have for every $u \in C_0^\infty(M)$ that

$$\int_M |\nabla_g u|^p \,\mathrm{d}v_g \geq \left(\frac{n-1}{p}\sqrt{-\kappa}\right)^p \int_M |u|^p \,\mathrm{d}v_g + 2\frac{(n-1)^p}{p^{p-1}}(\sqrt{-\kappa})^p \int_M \frac{|u|^p}{e^{2\sqrt{-\kappa}d_{g,x_0}}-1}\,\mathrm{d}v_g.$$

Another improvement of the McKean's spectral gap estimate will be provided in the next section.

We conclude this section by providing a short proof of the main result of Carvalho and Cavalcante [21, Theorem 1.1], which is valid on general Riemannian manifolds.

Theorem 3.26 (see [59])**.** *Let (M, g) be an n-dimensional Riemannian manifold, $n \geq 2$ and $\Omega \subseteq M$ be a domain. Given $p > 1$, we assume that there exists a function $\rho\colon \Omega \to \mathbb{R}$ such that $|\nabla_g \rho| \leq a$ and $\Delta_{g,p}\rho \geq b$ for some $a, b > 0$. Then*

$$\Lambda_{\mathrm{m},p}(\Omega) \geq \frac{b^p}{p^p a^{p(p-1)}}. \tag{3.32}$$

Proof. We apply the additive form of Theorem 3.1/(i) for the choices

$$w \equiv 1, \quad G \equiv c, \quad \text{and} \quad H(s) = \frac{|s|^p}{p}, \qquad \forall s \in \mathbb{R},$$

and for some $c > 0$, which will be determined later. Since all the assumptions of Theorem 3.1 are trivially verified, using (3.1) and the facts that

$$p > 1, \quad |\nabla_g \rho| \leq a, \quad \text{and} \quad \Delta_{g,p}\rho \geq b,$$

we obtain for every $u \in C_0^\infty(\Omega)$ that

$$\begin{aligned}\int_\Omega |\nabla_g u|^p \,\mathrm{d}v_g &\geq \int_\Omega |u|^p \left(c\Delta_{g,p}\rho + (1-p)c^{p'}|\nabla_g\rho|^p\right)\mathrm{d}v_g \\ &\geq \left(cb + (1-p)c^{p'}a^p\right)\int_\Omega |u|^p \,\mathrm{d}v_g.\end{aligned}$$

By simple computation

$$\max_{c>0}\left(cb + (1-p)c^{p'}a^p\right) = \frac{b^p}{p^p a^{p(p-1)}}$$

and it is achieved by

$$c = \left(\frac{b}{pa^p}\right)^{p-1},$$

hence the proof of inequality (3.32) is complete. □

3.2.4 Interpolation: Hardy inequality versus McKean spectral gap

The main result of this section is to prove an interpolation between the classical Hardy inequality and McKean's spectral gap, established first by Berchio, Ganguly, Grillo, and Pinchover [12, Theorem 2.1]. Although the authors stated their result on the hyperbolic space $\mathbb{H}^n_{-1}$, their approach also works on Cartan–Hadamard manifolds with sectional curvature $\mathbf{K} \leq \kappa < 0$. In the sequel, we provide here a simple proof of the same result by using our approach, based on Riccati pairs.

Theorem 3.27 (see [59]). *Let (M, g) be an $n \geq 3$-dimensional Cartan–Hadamard manifold, having sectional curvature $\mathbf{K} \leq \kappa < 0$, and $x_0 \in M$ be fixed. Then, for every $\lambda \in [n-2, \frac{(n-1)^2}{4}]$ and $u \in C_0^\infty(M \setminus \{x_0\})$ one has*

$$\begin{aligned}\int_M |\nabla_g u|^2 \, \mathrm{d}v_g \geq \lambda|\kappa| \int_M u^2 \, \mathrm{d}v_g + h_n^2(\lambda) \int_M \frac{u^2}{d_{g,x_0}^2} \, \mathrm{d}v_g \\ + |\kappa| \left(\frac{(n-2)^2}{4} - h_n^2(\lambda) \right) \int_M \frac{u^2}{\sinh^2(\sqrt{-\kappa} d_{g,x_0})} \, \mathrm{d}v_g \\ + h_n(\lambda)\gamma_n(\lambda) \int_M \frac{\mathbf{D}_\kappa(d_{g,x_0})}{d_{g,x_0}^2} u^2 \, \mathrm{d}v_g, \end{aligned} \tag{3.33}$$

where $\gamma_n(\lambda) = \sqrt{(n-1)^2 - 4\lambda}$ and $h_n(\lambda) = \frac{\gamma_n(\lambda)+1}{2}$.

Proof. Fix $\lambda \in [n-2, \frac{(n-1)^2}{4}]$. In Theorem 3.5 let us choose $\Omega = M \setminus \{x_0\}$, $w \equiv 1$, $\rho = d_{g,x_0} > 0$, as well as

$$\begin{aligned} L(t) &= (n-1)\mathbf{ct}_\kappa(t) = (n-1)\sqrt{-\kappa}\coth(\sqrt{-\kappa}t), \\ W_\lambda(t) &= \lambda|\kappa| + \frac{h_n^2(\lambda)}{t^2} + \frac{|\kappa|}{\sinh^2(\sqrt{-\kappa}t)} \left(\frac{(n-2)^2}{4} - h_n^2(\lambda) \right) + h_n(\lambda)\gamma_n(\lambda)\frac{\mathbf{D}_\kappa(t)}{t^2}, \end{aligned}$$

for all $t > 0$. It is immediate that W_λ is positive. The fundamental solution of the Riccati equation

$$G'(t) + L(t)G(t) - G(t)^2 = W_\lambda(t), \qquad \forall t > 0, \tag{3.34}$$

which is precisely the equality case of (3.8), is given by

$$G(t) = -\frac{h_n(\lambda)}{t} + \left(\frac{n-2}{2} + h_n(\lambda) \right) \mathbf{ct}_\kappa(t), \qquad \forall t > 0.$$

We easily observe that G is positive on $(0, \infty)$ and $(\mathbf{G})_{\rho,w}$ holds. Since

$$\Delta_g \rho \geq L(\rho)$$

by Theorem 2.1/(I)/(i), it turns out that (L, W_λ) is a $(2, \rho, 1)$-Riccati pair on $(0, \infty)$. Thus, Theorem 3.5 implies inequality (3.33) for every $u \in C_0^\infty(M \setminus \{x_0\})$. $\square$

A direct consequence of Theorem 3.27 can be stated as follows for the two marginal values of λ. More precisely, for $\lambda = n-2$ a Hardy improvement holds, while for $\lambda = \frac{(n-1)^2}{4}$ the McKean spectral gap is improved:

Corollary 3.28 (see [59]). *Let (M,g) be an $n \geq 3$-dimensional Cartan–Hadamard manifold with sectional curvature $\mathbf{K} \leq \kappa < 0$. If $x_0 \in M$ is fixed, then for every $u \in C_0^\infty(M \setminus \{x_0\})$ the following inequalities hold:*

(i) (Hardy improvement)

$$\int_M |\nabla_g u|^2 \,\mathrm{d}v_g \geq \frac{(n-2)^2}{4} \int_M \frac{u^2}{d_{g,x_0}^2} \,\mathrm{d}v_g + (n-2)|\kappa| \int_M u^2 \,\mathrm{d}v_g + \frac{(n-2)(n-3)}{2} \int_M \frac{\mathbf{D}_\kappa(d_{g,x_0})}{d_{g,x_0}^2} u^2 \,\mathrm{d}v_g. \tag{3.35}$$

(ii) (McKean spectral gap improvement)

$$\int_M |\nabla_g u|^2 \,\mathrm{d}v_g \geq \frac{(n-1)^2}{4}|\kappa| \int_M u^2 \,\mathrm{d}v_g + \frac{1}{4} \int_M \frac{u^2}{d_{g,x_0}^2} \,\mathrm{d}v_g + |\kappa| \frac{(n-1)(n-3)}{4} \int_M \frac{u^2}{\sinh^2(\sqrt{-\kappa}\, d_{g,x_0})} \,\mathrm{d}v_g. \tag{3.36}$$

Remark 3.29. (a) We note that inequality (3.35) appears in Berchio, Ganguly, Grillo, and Pinchover [12, Corollary 4.3]. Using this inequality and the estimate

$$t^2 \geq t \coth t - 1, \qquad \forall t > 0,$$

we obtain for every $u \in C_0^\infty(M \setminus \{x_0\})$ that (see Kristály [67, Theorem 4.1]):

$$\int_M |\nabla_g u|^2 \,\mathrm{d}v_g \geq \frac{(n-2)^2}{4} \int_M \frac{u^2}{d_{g,x_0}^2} \,\mathrm{d}v_g + \frac{(n-1)(n-2)}{2} \int_M \frac{\mathbf{D}_\kappa(d_{g,x_0})}{d_{g,x_0}^2} u^2 \,\mathrm{d}v_g.$$

Inequality (3.36) appears first in Akutagawa and Kumura [3, Theorem 1.3/(6)], which became a starting point of numerous further studies; see e.g., Berchio, Ganguly, and Grillo [11, Theorem 2.1], Berchio, Ganguly, Grillo, and Pinchover [12], and Flynn, Lam and Lu [45].

(b) It is worth mentioning that inequalities from Theorem 3.27 and Corollary 3.28 are known to be *critical* on $\mathbb{H}^n_{-1}$, i.e., the right-hand sides of the inequalities cannot be improved with weights that are strictly larger somewhere (see Devyver, Fraas, and Pinchover [34, Definition 2.1]).

Classical criticality proofs are usually formulated using the supersolutions approach; however, by Remarks 2.16 & 3.8, they can be adapted to Riccati pairs. In the sequel, we illustrate this through an example. Consider (3.36) on $\mathbb{H}^n_{-1}$ and recall the definition of W_λ from the proof of Theorem 3.27.

As we learned from [11], in order to prove the criticality, it is enough to show that the equation

$$\left(-\Delta_g - W_{\frac{(n-1)^2}{4}}\right)\varphi(x) = 0, \qquad \forall x \in \mathbb{H}^n_{-1} \setminus \{x_0\}, \tag{3.37}$$

admits two solutions $\varphi_\pm(x) = y_\pm(d_{g,x_0}(x))$ satisfying

$$\lim_{r\to\infty} \frac{y_+(r)}{y_-(r)} = 0.$$

Moreover, by Remark 2.16 it is enough to consider the ODE (3.15) from Remark 3.8. By simple computation we obtain that the following linearly independent functions meet the above conditions:

$$y_+(t) = \sqrt{t}\sinh(t)^{\frac{2-n}{2}} \quad \text{and} \quad y_-(t) = \sqrt{t}\log(t)\sinh(t)^{\frac{2-n}{2}}, \qquad \forall t > 0.$$

We note, however, that the above functions can be also obtained from two linearly independent solutions $G_\pm$ of the Riccati ODE (3.34) via the mechanism

$$G_\pm(t) = -\frac{y_\pm'(t)}{y_\pm(t)};$$

see Proposition 3.7. Using additional elements from criticality theory, one can also adapt the criticality proof of the general inequality (3.33).

(c) We notice that Berchio, Ganguly, and Grillo [11, Theorem 2.5] provided a more general version of the inequality (3.36) under a mild curvature assumption, which can be stated as follows. Let (M, g) be an n-dimensional Riemannian manifold ($n \geq 3$), $x_0 \in M$ be a point such that $\mathrm{Cut}_{x_0} = \emptyset$ (here, Cut_{x_0} stands for the cut locus of x_0), and the sectional curvature in the radial direction satisfies

$$\mathbf{K}_{\mathrm{rad}}(d_{g,x_0}(x)) \leq -\frac{\psi''(d_{g,x_0}(x))}{\psi(d_{g,x_0}(x))}, \qquad \forall x \in M, \tag{3.38}$$

where ψ is a positive, increasing C^2 function with $\psi(0) = \psi''(0) = 0$, $\psi'(0) = 1$ and

$$(n-2)\psi'(t) + (n-1)r\psi''(t) \geq 0, \qquad \forall t > 0. \tag{3.39}$$

Then for every $u \in C_0^\infty(M \setminus \{x_0\})$ one has

$$\begin{aligned}\int_M |\nabla_g u|^2 \,\mathrm{d}v_g \geq \frac{(n-1)}{4}\int_M \left(2\frac{\psi''(d_{g,x_0})}{\psi(d_{g,x_0})} + (n-3)\frac{(\psi'(d_{g,x_0})^2 - 1)}{\psi(d_{g,x_0})^2}\right) u^2 \,\mathrm{d}v_g \\ + \frac{1}{4}\int_M \frac{u^2}{d_{g,x_0}^2}\,\mathrm{d}v_g + \frac{(n-1)(n-3)}{4}\int_M \frac{u^2}{\psi^2(d_{g,x_0})}\,\mathrm{d}v_g.\end{aligned} \tag{3.40}$$

Observe that for $\kappa < 0$ and

$$\psi(t) = \frac{\sinh(\sqrt{-\kappa}t)}{\sqrt{-\kappa}},$$

inequality (3.40) reduces to (3.36).

As expected, inequality (3.40) can also be obtained via Riccati pairs. Indeed, the curvature assumption (3.38) implies the following refined Laplace comparison principle

$$\Delta d_{g,x_0} \leq (n-1)\frac{\psi'(d_{g,x_0})}{\psi(d_{g,x_0})},$$

see Greene and Wu [52]. Thus, inequality (3.40) follows from Theorem 3.5 by choosing

$$L(t) = (n-1)\frac{\psi'(t)}{\psi(t)} \quad \text{and} \quad G(t) = -\frac{1}{2t} + \frac{(n-1)}{2}\frac{\psi'(t)}{\psi(t)};$$

the positivity of G is guaranteed by the boundary conditions on ψ and relation (3.39).

This example shows that Riccati pairs can be efficiently used to extend various well-known Hardy inequalities to manifolds that satisfy – instead of a universal curvature bound – the *pointwise* curvature assumption (3.38).

We conclude this section with an alternative proof of an inequality by Akutagawa and Kumura [3, Theorem 1.3/(5)].

Theorem 3.30 (see [59]). *Let (M,g) be an $n \geq 2$-dimensional Cartan–Hadamard manifold with sectional curvature $\mathbf{K} \leq \kappa < 0$. Let $x_0 \in M$ be fixed, $R > 0$, and $\Omega = M \setminus B_{g,x_0}(R)$. Then for every and $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega |\nabla_g u|^2 \, dv_g \geq \int_\Omega \frac{(n-1)^2}{4}|\kappa| u^2 \, dv_g + \int_\Omega \frac{u^2}{4\left(d_{g,x_0} - R + \frac{1}{(n-1)\mathbf{ct}_\kappa(R)}\right)^2} \, dv_g$$
$$+ \int_\Omega |\kappa| \frac{(n-1)(n-3)}{4\sinh^2(\sqrt{-\kappa} d_{g,x_0})} u^2 \, dv_g.$$

Proof. The proof is similar to the one of Theorem 3.27, except for the choice of $W \colon (R,\infty) \to \mathbb{R}$, which is defined by

$$W(t) = \frac{(n-1)^2}{4}|\kappa| + \frac{1}{4\left(t - R + \frac{1}{(n-1)\mathbf{ct}_\kappa(R)}\right)^2} + |\kappa|\frac{(n-1)(n-3)}{4\sinh^2(\sqrt{-\kappa}t)}, \qquad \forall t > R.$$

One can check that the function

$$G(t) = -\frac{1}{2\left(t - R + \frac{1}{(n-1)\mathbf{ct}_\kappa(R)}\right)} + \frac{n-1}{2}\mathbf{ct}_\kappa(t), \qquad \forall t > R,$$

is positive and verifies both $(\mathbf{G})_{\rho,w}$ and the ODE

$$G'(t) + L(t)G(t) - G(t)^2 = W(t), \qquad \forall t > R,$$

which is precisely the equality case of (3.8), where

$$\rho = d_{g,x_0} \quad \text{and} \quad L(t) = (n-1)\mathbf{ct}_\kappa(t).$$

Thus, (L,W) is a $(2,\rho,1)$-Riccati pair on (R,∞), and Theorem 3.5 concludes the proof. $\square$

3.2.5 Ghoussoub–Moradifam-type inequalities

In this section, we provide an alternative proof of some inequalities established by Ghoussoub and Moradifam [50, Theorem 2.12] (see also [51]), where the weights are of the form

$$(a+b|x|^\alpha)^\beta/|x|^{2m}$$

for some parameters. Possible extensions of these inequalities to Cartan–Hadamard manifolds are also discussed; see Remark 3.32 and Theorem 3.33, where some technical difficulties arise.

Theorem 3.31 (see [59]). *Suppose that $a, b > 0$ and $\alpha, \beta, m \in \mathbb{R}$. The following statements hold:*

(i) *If $\alpha\beta > 0$ and $m \le \frac{n-2}{2}$, then for every $u \in C_0^\infty(\mathbb{R}^n)$ one has*

$$\int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m}} |\nabla u|^2 \,\mathrm{d}x \ge \left(\frac{n-2m-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m+2}} u^2 \,\mathrm{d}x. \tag{3.41}$$

(ii) *If $\alpha\beta < 0$ and $2m - \alpha\beta \le n-2$, then for every $u \in C_0^\infty(\mathbb{R}^n)$ one has*

$$\int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m}} |\nabla u|^2 \,\mathrm{d}x \ge \left(\frac{n-2m+\alpha\beta-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{(a+b|x|^\alpha)^\beta}{|x|^{2m+2}} u^2 \,\mathrm{d}x. \tag{3.42}$$

Proof. First, we prove that (i) and (ii) are equivalent, that is, they can be deduced from each other. Indeed, let us assume that (i) holds, i.e., by notation's convenience, if $\widetilde{\alpha}\widetilde{\beta} > 0$ and $\widetilde{m} \le \frac{n-2}{2}$, then for every $u \in C_0^\infty(\mathbb{R}^n)$ one has

$$\int_{\mathbb{R}^n} \frac{(b+a|x|^{\widetilde{\alpha}})^{\widetilde{\beta}}}{|x|^{2\widetilde{m}}} |\nabla u|^2 \,\mathrm{d}x \ge \left(\frac{n-2\widetilde{m}-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{(b+a|x|^{\widetilde{\alpha}})^{\beta}}{|x|^{2\widetilde{m}+2}} u^2 \,\mathrm{d}x. \tag{3.43}$$

Now, we fix $\alpha, \beta, m \in \mathbb{R}$ that verify the assumptions of (ii), namely,

$$\alpha\beta < 0 \quad \text{and} \quad 2m - \alpha\beta \le n-2.$$

Let us choose

$$\widetilde{\alpha} = -\alpha \quad \widetilde{\beta} = \beta \quad \text{and} \quad \widetilde{m} = m - \frac{\alpha\beta}{2}.$$

We observe that

$$\widetilde{\alpha}\widetilde{\beta} = -\alpha\beta > 0 \quad \text{and} \quad \widetilde{m} = m - \frac{\alpha\beta}{2} \le \frac{n-2+\alpha\beta}{2} - \frac{\alpha\beta}{2} = \frac{n-2}{2}.$$

Thus, we may apply the inequality (3.43) with the latter choices, obtaining after a simple computation precisely the inequality (3.42). The converse can be performed similarly. Thus, it is enough to prove (i). Consequently, we assume that

$$\alpha\beta > 0 \quad \text{and} \quad m \le \frac{n-2}{2}.$$

Let us define

$$w(t) = \frac{(a+bt^\alpha)^\beta}{t^{2m}}, \quad L(t) = \frac{n-1}{t}, \quad \text{and} \quad W_C(t) = \frac{C}{t^2}, \qquad \forall t > 0,$$

and for some $C > 0$, which will be determined later. Consider the ODE

$$(G(t)w(t))' + G(t)w(t)L(t) - |G(t)|^2 w(t) = W_C(t)w(t), \qquad \forall t > 0. \tag{3.44}$$

which is the equality case of (3.6). Let us introduce the numbers

$$K_0 \stackrel{\text{def}}{=} n - 2m - 2 \geq 0,$$
$$K_1 \stackrel{\text{def}}{=} n - 2m + \alpha\beta - 2 > 0.$$

The fundamental solution of (3.44) is given by

$$G(t) = \frac{K_0 - \sqrt{K_0^2 - 4C}}{2t} \cdot \left(1 - \beta \frac{bt^\alpha}{a} \mathcal{I}(t)\right), \qquad \forall t > 0, \tag{3.45}$$

where

$$\mathcal{I}(t) = \frac{{}_2F_1\left(1 + \frac{\beta}{2} + \frac{\sqrt{K_0^2-4C}-\sqrt{K_1^2-4C}}{2\alpha}, 1 + \frac{\beta}{2} + \frac{\sqrt{K_0^2-4C}+\sqrt{K_1^2-4C}}{2\alpha}; 2 + \frac{\sqrt{K_0^2-4C}}{2\alpha}; -\frac{bt^\alpha}{a}\right)}{{}_2F_1\left(\frac{\beta}{2} + \frac{\sqrt{K_0^2-4C}-\sqrt{K_1^2-4C}}{2\alpha}, \frac{\beta}{2} + \frac{\sqrt{K_0^2-4C}+\sqrt{K_1^2-4C}}{2\alpha}; 1 + \frac{\sqrt{K_0^2-4C}}{2\alpha}; -\frac{bt^\alpha}{a}\right)}$$

and ${}_2F_1$ stands for the *regularized Gaussian hypergeometric function*. More precisely, for $q, r, s \in \mathbb{C}$ with $s \notin \mathbb{Z}_-$,

$${}_2F_1(q, r; s; z) = \frac{1}{\Gamma(1+s)} \sum_{k \geq 0} \frac{(q)_k (r)_k}{(s)_k} \frac{z^k}{k!}, \qquad \forall |z| < 1, \tag{3.46}$$

extended by analytic continuation elsewhere, while

$$(q)_k = q(q+1)\dots(q+k-1) = \frac{\Gamma(q+k)}{\Gamma(q)}$$

denotes the *Pochhammer symbol*, where $k \in \mathbb{N}$.

In order to have a well-defined function G, we should first guarantee that

$$C \leq \frac{1}{4} \min\left(K_0^2, K_1^2\right) = \left(\frac{n-2m-2}{2}\right)^2.$$

Thus, in (3.45) we put the best possible value, which is $C = \left(\frac{n-2m-2}{2}\right)^2$. With this choice, for every $t > 0$ the expression of $G(t)$ reduces to

$$G(t) = \frac{K_0}{2t} \cdot \left(1 - \beta \frac{bt^\alpha}{a} \frac{{}_2F_1\left(1 + \frac{\beta}{2} - \frac{\sqrt{K_1^2-4C}}{2\alpha}, 1 + \frac{\beta}{2} + \frac{\sqrt{K_1^2-4C}}{2\alpha}; 2; -\frac{bt^\alpha}{a}\right)}{{}_2F_1\left(\frac{\beta}{2} - \frac{\sqrt{K_1^2-4C}}{2\alpha}, \frac{\beta}{2} + \frac{\sqrt{K_1^2-4C}}{2\alpha}; 1; -\frac{bt^\alpha}{a}\right)}\right). \tag{3.47}$$

We observe that the denominator in the latter expression does not vanish. To see this, let

$$A \stackrel{\text{def}}{=} \frac{\beta}{2} \quad \text{and} \quad B \stackrel{\text{def}}{=} \frac{\sqrt{K_1^2 - 4C}}{2\alpha} = \frac{\sqrt{\alpha\beta(\alpha\beta + 2K_0)}}{2\alpha}. \tag{3.48}$$

At this point one may apply the Euler–Pfaff transformations (see e.g., Olver, Lozier, Boisvert, and Clark [91, rel. 15.8.1]). We distinguish the following two cases.

- If $A > 0$ (thus $\alpha > 0$), then $B \geq A > 0$ and for every $z > 0$ we have

$${}_2F_1\left(A - B, A + B; 1; -z\right) = (1 + z)^{-A-B}{}_2F_1\left(1 - A + B, A + B; 1; \frac{z}{1+z}\right) > 0.$$

- If $A < 0$ (thus $\alpha < 0$), one has $B \leq A < 0$, then for every $z > 0$ it follows that

$${}_2F_1\left(A - B, A + B; 1; -z\right) = (1 + z)^{B-A}{}_2F_1\left(A - B, 1 - A - B; 1; \frac{z}{1+z}\right) > 0.$$

Now, we shall apply Theorem 3.5 for $p = 2$, $\Omega = \mathbb{R}^n \setminus \{0\}$, $\rho(x) = |x| > 0$ for $x \in \Omega$, as well as

$$w(t) = \frac{(a + bt^\alpha)^\beta}{t^{2m}}, \quad L(t) = \frac{n-1}{t}, \quad \text{and} \quad W(t) = \left(\frac{n - 2m - 2}{2t}\right)^2, \qquad \forall t > 0.$$

One can easily see that the integrability assumptions of $(\mathbf{G})_{\rho,w}$ are verified. By the above arguments, it follows that (L, W) is a $(2, \rho, w)$-Riccati pair in $(0, \infty)$. Therefore, by (3.9) we obtain the validity of (3.41) for every $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. It remains to extend (3.41) to the space $C_0^\infty(\mathbb{R}^n)$. $\square$

Remark 3.32. One could expect a similar proof on Cartan–Hadamard manifolds as in Theorem 3.31. However, a subtle technical difficulty shows up that comes from the fact that – despite several confirming numerical tests – there is no evidence on *positiveness* of the function G of (3.47) for the full range of parameters. Note that in the Euclidean case such a sign property is *not* necessary, as $\Delta|x| = (n-1)/|x|$ for all $x \neq 0$; see Remark 3.6/(b) and Condition (**c2**).

Closely related to the latter observation, we highlight that G contains the expression

$$zf'(z)/f(z), \quad \text{where} \quad f(z) = {}_2F_1\left(a, b; 1; -z\right), \qquad \forall z > 0,$$

which is used to characterize the *order of starlikeness* (with respect to zero) of the function f (see e.g., Küstner [76]). Although one can find various results in the literature, to the best of our knowledge, there is no full characterization of the order of starlikeness for Gaussian hypergeometric functions (with respect to the *full* spectrum of parameters). However, under specific constraints on the parameters, we provide the following partial result on Cartan–Hadamard manifolds. For simplicity, we only focus on the inequality (3.41); by equivalence, it can also be formulated in terms of its 'dual' (3.42).

Theorem 3.33 (see [59]). *Let (M,g) be an $n \geq 2$-dimensional Cartan–Hadamard manifold, and $x_0 \in M$ be a point. Let $a, b, \alpha, \beta > 0$ and $m \in \mathbb{R}$, with $m \leq \frac{n-2}{2}$ and*

$$\alpha\beta + \sqrt{\alpha\beta(\alpha\beta + 2(n-2m-2))} \leq 2. \tag{3.49}$$

Then for every $u \in C_0^\infty(M)$ the following inequality holds:

$$\int_M \frac{(a+bd_{g,x_0}^\alpha)^\beta}{d_{g,x_0}^{2m}} |\nabla_g u|^2 \,\mathrm{d}v_g \geq \left(\frac{n-2m-2}{2}\right)^2 \int_M \frac{(a+bd_{g,x_0}^\alpha)^\beta}{d_{g,x_0}^{2m+2}} u^2 \,\mathrm{d}v_g. \tag{3.50}$$

Proof. We apply Theorem 3.5 for $p=2$, $\Omega = M \setminus \{x_0\}$, $\rho = d_{g,x_0} > 0$ in Ω, as well as

$$w(t) = \frac{(a+bt^\alpha)^\beta}{t^{2m}}, \quad L(t) = \frac{n-1}{t}, \quad \text{and} \quad W(t) = \left(\frac{n-2m-2}{2t}\right)^2, \qquad \forall t > 0.$$

Note that the function G of (3.47) verifies the ODI (3.6) with equality. It is also clear that $(\mathbf{G})_{\rho,w}$ holds. Once we can prove that G is positive on $(0,\infty)$ (see also Remark 3.32), it turns out that (L,W) is a $(2,\rho,w)$-Riccati pair in $(0,\infty)$, and we can apply Theorem 3.5 concluding the proof of (3.50).

To prove the positiveness of G on $(0,\infty)$, it is enough to show that

$$2Az \cdot \frac{{}_2F_1\left(1+A-B, 1+A+B; 2; -z\right)}{{}_2F_1\left(A-B, A+B; 1; -z\right)} \leq 1, \qquad \forall z > 0,$$

where A and B are as in (3.48); by our assumption, we have $B \geq A > 0$. The definition of the hypergeometric function implies that the above inequality is equivalent to

$$\frac{2A}{A+B}\left(1 - \frac{{}_2F_1\left(1+A-B, A+B; 1; -z\right)}{{}_2F_1\left(A-B, A+B; 1; -z\right)}\right) \leq 1, \qquad \forall z > 0.$$

In particular, since $B \geq A > 0$, it is enough to prove that

$$\frac{{}_2F_1\left(1+A-B, A+B; 1; -z\right)}{{}_2F_1\left(A-B, A+B; 1; -z\right)} \geq 0, \qquad \forall z > 0.$$

By assumption (3.49), we have that $|A \pm B| \leq 1$. In this case, we have the integral representation

$$\frac{{}_2F_1\left(1+A-B, A+B; 1; -z\right)}{{}_2F_1\left(A-B, A+B; 1; -z\right)} = \int_0^1 \frac{1}{1+tz} \,\mathrm{d}\mu_0(t), \qquad \forall z > 0,$$

where $\mu_0\colon [0,1] \to [0,1]$ is a non-decreasing function satisfying $\mu_0(1) - \mu_0(0) = 1$ (see Küstner [76]). Clearly, by the latter representation, our claim easily follows. □

Remark 3.34. As we already pointed out in Remark 3.32, numerical tests confirm the positiveness of G for every $a, b > 0$ and $\alpha, \beta, m \in \mathbb{R}$, whose proof requires some specific arguments from the theory of special functions. At this moment, such an approach is not available. In particular, we expect to cancel the additional hypothesis from Theorem 3.33.

3.3 Applications II: Multiplicative Hardy-type inequalities

3.3.1 Sharp uncertainty principles

Let (M,g) be a complete, non-compact n-dimensional Riemannian manifold ($n \geq 2$), and $x_0 \in M$ be fixed. Suppose that $p, \alpha \in \mathbb{R}$ such that

$$n > p > 1 \quad \text{and} \quad -p+1 < \alpha \leq 1. \tag{3.51}$$

We investigate the following *uncertainty principle*: for every $u \in C_0^\infty(M)$, one has

$$\left(\int_M |\nabla_g u|^p \,\mathrm{d}v_g\right)^{\frac{1}{p}} \left(\int_M d_{g,x_0}^{p'\alpha} |u|^p \,\mathrm{d}v_g\right)^{\frac{1}{p'}} \geq \frac{n+\alpha-1}{p} \int_M \left(1 + \frac{n-1}{n+\alpha-1}\mathbf{D}_\kappa(d_{g,x_0})\right) d_{g,x_0}^{\alpha-1} |u|^p \,\mathrm{d}v_g. \tag{$\mathbf{UP}_\kappa$}$$

We observe that ($\mathbf{UP}_\kappa$) formally reduces to the:

- *Heisenberg–Pauli–Weyl uncertainty principle*, whenever $\alpha = 1$ (see Kombe and Özaydin [64, 65] for $p = 2$ and $\kappa = 0$, Kristály [67] for $p = 2$ and $\kappa \leq 0$, and Nguyen [89] for generic $p > 1$ and $\kappa \leq 0$);
- *Hydrogen uncertainty principle*, whenever $\alpha = 0$ (see Cazacu, Flynn, and Lam [23] and Frank [47] in $\mathbb{R}^n$, thus, for $\kappa = 0$);
- *Hardy inequality* in the limit case when $\alpha \to -p+1$ (see Corollary 3.15).

Our first result shows the validity of ($\mathbf{UP}_\kappa$) on Cartan–Hadamard manifolds, which can be stated as follows.

Theorem 3.35 (see [59]). *Let (M,g) be an n-dimensional Cartan–Hadamard manifold ($n \geq 2$), such that $\mathbf{K} \leq \kappa \leq 0$. If the conditions of (3.51) hold, then ($\mathbf{UP}_\kappa$) holds as well; moreover, the constant $\frac{n+\alpha-1}{p}$ is sharp.*

Proof. In Theorem 3.1/(ii) let us choose $w \equiv 1$, $\rho = d_{g,x_0}$, as well as

$$G(t) = t^\alpha \quad \text{and} \quad H(s) = \frac{|s|^p}{p}, \qquad \forall t > 0,\ \forall s \in \mathbb{R}.$$

A simple computation shows that $(\mathbf{G})_{\rho,w}$ holds, due to our assumptions on α, p and n. By Laplace comparison (see Theorem 2.1/(I)/(i)) we have

$$\Delta_{g,p}\rho = \Delta_g \rho \geq (n-1)\mathbf{ct}_\kappa(\rho).$$

Using Theorem 3.1/(ii), yields that inequality ($\mathbf{UP}_\kappa$) holds for every $u \in C_0^\infty(M \setminus \{x_0\})$. It remains to extend ($\mathbf{UP}_\kappa$) to the whole space $C_0^\infty(M)$.

We assume by contradiction that there exists $C > \frac{n+\alpha-1}{p}$ such that

$$\left(\int_M |\nabla_g u|^p \,\mathrm{d}v_g\right)^{\frac{1}{p}} \left(\int_M d_{g,x_0}^{p'\alpha}|u|^p \,\mathrm{d}v_g\right)^{\frac{1}{p'}} \geq C \int_M \left(1 + \frac{n-1}{n+\alpha-1}\mathbf{D}_\kappa(d_{g,x_0})\right) d_{g,x_0}^{\alpha-1}|u|^p \,\mathrm{d}v_g$$

for all $u \in C_0^\infty(M)$. Since $\mathbf{D}_\kappa \geq 0$, the latter inequality implies that

$$\left(\int_M |\nabla_g u|^p \,\mathrm{d}v_g\right)^{\frac{1}{p}} \left(\int_M d_{g,x_0}^{p'\alpha}|u|^p \,\mathrm{d}v_g\right)^{\frac{1}{p'}} \geq C \int_M d_{g,x_0}^{\alpha-1}|u|^p \,\mathrm{d}v_g, \tag{3.52}$$

for all $u \in C_0^\infty(M)$. Fix $\varepsilon > 0$ arbitrarily. Then there exist a number $r > 0$ and a local diffeomorphism $\phi\colon B_O(r) \to M$ such that $\phi(0) = x_0$ and in the sense of bilinear forms. The components g_{ij} of the metric g satisfy on $\phi(B_O(r))$ the estimates

$$(1-\varepsilon)\delta_{ij} \leq g_{ij} \leq (1+\varepsilon)\delta_{ij}; \tag{3.53}$$

see Hebey [54]. As before, $B_O(r)$ denotes the n-dimensional Euclidean ball centered at the origin O, with radius $r > 0$.

Due to relations (3.52) and (3.53), for $0 < \varepsilon \ll 1$ one can find $\tilde{r} > 0$ with the property that for every $r \in (0, \tilde{r})$ and $w \in C_0^\infty(B_O(r))$, the following inequality holds

$$\left(\int_{B_O(r)} |\nabla w|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{B_O(r)} |x|^{p'\alpha}|w|^p \,\mathrm{d}x\right)^{\frac{1}{p'}} \geq C \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{n+\alpha+1} \int_{B_O(r)} |x|^{\alpha-1}|w|^p \,\mathrm{d}x. \tag{3.54}$$

By assumption, $C > \frac{n+\alpha-1}{p}$, thus, we may choose $0 < \varepsilon \ll 1$ such that

$$C_\varepsilon \stackrel{\text{def}}{=} C \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{n+\alpha+1} > \frac{n+\alpha-1}{p}. \tag{3.55}$$

Let $u \in C_0^\infty(\mathbb{R}^n)$ be fixed arbitrarily and define the scaled function

$$w_\lambda(x) = u(\lambda x), \qquad \forall \lambda > 0.$$

For large enough $\lambda > 0$, it follows that $w_\lambda \in C_0^\infty(B_O(r))$; accordingly, we may use w_λ as a test function in inequality (3.54), obtaining

$$\left(\int_{B_O(r)} |\nabla w_\lambda|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{B_O(r)} |x|^{p'\alpha}|w_\lambda|^p \,\mathrm{d}x\right)^{\frac{1}{p'}} \geq C_\varepsilon \int_{B_O(r)} |x|^{\alpha-1}|w_\lambda|^p \,\mathrm{d}x.$$

Making the suitable change of variables, we have the scaling properties

$$\int_{B_O(r)} |\nabla w_\lambda|^p \mathrm{d}x = \lambda^{p-n} \int_{\mathbb{R}^n} |\nabla u|^p \mathrm{d}x,$$
$$\int_{B_O(r)} |x|^{p'\alpha}|w_\lambda|^p \,\mathrm{d}x = \lambda^{-p'\alpha-n} \int_{\mathbb{R}^n} |x|^{p'\alpha}|u|^p \,\mathrm{d}x,$$
$$\int_{B_O(r)} |x|^{\alpha-1}|w_\lambda|^p \,\mathrm{d}x = \lambda^{-\alpha+1-n} \int_{\mathbb{R}^n} |x|^{\alpha-1}|w_\lambda|^p \,\mathrm{d}x.$$

In particular, by the scaling properties and the latter inequality, we obtain the following λ-independent inequality:

$$\left(\int_{\mathbb{R}^n} |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |x|^{p'\alpha}|u|^p \,\mathrm{d}x\right)^{\frac{1}{p'}} \geq C_\varepsilon \int_{\mathbb{R}^n} |x|^{\alpha-1}|u|^p \,\mathrm{d}x.$$

Since $-p+1<\alpha\leq 1$, one can approximate

$$u(x) = e^{-\frac{|x|^{1+\frac{\alpha}{p-1}}}{p}}, \qquad \forall x \in \mathbb{R}^n,$$

by functions from $C_0^\infty(\mathbb{R}^n)$, and we may use it as a test function in the latter inequality. Taking polar coordinates and using the relation

$$\int_0^\infty e^{-s^\beta} s^\gamma \,\mathrm{d}s = \frac{1}{\beta}\Gamma\left(\frac{\gamma+1}{\beta}\right), \qquad \forall \beta>0,\ \forall\gamma>-1,$$

we obtain that

$$\frac{n+\alpha-1}{p} \geq C_\varepsilon,$$

which contradicts (3.55), showing the sharpness of $\frac{n+\alpha-1}{p}$ in ($\mathbf{UP}_\kappa$). $\square$

Remark 3.36. If equality holds in ($\mathbf{UP}_\kappa$) for some positive function $u \in W^{1,p}(M)$, then equality $\Delta_g d_{g,x_0} = (n-1)\mathbf{ct}_\kappa(d_{g,x_0})$ also holds, which implies that the manifold (M,g) is isometric to the model space form $\mathbf{M}_\kappa^n$ (see Theorem 2.1). This rigidity result is known for $\kappa=0$ from Kristály [67] for $p=2$, and from Nguyen [89] for $p>1$.

The following result is a counterpart of Theorem 3.35, providing the *rigidity* of Riemannian manifolds with $\mathbf{Ric} \geq \kappa(n-1)g$ for some $\kappa \leq 0$ supporting the uncertainty principle ($\mathbf{UP}_\kappa$). Similar results have been obtained first by Kristály [67] for $p=2$ and $\alpha=1$, and then by Nguyen [89] for generic $p>1$, both considering only the case $\kappa=0$. Now, we have a more general result, valid for every $\kappa \leq 0$:

Theorem 3.37 (see [59])**.** *Let (M,g) be an $n \geq 2$-dimensional complete, non-compact Riemannian manifold, with* $\mathbf{Ric} \geq \kappa(n-1)g$*, for some $\kappa \leq 0$. Suppose that* ($\mathbf{UP}_\kappa$) *holds for some $x_0 \in M$ and the parameters α, p, n verify either $-p+1<\alpha\leq 1<p<n$, when $\kappa=0$, or $0<\alpha\leq 1<p<n$, when $\kappa<0$. Then (M,g) is isometric to the model space form $\mathbf{M}_\kappa^n$.*

Proof. For simplicity of notations, let $\gamma = 1+\frac{\alpha}{p-1}>0$ and define $u_0\colon M \to (0,\infty)$ by

$$u_0 = e^{-\frac{d_{g,x_0}^\gamma}{p}}.$$

Clearly, we may approximate u_0 by elements of $C_0^\infty(M)$; moreover, by using the eikonal equation (2.1), it turns out that

$$\int_M |\nabla_g u_0|^p \,\mathrm{d}v_g = \left(\frac{\gamma}{p}\right)^p \int_M d_{g,x_0}^{p'\alpha} u_0^p \,\mathrm{d}v_g.$$

We first claim that the latter integrals are well defined. To see this, using (2.9), we observe that $\kappa = 0$ implies

$$\gamma > 0 \quad \text{and} \quad \mathrm{VB}_\kappa(R) = \omega_n R^n,$$

while $\kappa < 0$ implies

$$\gamma > 1 \quad \text{and} \quad \mathrm{VB}_\kappa(R) \sim e^{(n-1)\sqrt{|\kappa|}R}, \qquad \text{as } R \to \infty,$$

up to a multiplicative constant.

At this point, by the layer cake representation and the volume comparison (see Theorem 2.1/(II)/(ii)), we conclude the claim.

By using u_0 as a test function in ($\mathbf{UP}_\kappa$), it follows that

$$\gamma \int_M d_{g,x_0}^{p'\alpha} u_0^p \,\mathrm{d}v_g \geq (n+\alpha-1) \int_M \left(1 + \frac{n-1}{n+\alpha-1}\mathbf{D}_\kappa(d_{g,x_0})\right) d_{g,x_0}^{\alpha-1} u_0^p \,\mathrm{d}v_g. \tag{3.56}$$

By the Laplace comparison (see Theorem 2.1/(II)/(i)), we have that

$$\Delta_g d_{g,x_0} \leq (n-1)\mathbf{ct}_\kappa(d_{g,x_0}).$$

Thus, relation (3.56), an integration by parts (2.3) and the eikonal equation (2.1) yield

$$\begin{aligned}
\gamma \int_M d_{g,x_0}^{p'\alpha} e^{-d_{g,x_0}^\gamma} \,\mathrm{d}v_g &\geq \int_M \left(n+\alpha-1+(n-1)\mathbf{D}_\kappa(d_{g,x_0})\right) d_{g,x_0}^{\alpha-1} e^{-d_{g,x_0}^\gamma} \,\mathrm{d}v_g \\
&= \int_M \left(\alpha + (n-1) d_{g,x_0}\mathbf{ct}_\kappa(d_{g,x_0})\right) d_{g,x_0}^{\alpha-1} e^{-d_{g,x_0}^\gamma} \,\mathrm{d}v_g \\
&\geq \int_M \left(\alpha + d_{g,x_0}\Delta_g d_{g,x_0}\right) d_{g,x_0}^{\alpha-1} e^{-d_{g,x_0}^\gamma} \,\mathrm{d}v_g \\
&= \alpha \int_M d_{g,x_0}^{\alpha-1} e^{-d_{g,x_0}^\gamma} \,\mathrm{d}v_g + \int_M d_{g,x_0}^{\alpha} e^{-d_{g,x_0}^\gamma} \Delta_g d_{g,x_0} \,\mathrm{d}v_g \\
&= \gamma \int_M d_{g,x_0}^{\alpha+\gamma-1} e^{-d_{g,x_0}^\gamma} \,\mathrm{d}v_g \\
&= \gamma \int_M d_{g,x_0}^{p'\alpha} e^{-d_{g,x_0}^\gamma} \,\mathrm{d}v_g,
\end{aligned}$$

where we used the fact that $\alpha + \gamma - 1 = p'\alpha$.

Since the two marginal terms are equal (and finite), we should have equality in each step. In particular, we have

$$\Delta_g d_{g,x_0} = (n-1)\mathbf{ct}_\kappa(d_{g,x_0})$$

on $M \setminus \{x_0\}$, therefore, by Theorem 2.1 it follows that (M, g) is isometric to the model space form $\mathbf{M}_\kappa^n$. $\square$

Remark 3.38. The proof of Theorem 3.37 is much simpler than the original rigidity argument performed in Kristály [67] and Nguyen [89], where the comparison of certain ordinary differential inequalities/equations is used.

Remark 3.39. By using additive-type inequalities we can produce multiplicative inequalities in the spirit of Berchio, Ganguly, Grillo, and Pinchover [12, Section 3]. For example, inequality (3.35) combined with the Cauchy–Schwarz inequality (2.10) imply (in the same geometric context as in Corollary 3.28) for every $u \in C_0^\infty(M)$ that

$$\left(\int_M |\nabla_g u|^2 \,\mathrm{d}v_g - (n-2)|\kappa| \int_M u^2 \,\mathrm{d}v_g\right)\left(\int_M d_{g,x_0}^2 u^2 \,\mathrm{d}v_g\right) \geq \frac{(n-2)^2}{4}\left(\int_M u^2 \,\mathrm{d}v_g\right)^2.$$

In the special case when $M = \mathbb{H}^n_{-1}$, the latter inequality reduces to [12, Corollary 3.1]. Similarly, further inequalities can be obtained from the results of Section 3.2.4.

3.3.2 Caffarelli–Kohn–Nirenberg inequalities

In this section, we prove a version of the *Caffarelli–Kohn–Nirenberg inequality* (see [19]), which easily follows from the multiplicative inequality of Theorem 3.1/(ii).

Theorem 3.40 (see [59]). *Let (M,g) be an n-dimensional Cartan–Hadamard manifold with $n \geq 2$ such that $\mathbf{K} \leq \kappa$ for some $\kappa \leq 0$. Fix $x_0 \in M$ and suppose that $p, \alpha, r \in \mathbb{R}$ satisfy*

$$r > p > 1, \quad \alpha + p > 1, \quad \text{and} \quad p(n+\alpha-1) > r(n-p) > 0.$$

Then for every $u \in C_0^\infty(M)$ one has

$$\left(\int_M |\nabla_g u|^p \,\mathrm{d}v_g\right)^{\frac{1}{p}}\left(\int_M d_{g,x_0}^{p'\alpha}|u|^{p'(r-1)} \,\mathrm{d}v_g\right)^{\frac{1}{p'}}$$
$$\geq \frac{n+\alpha-1}{r}\int_M \left(1 + \frac{n-1}{n+\alpha-1}\mathbf{D}_\kappa(d_{g,x_0})\right) d_{g,x_0}^{\alpha-1}|u|^r \,\mathrm{d}v_g.$$

Moreover, the constant $\frac{n+\alpha-1}{r}$ is sharp.

Proof. The proof is similar to Theorem 3.35, the difference is that we choose

$$H(s) = \frac{|s|^r}{r}, \qquad \forall s \in \mathbb{R}.$$

The sharpness of $\frac{n+\alpha-1}{r}$ is based on the same scaling argument as before, the test function (in $\mathbb{R}^n$) in the last step being the Talenti-type function

$$u(x) = \left(1 + |x|^{1+\frac{\alpha}{p-1}}\right)^{\frac{p-1}{p-r}}, \qquad \forall x \in \mathbb{R}^n.$$

The rest of the proof is similar as before. □

For further Caffarelli–Kohn–Nirenberg inequalities, see e.g., Catrina and Wang [22], Kristály and Ohta [72], Wang and Willem [103], and Xia [106].

As we already pointed out, various choices of H and G in Theorem 3.1 produce well-known or new functional inequalities. In this spirit, we conclude the section with an unusual Caffarelli–Kohn–Nirenberg-type inequality, which can be the starting point to build further functional inequalities through Theorem 3.1.

Theorem 3.41 (see [59]). *Let (M,g) be an n-dimensional Cartan–Hadamard manifold $(n \geq 2)$, such that $\mathbf{K} \leq \kappa < 0$. Then for every $u \in C_0^\infty(M) \setminus \{0\}$ and $c \in \mathbb{R}$, we have*

$$\int_M |\nabla_g u|^2 \,\mathrm{d}v_g \geq \frac{\left(\int_M \mathbf{s}_c^2(u)\,\mathrm{d}v_g\right)^2}{\int_M \mathbf{s}_c^2(2u)\,\mathrm{d}v_g}(n-1)^2|\kappa|. \tag{3.57}$$

In particular, we also have the McKean spectral gap estimate

$$\lambda_{\mathrm{m}}(M) \geq \frac{(n-1)^2}{4}|\kappa|.$$

Proof. Fix $x_0 \in M$ and let $w \equiv 1$, $\rho = d_{g,x_0}$, $p = 2$, as well as

$$G \equiv 1 \quad \text{and} \quad H(s) = \mathbf{s}_c^2(s), \qquad \forall s \in \mathbb{R}.$$

By Laplace comparison (see Theorem 2.1/(I)/(i)), we have

$$\Delta_g d_{g,x_0} \geq (n-1)\mathbf{ct}_\kappa(d_{g,x_0}) \geq (n-1)\sqrt{-\kappa},$$

hence inequality (3.57) follows at once from Theorem 3.1/(ii). For $c = 0$, we obtain the classical L^2-McKean spectral estimate. $\square$

Chapter 4

Rellich inequalities on general Riemannian manifolds

In this chapter, we present an elegant approach for proving Rellich-type inequalities on Riemannian manifolds. Similarly to the previous chapter, the main ingredients are general functional inequalities built upon simple convexity and comparison arguments. The proofs do not involve spherical harmonics decompositions; moreover, they are symmetrization-free, thus, the validity of the generalized Cartan–Hadamard conjecture (or sharp isoperimetric inequality) is not required.

Our method is extremely useful for providing higher-order spectral gap estimates for problems involving higher-order operators on general Cartan–Hadamard manifolds. These include both the *clamped plate problem* (2.38) and *buckling problem* (2.43) as well as their higher-order variants. Furthermore, it can also be used to establish additional Rellich-type inequalities in the same geometric setting by either extending classical Euclidean inequalities or providing genuinely new inequalities.

In Section 4.1 we present two general functional inequalities. The first inequality (see Theorem 4.1) connects $|\Delta_g u|^p$ and $|u|^p$ for $p > 1$, and it is tailored to provide a sharp spectral gap estimate for the clamped plate problem, for general $p > 1$. The second inequality (see Theorem 4.2) connects $|\Delta_g u|^2$ and $|\nabla_g u|^2$, and it is designed to provide a sharp spectral gap estimate for the buckling problem. We notice that the first inequality involves an ODI, while the second involves a PDI, which turn out to be more problematic to deal with. However, for special parameter choices, the PDI reduces to an ODI (see Remark 4.3/(a)). We also note that the second inequality cannot be extended for general $p > 1$, due to some technical limitations (see Remark 4.3/(b)).

In Section 4.2 we prove various spectral gap results and their *sharpness* on Cartan–Hadamard manifolds. More specifically, the following will be discussed.

- In Section 4.2.1 we prove a sharp spectral gap estimate concerning the clamped plate problem, for general $p > 1$ (see Theorem 4.4).
- In Section 4.2.2 we demonstrate a sharp spectral gap estimate concerning the buckling problem, for $p = 2$ (see Theorem 4.5).
- In Section 4.2.3 we establish higher-order variants of the previous results (see Theorems 4.6 & 4.7).

In Section 4.3 we give alternative proofs for further Rellich inequalities, namely:

- In Section 4.3.1 we extend the weighted Euclidean Rellich inequality to Cartan–Hadamard manifolds (see Theorem 4.8); as a consequence, we obtain an extension of the classical Rellich inequality as well (see Corollary 4.9).
- In Section 4.3.2 we establish higher-order versions of the previous results (see Theorem 4.10).
- In Section 4.3.3 we provide further applications concerning improved Rellich-type inequalities (see Theorems 4.11 & 4.12 & 4.14).

4.1 General functional inequalities

In this section, we present two general functional inequalities. The first inequality can be stated as follows.

Theorem 4.1 (see [39]). *Let (M,g) be an $n\geq 2$-dimensional, complete, non-compact Riemannian manifold. Let $\Omega\subseteq M$ be a domain, $x_0\in\Omega$, and $\rho=d_{g,x_0}$. Let $p>1$ and suppose that $L,W,w,G,H\colon(0,\sup_\Omega\rho)\to(0,\infty)$ satisfy the following conditions:*

(C1) *L,W are continuous, w,G are of class C^2, and H is of class C^1;*

(C2) *$\Delta_g\rho\geq L(\rho)$ in the distributional sense and $(wG)'\leq 0$;*

(C3) *the ordinary differential inequality*

$$(p-1)\left[2(wGH)'+2wGHL-pwGH^2-w|G|^{p'}\right]-(wG)''-(wG)'L\geq W \quad (4.1)$$

holds for the functions $L(t),W(t),w(t),G(t),H(t)$, for all $t\in(0,\sup_\Omega\rho)$.

Then for every $u\in C_0^\infty(\Omega)$ one has

$$\int_\Omega w(\rho)|\Delta_g u|^p\,\mathrm{d}v_g\geq\int_\Omega W(\rho)|u|^p\,\mathrm{d}v_g.$$

Proof. The convexity of $\xi\mapsto|\xi|^p$ implies

$$|\xi|^p\geq|\eta|^p+p|\eta|^{p-2}(\xi-\eta)\eta=p|\eta|^{p-2}\xi\eta+(1-p)|\eta|^p,\qquad\forall\xi,\eta,\quad (4.2)$$

where ξ and η are both scalars or both vector fields. Fix $u\in C_0^\infty(\Omega)$ arbitrarily and choose

$$\xi=\Delta_g u\quad\text{and}\quad\eta=-|G(\rho)|^{\frac{2-p}{p-1}}G(\rho)u$$

to obtain

$$|\Delta_g u|^p\geq-pG(\rho)|u|^{p-2}u\Delta_g u+(1-p)|G(\rho)|^{p'}|u|^p.$$

Multiplying both sides by $w(\rho)>0$ and integrating over Ω yield

$$\int_\Omega w(\rho)|\Delta_g u|^p\,\mathrm{d}v_g\geq-p\int_\Omega w(\rho)G(\rho)|u|^{p-2}u\Delta_g u\,\mathrm{d}v_g+(1-p)\int_\Omega w(\rho)|G(\rho)|^{p'}|u|^p\,\mathrm{d}v_g.$$

We shall focus on the second term. On the one hand, using the relation

$$\Delta_g \frac{|u|^p}{p} = (p-1)|u|^{p-2}|\nabla_g u|^2 + |u|^{p-2}u\Delta_g u,$$

and Green's second identity (2.4) leads to

$$\begin{aligned} -p\int_\Omega w(\rho)G(\rho)|u|^{p-2}u\Delta_g u \,\mathrm{d}v_g = p(p-1)&\int_\Omega w(\rho)G(\rho)|u|^{p-2}|\nabla_g u|^2 \,\mathrm{d}v_g \\ &- \int_\Omega |u|^p \Delta[w(\rho)G(\rho)] \,\mathrm{d}v_g. \end{aligned}$$

On the other hand, choose

$$p = 2, \quad \xi = \nabla_g u, \quad \text{and} \quad \eta = -uH(\rho)\nabla_g \rho$$

in inequality (4.2) to obtain

$$|\nabla_g u|^2 \geq -2H(\rho)u\nabla_g u \nabla_g \rho - H(\rho)^2|u|^2,$$

hence

$$\begin{aligned} \int_\Omega w(\rho)G(\rho)|u|^{p-2}|\nabla_g u|^2 \,\mathrm{d}v_g \geq -2&\int_\Omega w(\rho)G(\rho)H(\rho)|u|^{p-2}u\nabla_g u\nabla_g\rho \,\mathrm{d}v_g \\ &- \int_\Omega w(\rho)G(\rho)H(\rho)^2|u|^p \,\mathrm{d}v_g. \end{aligned}$$

Finally, integration by parts (2.3) yields

$$\begin{aligned} -2\int_\Omega w(\rho)G(\rho)H(\rho)|u|^{p-2}u\nabla_g u\nabla_g\rho \,\mathrm{d}v_g &= -\frac{2}{p}\int_\Omega w(\rho)G(\rho)H(\rho)\nabla_g|u|^p\nabla_g \,\mathrm{d}v_g\rho \\ &= \frac{2}{p}\int_\Omega |u|^p \operatorname{div}_g(w(\rho)G(\rho)H(\rho)\nabla_g\rho)\,\mathrm{d}v_g. \end{aligned}$$

By the above computations, for every $u \in C_0^\infty(\Omega)$ one has

$$\int_\Omega w(\rho)|\Delta_g u|^p \,\mathrm{d}v_g \geq \int_\Omega |u|^p W(\rho)\,\mathrm{d}v_g,$$

provided that

$$\begin{aligned} W(\rho) &\leq (p-1)(2\operatorname{div}_g(w(\rho)G(\rho)H(\rho)\nabla_g\rho) - pw(\rho)G(\rho)H(\rho)^2 - w(\rho)|G(\rho)|^{p'}) \\ &\quad - \Delta(w(\rho)G(\rho)) \\ &\leq (p-1)\,(2(w(\rho)G(\rho)H(\rho))' + 2w(\rho)G(\rho)H(\rho)\Delta_g\rho) \\ &\quad + (p-1)\left(-pw(\rho)G(\rho)H(\rho)^2 - w(\rho)|G(\rho)|^{p'}\right) \\ &\quad - (w(\rho)G(\rho))'' - (w(\rho)G(\rho))'\Delta_g\rho, \end{aligned}$$

which easily follows from **(C2)** and **(C3)**. □

Theorem 4.2 (see [39]). *Let (M,g) be an $n \geq 2$-dimensional complete, non-compact Riemannian manifold. Let $\Omega \subseteq M$ be a domain, $x_0 \in \Omega$, and $\rho = d_{g,x_0}$. Suppose that $L, W, G, H \colon (0, \sup_\Omega \rho) \to (0,\infty)$ satisfy the following conditions:*

(**C1'**) *L, W are continuous, G is of class C^2, and H is of class C^1;*

(**C2'**) *$\Delta_g \rho \geq L(\rho)$ in the distributional sense;*

(**C3'**) *the following PDI holds for $\rho = d_{g,x_0}(x)$ and for every $x \in \Omega$:*

$$(W(\rho)H(\rho))' + W(\rho)H(\rho)L(\rho) - W(\rho)H(\rho)^2 \geq \Delta_g G(\rho) + G(\rho)^2. \tag{4.3}$$

Then for every $u \in C_0^\infty(\Omega)$ one has

$$\int_\Omega |\Delta_g u|^2 \, \mathrm{d}v_g \geq \int_\Omega (2G(\rho) - W(\rho))|\nabla_g u|^2 \, \mathrm{d}v_g.$$

Proof. For $p = 2$, the convexity inequality (4.2) reads as

$$\xi^2 \geq 2\xi\eta - \eta^2, \qquad \forall \xi, \eta \in \mathbb{R}. \tag{4.4}$$

On the one hand, by choosing

$$\xi = \Delta_g u \quad \text{and} \quad \eta = -G(\rho)u,$$

we obtain that

$$(\Delta_g u)^2 \geq -2G(\rho)u\Delta_g u - G(\rho)^2 u^2.$$

Integrating over Ω yields

$$\int_\Omega (\Delta_g u)^2 \, \mathrm{d}v_g \geq -2\int_\Omega G(\rho)u\Delta_g u \, \mathrm{d}v_g - \int_\Omega G(\rho)^2 u^2 \, \mathrm{d}v_g. \tag{4.5}$$

By using relation

$$-2u\Delta_g u = 2|\nabla_g u|^2 - \Delta_g(u^2)$$

and Green's second identity (2.4) in the second term of (4.5) leads to

$$\int_\Omega (\Delta_g u)^2 \, \mathrm{d}v_g \geq 2\int_\Omega G(\rho)|\nabla_g u|^2 \, \mathrm{d}v_g - \int_\Omega (\Delta_g G(\rho))u^2 \, \mathrm{d}v_g - \int_\Omega G(\rho)^2 u^2 \, \mathrm{d}v_g.$$

Thus, to finish our proof, it is enough to show that

$$\int_\Omega W(\rho)|\nabla_g u|^2 \, \mathrm{d}v_g \geq \int_\Omega (\Delta_g G(\rho) + G(\rho)^2)u^2 \, \mathrm{d}v_g. \tag{4.6}$$

On the other hand, by choosing

$$\xi = \nabla_g u \quad \text{and} \quad \eta = -uH(\rho)\nabla_g \rho$$

in inequality (4.4), we infer that

$$|\nabla_g u|^2 \geq -2H(\rho)u\nabla_g u \nabla_g \rho - u^2 H(\rho)^2.$$

Multiplying both sides with $W(\rho) > 0$ and integrating over Ω yield

$$\int_\Omega W(\rho)|\nabla_g u|^2 \,\mathrm{d}v_g \geq -2\int_\Omega W(\rho)H(\rho)u\nabla_g u\nabla_g\rho \,\mathrm{d}v_g - \int_\Omega W(\rho)H(\rho)^2u^2 \,\mathrm{d}v_g.$$

Integration by parts (2.3) and Condition **(C2')** imply that

$$\begin{aligned}
-2\int_\Omega W(\rho)H(\rho)u\nabla_g u\nabla_g\rho \,\mathrm{d}v_g &= -\int_\Omega W(\rho)H(\rho)\nabla_g\rho\nabla_g(u^2) \,\mathrm{d}v_g \\
&= \int_\Omega \mathrm{div}_g[W(\rho)H(\rho)\nabla_g\rho]u^2 \,\mathrm{d}v_g \\
&= \int_\Omega [(W(\rho)H(\rho))' + W(\rho)H(\rho)\Delta_g\rho]u^2 \,\mathrm{d}v_g \\
&\geq \int_\Omega [(W(\rho)H(\rho))' + W(\rho)H(\rho)L(\rho)]u^2 \,\mathrm{d}v_g.
\end{aligned}$$

Finally, Condition **(C3')** yields (4.6), which finishes the proof. □

We conclude this section by presenting several comments concerning the above result and its possible alternatives.

Remark 4.3. (a) We highlight that Condition **(C3)** involves an ordinary differential inequality, while **(C3')** involves a partial differential inequality on the manifold; the latter is due to the dependence of Δ_g on ρ. For a radial function $G(\rho)$ one has

$$\Delta_g G(\rho) = G''(\rho) + G'(\rho)\Delta_g\rho.$$

Hence **(C3')** is genuinely harder to verify than **(C3)**. However, when G is constant, ρ can be simply replaced by a scalar t, and the partial differential inequality reduces to an ordinary one.

(b) The technique presented in the proof of Theorem 4.2 works only for $p = 2$. For general $p > 1$ the second term of the convexity inequality (4.2) contains the term

$$|u|^{p-2}u\Delta_g u,$$

which cannot be transformed into $|\nabla_g u|^p$.

(c) Later, in Theorem 5.3/(ii) we present an alternative functional inequality, which connects $|\Delta_\kappa u|^2$ and $|\nabla_\kappa u^2|$ on space forms $\mathbf{M}^n_\kappa$, for some $\kappa \leq 0$. The proof again relies on the convexity inequality (4.2), using alternative choices of ξ and η. However, due to some other technical limitations, that method remains valid for $p = 2$ as well.

(d) It is an open question if there exists a convexity argument involving arbitrary choices of ξ and η and possibly multiple uses of integration by parts, which provides a general functional inequality on general Riemannian manifolds connecting the integrals of $|\Delta_g u|^p$ and $|\nabla_g u|^p$, for general $p > 1$.

4.2 Applications I: Spectral gap estimates

In this section, we establish sharp spectral estimates on Cartan–Hadamard manifolds. First, in Theorem 4.4 we prove a spectral gap estimate and its sharpness for the clamped plate problem (2.38), for general $p > 1$. Next, in Theorem 4.5 we establish a sharp spectral gap estimate for the buckling problem (2.43), for $p = 2$. Finally, by iterative applications of these results and using the spectral gap estimate (2.37) for the fixed membrane problem, we also establish higher-order estimates (see Theorems 4.6 & 4.7). All the proofs are built upon convexity and comparison arguments; moreover, they are symmetrization-free.

4.2.1 Clamped plate problem

Our spectral gap result concerning the clamped plate problem reads as follows.

Theorem 4.4 (see [39]). *Let (M, g) be an n-dimensional Cartan–Hadamard manifold with $n \geq 2$ and sectional curvature $\mathbf{K} \leq \kappa$, for some $\kappa < 0$. Let $p > 1$ and $\Omega \subseteq M$ be a domain. Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega |\Delta_g u|^p \, \mathrm{d}v_g \geq \left(\frac{(n-1)^2|\kappa|(p-1)}{p^2} \right)^p \int_\Omega |u|^p \, \mathrm{d}v_g. \tag{4.7}$$

Moreover, the constant in (4.7) is sharp.

Proof. For simplicity denote $\vartheta = \sqrt{-\kappa}$. Fix $x_0 \in \Omega$ and let $\rho = d_{g,x_0}$. In Theorem 4.1 let us choose

$$L \equiv (n-1)\vartheta, \quad W \equiv C, \quad w \equiv 1, \quad G \equiv a, \quad \text{and} \quad H \equiv b,$$

for some constants $C, a, b > 0$, which will be determined later. Condition **(C1)** clearly holds. By Laplace comparison (see Theorem 2.1/(I)/(i)), one has

$$\Delta_g \rho \geq (n-1)\mathbf{ct}_\kappa(\rho) = (n-1)\vartheta \coth(\vartheta\rho) \geq (n-1)\vartheta.$$

Additionally, since G is constant, Condition **(C2)** holds as well. Inequality (4.1) from Condition **(C3)** is equivalent to

$$f(a,b) \stackrel{\text{def}}{=} (p-1)\left(2ab\vartheta(n-1) - a^{\frac{p}{p-1}} - ab^2p\right) \geq C.$$

The best choice for the constant C is obtained as

$$\max_{a,b} f(a,b) = f\left(\left(\frac{(n-1)^2\vartheta^2(p-1)}{p^2}\right)^{p-1}, \frac{(n-1)\vartheta}{p}\right) = \left(\frac{(n-1)^2\vartheta^2(p-1)}{p^2}\right)^p,$$

which yields the inequality (4.7).

To prove sharpness, fix $\delta > 4$ and define the *truncation function*

$$\phi(t) = \begin{cases} t - \frac{\delta}{2}, & \text{if } t \in \left[\frac{\delta}{2}, \frac{\delta}{2}+1\right], \\ 1, & \text{if } t \in \left[\frac{\delta}{2}+1, \delta-1\right], \\ \delta - t, & \text{if } t \in [\delta-1, \delta], \\ 0, & \text{otherwise}. \end{cases} \tag{4.8}$$

Choose

$$\Omega = \mathbf{M}^n_\kappa, \quad \rho = d_{\kappa,x_0}, \quad s = \frac{(n-1)\vartheta}{p}, \quad \text{and} \quad u_\delta = \phi(\rho)e^{-s\rho}.$$

Using the properties of ρ (see formulas (2.11) & (2.12)), we have

$$|\nabla_\kappa \rho| = 1 \quad \text{and} \quad \Delta_\kappa \rho = (n-1)\vartheta \coth(\vartheta\rho) = ps\coth(\vartheta\rho).$$

By using the fact that $\phi'' = 0$ (except a finite number of points), one has $\mathrm{d}v_\kappa$-a.e. that

$$\begin{aligned}
\nabla_\kappa u_\delta &= (\phi'(\rho) - s\phi(\rho))e^{-s\rho}\nabla_\kappa \rho,\\
\Delta_\kappa u_\delta &= \left[-2s\phi'(\rho) + s^2\phi(\rho) + (\phi'(\rho) - s\phi(\rho))ps\coth(\vartheta\rho)\right]e^{-s\rho}\\
&= \left[s(p\coth(\vartheta\rho) - 2)\phi'(\rho) + s^2(1 - p\coth(\vartheta\rho))\phi(\rho)\right]e^{-s\rho}.
\end{aligned}$$

On the one hand, using a polar coordinate transform and the second branch of (4.8) we have

$$\int_\Omega |u_\delta|^p \,\mathrm{d}v_\kappa = \int_{\frac{\delta}{2}}^{\delta} \phi(t)^p e^{-pst}\frac{\sinh^{n-1}(\vartheta t)}{\vartheta^{n-1}}\,\mathrm{d}t \geq \frac{1}{\vartheta^{n-1}}\int_{\frac{\delta}{2}+1}^{\delta-1} e^{-pst}\sinh^{n-1}(\vartheta t)\,\mathrm{d}t.$$

Observe that

$$e^{-pst}\sinh^{n-1}(\vartheta t) = (e^{-\vartheta t}\sinh(\vartheta t))^{n-1} = \left(\frac{1}{2} - \frac{e^{-2\vartheta t}}{2}\right)^{n-1}$$

is strictly increasing in t, thus, we have the following estimate

$$\int_\Omega |u_\delta|^p\,\mathrm{d}v_\kappa \geq \frac{1}{\vartheta^{n-1}}\left(\frac{\delta}{2} - 2\right)\left(\frac{1}{2} - \frac{e^{-\vartheta\delta - 2\vartheta}}{2}\right)^{n-1} \stackrel{\text{def}}{=} E_1(\delta). \tag{4.9}$$

On the other hand, similarly to the previous computations, one has

$$\int_\Omega |\Delta_\kappa u_\delta|^p\,\mathrm{d}v_\kappa = \int_{\frac{\delta}{2}}^{\delta}\frac{1}{\vartheta^{n-1}}\Phi(t)\,\mathrm{d}t,$$

where

$$\Phi(t) = \left|s(p\coth(\vartheta t) - 2)\phi'(t) + s^2(1 - p\coth(\vartheta t))\phi(t)\right|^p\left(e^{-\vartheta t}\sinh(\vartheta t)\right)^{n-1}.$$

Observe that $\Phi(t)$ is bounded. Let M_1 and M_2 be the maximum of $\Phi(t)$ on $\left[\frac{\delta}{2}, \frac{\delta}{2} + 1\right]$ and $[\delta - 1, \delta]$, respectively. Thus, using again (4.8) we have

$$\int_\Omega |\Delta_\kappa u_\delta|^p\,\mathrm{d}v_\kappa \leq M_1 + M_2 + \frac{s^{2p}}{\vartheta^{n-1}}\int_{\frac{\delta}{2}+1}^{\delta-1}(p\coth(\vartheta t) - 1)^p\left(e^{-\vartheta t}\sinh(\vartheta t)\right)^{n-1}\,\mathrm{d}t.$$

Since $(p\coth(\vartheta t) - 1)^p$ is decreasing, $e^{-\vartheta t}\sinh(\vartheta t)$ is increasing, and both expressions are positive, we get

$$\begin{aligned}
\int_\Omega |\Delta_\kappa u_\delta|^p\,\mathrm{d}v_\kappa &\leq M_1 + M_2\\
&\quad + \frac{s^{2p}}{\vartheta^{n-1}}\left(\frac{\delta}{2} - 2\right)\left(p\coth\left(\left(\frac{\delta}{2}+1\right)\vartheta\right) - 1\right)^p\left(\frac{1}{2} - \frac{e^{-2\vartheta(\delta-1)}}{2}\right)^{n-1}\\
&\stackrel{\text{def}}{=} E_2(\delta).
\end{aligned} \tag{4.10}$$

Using the two estimates, we have

$$\lim_{\delta\to\infty}\frac{\displaystyle\int_\Omega |\Delta_\kappa u_\delta|^p \,\mathrm{d}v_\kappa}{\displaystyle\int_\Omega |u_\delta|^p \,\mathrm{d}v_\kappa} \leq \lim_{\delta\to\infty}\frac{E_2(\delta)}{E_1(\delta)} = s^{2p}(p-1)^p = \left(\frac{(n-1)^2|\kappa|(p-1)}{p^2}\right)^p,$$

hence the inequality is sharp. □

We highlight that the estimate from Theorem 4.4 is a novel result that has not yet been established in such a general context. For a summary of milestone results related to the clamped plate problem, see Section 2.3.2.

4.2.2 Buckling problem

Our spectral gap result concerning the buckling problem reads as follows.

Theorem 4.5 (see [39]). *Let (M,g) be an n-dimensional Cartan–Hadamard manifold, with $n \geq 2$ and sectional curvature $\mathbf{K} \leq \kappa$, for some $\kappa < 0$. If $\Omega \subseteq M$ is a domain, then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega |\Delta_g u|^2 \,\mathrm{d}v_g \geq \frac{(n-1)^2|\kappa|}{4}\int_\Omega |\nabla_g u|^2 \,\mathrm{d}v_g. \tag{4.11}$$

Moreover, the constant in (4.11) is sharp.

Proof. For simplicity denote $\vartheta = \sqrt{-\kappa}$. Fix $x_0 \in \Omega$ and let $\rho = d_{g,x_0}$. In Theorem 4.2 let us choose

$$L \equiv (n-1)\vartheta, \quad W \equiv C, \quad G \equiv a, \quad \text{and} \quad H \equiv b,$$

for some constants $C, a, b > 0$, which will be determined later. Condition **(C1')** clearly holds. By Laplace comparison (see Theorem 2.1/(I)/(i)), one has

$$\Delta_g \rho \geq (n-1)\mathbf{ct}_\kappa(\rho) = (n-1)\vartheta\coth(\vartheta\rho) \geq (n-1)\vartheta,$$

implying Condition **(C2')**. Inequality (4.3) from Condition **(C3')** is equivalent to

$$Cb(n-1)\vartheta - Cb^2 \geq a^2.$$

Provided that the above inequality holds, Theorem 4.2 implies

$$\int_\Omega |\Delta_g u|^2 \,\mathrm{d}v_g \geq (2a - C)\int_\Omega |\nabla_g u|^2 \,\mathrm{d}v_g, \qquad \forall u \in C_0^\infty(\Omega).$$

To obtain the best spectral gap estimate we need to maximize

$$f(b,C) \stackrel{\text{def}}{=} 2\sqrt{Cb(n-1)\vartheta - Cb^2} - C,$$

which can be obtained as

$$\max_{b,C} f(b,C) = f\left(\frac{(n-1)\vartheta}{2}, \frac{(n-1)^2\vartheta^2}{4}\right) = \frac{(n-1)^2\vartheta^2}{4}.$$

The proof of the sharpness is similar as before. Fix $\delta > 4$ and choose

$$\Omega = \mathbf{M}^n_\kappa, \quad \rho = d_{\kappa,x_0}, \quad s = \frac{(n-1)\vartheta}{p}, \quad \text{and} \quad u_\delta = \phi(\rho)e^{-s\rho},$$

where ϕ is the truncation function of (4.8). Since

$$\nabla_\kappa u_\delta = (\phi'(\rho) - s\phi(\rho))e^{-s\rho}\nabla_\kappa \rho,$$

using a polar coordinate transform and the second branch of (4.8) we have

$$\begin{aligned}
\int_\Omega |\nabla_\kappa u_\delta|^2 \,\mathrm{d}v_\kappa &= \int_{\frac{\delta}{2}}^{\delta} (\phi'(t) - s\phi(t))^2 e^{-2st} \frac{\sinh^{n-1}(\vartheta t)}{\vartheta^{n-1}} \,\mathrm{d}t \\
&\geq \frac{s^2}{\vartheta^{n-1}} \int_{\frac{\delta}{2}+1}^{\delta-1} e^{-2st} \sinh^{n-1}(\vartheta t)\,\mathrm{d}t \\
&= \frac{s^2}{\vartheta^{n-1}} \int_{\frac{\delta}{2}+1}^{\delta-1} \left(\frac{1}{2} - \frac{e^{-2\vartheta t}}{2}\right)^{n-1} \mathrm{d}t \\
&\geq \frac{s^2}{\vartheta^{n-1}} \left(\frac{\delta}{2} - 2\right)\left(\frac{1}{2} - \frac{e^{-\vartheta\delta - 2\vartheta}}{2}\right)^{n-1} \\
&\stackrel{\text{def}}{=} E_1(\delta).
\end{aligned}$$

Recall the estimate (4.10) for $p = 2$ to obtain

$$\begin{aligned}
\int_\Omega |\Delta_\kappa u_\delta|^2 \,\mathrm{d}v_\kappa &\leq M_1 + M_2 \\
&\quad + \frac{s^{2p}}{\vartheta^{n-1}} \left(\frac{\delta}{2} - 2\right)\left(p \coth\left(\left(\frac{\delta}{2}+1\right)\vartheta\right) - 1\right)^p \left(\frac{1}{2} - \frac{e^{-2\vartheta(\delta-1)}}{2}\right)^{n-1} \\
&\stackrel{\text{def}}{=} E_2(\delta),
\end{aligned}$$

where M_1 and M_2 is the maximum of the bounded function

$$\Phi(t) = \left(s(2\coth(\vartheta t) - 2)\phi'(t) + s^2(1 - 2\coth(\vartheta t))\phi(t)\right)^2 \left(e^{-\vartheta t}\sinh(\vartheta t)\right)^{n-1},$$

on the intervals $\left[\frac{\delta}{2}, \frac{\delta}{2}+1\right]$ and $[\delta - 1, \delta]$, respectively.

Using the two estimates, we have

$$\lim_{\delta\to\infty} \frac{\displaystyle\int_\Omega |\Delta_\kappa u_\delta|^2 \,\mathrm{d}v_\kappa}{\displaystyle\int_\Omega |\nabla_\kappa u_\delta|^2 \,\mathrm{d}v_\kappa} \leq \lim_{\delta\to\infty} \frac{E_2(\delta)}{E_1(\delta)} = s^2 = \frac{(n-1)^2|\kappa|}{4},$$

hence the inequality is sharp. □

We note that Theorem 4.5 is a novel result, which provides the spectral gap estimate in general Cartan–Hadamard manifolds for $p = 2$. For a short summary of alternative results regarding the buckling problem, see Section 2.3.3.

4.2.3 Higher-order estimates

In the sequel, we present higher-order estimates concerning both the clamped plate problem and the buckling problem. The proofs are built upon Theorems 4.4 & 4.5 and inequality (2.37), hence they are symmetrization-free.

In case of the clamped plate problem, the following higher-order estimates hold.

Theorem 4.6 (see [39]). *Let (M, g) be an n-dimensional Cartan–Hadamard manifold with $n \geq 2$ and sectional curvature $\mathbf{K} \leq \kappa$ for some $\kappa < 0$. Let $p > 1$ and $\Omega \subseteq M$ be a domain. Then for every $u \in C_0^\infty(\Omega)$ and $k \geq 1$ with $k \in \mathbb{N}$, one has*

$$\int_\Omega |\Delta_g^k u|^p \, \mathrm{d}v_g \geq \left(\frac{(n-1)^2(p-1)\kappa^2}{p^2} \right)^{kp} \int_\Omega |u|^p \, \mathrm{d}v_g, \tag{4.12}$$

$$\int_\Omega |\nabla_g \Delta_g^k u|^p \, \mathrm{d}v_g \geq \left(\frac{(n-1)\kappa}{p} \right)^p \left(\frac{(n-1)^2(p-1)\kappa^2}{p^2} \right)^{kp} \int_\Omega |u|^p \, \mathrm{d}v_g. \tag{4.13}$$

Moreover, the constants in (4.12) and (4.13) are sharp.

Proof. Inequality (4.12) can be obtained by iterative applications of Theorem 4.4 for the functions

$$u \rightarrowtail \Delta^l u, \qquad \text{for every} \quad l \in \{0, 1, \ldots, k-1\}.$$

To obtain (4.13), also apply the inequality (2.35) for the function $u \rightarrowtail \Delta^k u$.

To prove sharpness, as before, fix $\delta > 4$ and choose

$$\Omega = \mathbf{M}_\kappa^n, \quad \rho = d_{\kappa, x_0}, \quad s = \frac{(n-1)\vartheta}{p}, \quad \text{and} \quad u_\delta = \phi(\rho) e^{-s\rho},$$

where $\vartheta = \sqrt{-\kappa}$ and ϕ is the truncation function of (4.8). For simplicity, denote

$$L(\delta) = (n-1)\mathbf{ct}_\kappa(\delta) = (n-1)\vartheta \coth(\vartheta\delta) = ps \coth(\vartheta\delta).$$

To obtain the proof for general $k \geq 1$, we have to compute $\Delta_\kappa^k u_\delta$ and give an appropriate lower bound for it. This computation becomes increasingly involved for higher values of k; however, based on the ideas used in case $k = 1$, we can significantly simplify them.

The first observation is that the branches when $t \in [\frac{\delta}{2}, \frac{\delta}{2} + 1]$ and $t \in [\delta - 1, \delta]$ do not have any contribution to the final limit. This is due to the fact that the integrands are bounded, and the integration interval is of unit length; therefore, these integrals are dominated by the leading term provided by the branch when $t \in [\frac{\delta}{2} + 1, \delta - 1]$. The same phenomenon occurs when $k \geq 1$. We can restrict our attention only to this case, and technically we can assume in the sequel that $\phi = 1$.

The second observation is that since u is radially symmetric, we have

$$\Delta_\kappa u_\delta = u_\delta'' + L u_\delta' = s(s - L)e^{-s\rho}.$$

Based on the computation for the case $k = 1$, we are only interested in the asymptotic behavior when $\delta \to \infty$. One can easily verify that the k-th derivatives of $L(\delta)$ satisfy

$$\lim_{\delta \to \infty} L^{(k)}(\delta) = \begin{cases} ps, & \text{if } k = 0, \\ 0, & \text{if } k \geq 1. \end{cases}$$

Using this fact, for the bi-Laplacian one has

$$\begin{aligned}\Delta_\kappa^2 u_\delta &= u_\delta^{(4)} + 2Lu_\delta^{(3)} + L^2 u_\delta'' + L'' u_\delta' + 2L' u_\delta'' + LL' u_\delta' \\ &\sim u_\delta^{(4)} + 2Lu_\delta^{(3)} + L^2 u_\delta'' = s^2(s-L)^2 e^{-s\rho} \sim s^4(1-p)^2 e^{s\rho}, \qquad \text{as } \delta \to \infty.\end{aligned}$$

By a similar argument, for general $k \geq 1$ one has

$$\Delta_\kappa^k u_\delta \sim s^{2k}(1-p)^k e^{-s\rho}, \qquad \text{as } \delta \to \infty. \tag{4.14}$$

Using the estimate (4.9) from the proof of the case when $k = 1$, we obtain

$$\lim_{\delta\to\infty} \frac{\displaystyle\int_\Omega |\Delta_\kappa^k u_\delta|^p \, dv_\kappa}{\displaystyle\int_\Omega |u_\delta|^p \, dv_\kappa} = s^{2kp}(1-p)^{kp} = \left(\frac{(n-1)^2(p-1)\kappa^2}{p^2}\right)^{kp}.$$

Taking the gradient in relation (4.14) implies

$$\nabla_\kappa \Delta_\kappa^k u_\delta \sim -s \cdot s^{2k}(1-p)^k e^{-s\rho}, \qquad \text{as } \delta \to \infty,$$

hence we obtain

$$\lim_{\delta\to\infty} \frac{\displaystyle\int_\Omega |\nabla_\kappa \Delta_\kappa^k u_\delta|^p \, dv_\kappa}{\displaystyle\int_\Omega |u_\delta|^p \, dv_\kappa} = s^p s^{2kp}(1-p)^{kp} = \left(\frac{(n-1)\kappa}{p}\right)^p \left(\frac{(n-1)^2(p-1)\kappa^2}{p^2}\right)^{kp},$$

which concludes the proof. □

In case of the buckling problem, the following higher-order estimates hold.

Theorem 4.7 (see [39]). *Let (M, g) be an n-dimensional Cartan–Hadamard manifold with $n \geq 2$ and sectional curvature $\mathbf{K} \leq \kappa$ for some $\kappa < 0$. If $\Omega \subseteq M$ is a domain, then for every $u \in C_0^\infty(\Omega)$ and $k \geq 1$ one has*

$$\int_\Omega |\Delta_g^k u|^2 \, dv_g \geq \left(\frac{(n-1)\kappa}{2}\right)^{4k-2} \int_\Omega |\nabla_g u|^2 \, dv_g, \tag{4.15}$$

$$\int_\Omega |\nabla_g \Delta_g^k u|^2 \, dv_g \geq \left(\frac{(n-1)\kappa}{2}\right)^{4k} \int_\Omega |\nabla_g u|^2 \, dv_g. \tag{4.16}$$

Proof. Inequality (4.15) can be obtained using inequality (4.12) for

$$p \rightarrowtail 2, \quad u \rightarrowtail \Delta_g u, \quad \text{and} \quad k \rightarrowtail k-1,$$

and applying Theorem 4.5. To obtain the inequality (4.16), combine the previous result with the inequality (2.37) for

$$p \rightarrowtail 2 \quad \text{and} \quad u \rightarrowtail \Delta^k u.$$

The sharpness can be proved similarly as before: choose

$$\Omega = \mathbf{M}_\kappa^n, \quad \rho = d_{\kappa,x_0}, \quad s = \frac{(n-1)\vartheta}{p}, \quad u_\delta = \phi(\rho)e^{-s\rho},$$

where $\vartheta = \sqrt{-\kappa}$ and ϕ is the truncation function of (4.8). By letting $\delta \to \infty$, we obtain the desired result. □

4.3 Applications II: Rellich inequalities

In this section, we present additional utilizations of our general functional inequalities to obtain various Rellich inequalities on Cartan–Hadamard manifolds. First, we use Theorem 4.1 to extend the weighted Rellich inequality to Cartan–Hadamard manifolds. Next, based on this result, we state higher-order Rellich inequalities. Finally, we present short proofs of some, formally well-known Rellich-type inequalities, suggesting further applicability of Theorems 4.1 & 4.2.

4.3.1 Classical and weighted Rellich inequalities

The extension of the weighted Rellich inequality reads as follow; for the Euclidean version, see Mitidieri [83, Theorem 3.1].

Theorem 4.8 (see [39]). *Let (M, g) be an n-dimensional Cartan–Hadamard manifold, with $n \geq 5$. Let $\Omega \subseteq M$ be a domain, and $p, \gamma \in \mathbb{R}$ such that*

$$1 < p < n/2 \quad \text{and} \quad 2 - \frac{n}{p} < \gamma < \frac{n(p-1)}{p}.$$

Fix $x_0 \in \Omega$ and let $\rho = d_{g,x_0}$. Then for every $u \in C_0^\infty(\Omega)$ one has

$$\int_\Omega \rho^{\gamma p} |\Delta_g u|^p \,\mathrm{d}v_g \geq \left(\frac{n}{p} - 2 + \gamma\right)^p \left(\frac{n(p-1)}{p} - \gamma\right)^p \int_\Omega \frac{|u|^p}{\rho^{(2-\gamma)p}} \,\mathrm{d}v_g. \tag{4.17}$$

Proof. In Theorem 4.1 we choose

$$L(t) = \frac{n-1}{t}, \quad W(t) = \frac{C}{t^{(2-\gamma)p}}, \quad w = t^{\gamma p}, \quad G(t) = \frac{a}{t^{2p-2}}, \quad \text{and} \quad H(t) = \frac{b}{t},$$

for all $t \in (0, \sup_\Omega \rho)$ and for some constants $C, a, b > 0$, which will be determined later. Condition **(C1)** clearly holds, while a straightforward computation and the Laplace comparison (see Theorem 2.1/(I)/(i)) imply Condition **(C2)**. Inequality (4.1) from Condition **(C3)** is equivalent to

$$\begin{aligned} f(a,b) &\stackrel{\text{def}}{=} a(p-1)(2(b+1)n - (2+b)^2 p)ap(2b(p-1) + 4p - n - 2)\gamma \\ &\quad - ap^2\gamma^2 - (p-1)a^{\frac{p}{p-1}} \\ &\geq C. \end{aligned}$$

The best value for the constant C is obtained as

$$\begin{aligned} \max_{a,b} f(a,b) &= f\left(\left(\frac{n}{p} - 2 + \gamma\right)^{p-1} \left(\frac{n(p-1)}{p} - \gamma\right)^{p-1}, \frac{n}{p} - 2 + \gamma\right) \\ &= \left(\frac{n}{p} - 2 + \gamma\right)^p \left(\frac{n(p-1)}{p} - \gamma\right)^p, \end{aligned}$$

which provides the desired inequality (4.17). □

Corollary 4.9 (see [39]). *By choosing $\gamma = 0$ in Theorem 4.8, we obtain the extension of the classical Rellich inequality; namely, one has for every $u \in C_0^\infty(\Omega)$ that*

$$\int_\Omega |\Delta_g u|^p \, \mathrm{d}v_g \geq \left(\frac{n}{p} - 2\right)^p \left(\frac{n(p-1)}{p}\right)^p \int_\Omega \frac{|u|^p}{\rho^{2p}} \, \mathrm{d}v_g.$$

In particular, for $p = 2$ one has for every $u \in C_0^\infty(\Omega)$ that

$$\int_\Omega |\Delta_g u|^2 \, \mathrm{d}v_g \geq \frac{n^2(n-4)^2}{16} \int_\Omega \frac{|u|^2}{\rho^4} \, \mathrm{d}v_g. \tag{4.18}$$

4.3.2 Higher-order variants of the classical Rellich inequality

Extension of higher-order Rellich inequalities can be stated as follows; for the Euclidean versions, see Mitidieri [83, Theorem 3.3].

Theorem 4.10 (see [39]). *Let (M, g) be an n-dimensional Cartan–Hadamard manifold, with $n \geq 5$. Let $\Omega \subseteq M$ be a domain, fix $x_0 \in \Omega$ and define $\rho = d_{g,x_0}$. The following inequalities hold:*

(i) *If $k \geq 1$ and $n > 2kp$, then*

$$\int_\Omega |\Delta_g^k u|^p \, \mathrm{d}v_g \geq \Lambda_{\mathrm{r};1}(k,p) \int_\Omega \frac{|u|^p}{\rho^{2kp}} \, \mathrm{d}v_g, \qquad \forall u \in C_0^\infty(\Omega),$$

where

$$\Lambda_{\mathrm{r};1}(k,p) = \prod_{s=1}^{k} \left(\frac{n}{p} - 2s\right)^p \left(\frac{n(p-1)}{p} + 2s - 2\right)^p.$$

(ii) *If $k \geq 1$ and $n > (2k+1)p$, then*

$$\int_\Omega |\nabla_g \Delta_g^k u|^p \, \mathrm{d}v_g \geq \Lambda_{\mathrm{r};2}(k,p) \int_\Omega \frac{|u|^p}{\rho^{(2k+1)p}} \, \mathrm{d}v_g, \qquad \forall u \in C_0^\infty(\Omega)$$

where

$$\Lambda_{\mathrm{r};2}(k,p) = \left(\frac{n-p}{p}\right)^p \prod_{s=1}^{k} \left(\frac{n}{p} - 2s - 1\right)^p \left(\frac{n(p-1)}{p} + 2s - 1\right)^p.$$

Proof. To obtain (i), we apply Theorem 4.8 with the choices

$$u \rightarrowtail \Delta_g^{k-s} u \quad \text{and} \quad \gamma \rightarrowtail 2 - 2s, \qquad \forall s \in \{1, \ldots, k\}.$$

To prove (ii), first apply Theorem 4.8 with the choices

$$u \rightarrowtail \Delta_g^{k-s} u \quad \text{and} \quad \gamma \rightarrowtail 1 - 2s, \qquad \forall s \in \{1, \ldots, k\},$$

and then choose $u \rightarrowtail \Delta_g^k u$ in the following classical Hardy inequality

$$\int_\Omega |\nabla_g u|^p \, \mathrm{d}v_g \geq \left(\frac{n-p}{p}\right)^p \int_\Omega |u|^p \, \mathrm{d}v_g, \qquad \forall u \in C_0^\infty(\Omega),$$

where $1 < p < n$. The latter inequality follows e.g., from Corollary 3.15 for $\alpha = 0$.

The sharpness of $\Lambda_{\mathrm{r};1}(k,p)$ and $\Lambda_{\mathrm{r};2}(k,p)$ can be proved in the usual way. □

4.3.3 Further Rellich inequalities

In this section we present further applications to our general functional inequalities providing short proofs for additional Rellich-type inequalities.

Theorem 4.11 (see [39]). *Let (M,g) be an n-dimensional Cartan–Hadamard manifold $(n \geq 5)$. Let $\Omega = B_{g,x_0}(1) \subseteq M$ be a ball centered at $x_0 \in M$ with unit radius. Define $\rho = d_{g,x_0}$. Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega |\Delta_g u|^2 \, dv_g \geq \frac{n^2(n-4)^4}{16} \int_\Omega \frac{u^2}{\rho^4} \, dv_g + \frac{n(n-4)j_{0,1}^2}{2} \int_\Omega \frac{u^2}{\rho^2} \, dv_g,$$

where $j_{0,1}$ denotes the first positive zero of the Bessel function J_0.

Proof. Apply Theorem 4.1 for $p = 2$ with the following choices:

$$L(t) = \frac{n-1}{t}, \quad G(t) = \frac{n(n-4)}{4t^2}, \quad \text{and} \quad H(t) = \frac{n-4}{2t} + \frac{j_{0,1} \cdot J_1(j_{0,1}t)}{J_0(j_{0,1}t)},$$

for all $t \in (0,1)$. A simple computation yields the desired inequality. □

The second result deals with the case when $\mathbf{K} \leq \kappa$ for some $\kappa < 0$, and can be formulated as follows.

Theorem 4.12 (see [39]). *Let (M,g) be an n-dimensional Cartan–Hadamard manifold with $n \geq 5$ and sectional curvature $\mathbf{K} \leq \kappa$ for some $\kappa < 0$. Let $\Omega \subseteq M$ be a domain, fix $x_0 \in \Omega$ and define $\rho = d_{g,x_0}$. Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\begin{aligned} \int_\Omega |\Delta_g u|^2 \, dv_g \geq \frac{(n-1)^4|\kappa|^2}{16} \int_\Omega u^2 \, dv_g + \frac{(n-1)^2|\kappa|}{8} \int_\Omega \frac{u^2}{\rho^2} \, dv_g \\ + \frac{(n-1)^3(n-3)|\kappa|^2}{8} \int_\Omega \frac{u^2}{\sinh^2(\kappa\rho)} \, dv_g. \end{aligned}$$

Proof. We apply Theorem 4.1 by choosing

$$\begin{aligned} L(t) &= (n-1)\sqrt{-\kappa}\coth(\sqrt{-\kappa}t), \quad G(t) = \frac{(n-1)^2|\kappa|}{4}, \quad \text{and} \\ H(t) &= \frac{(n-1)\sqrt{-\kappa}\coth(\sqrt{-\kappa}t)}{2} - \frac{1}{2t}, \end{aligned}$$

for all $t > 0$. The required inequality follows from a simple computation. □

Remark 4.13. The inequality of Theorem 4.12 shall be compared to the main results of Berchio, Ganguly, and Roychowdhury [14], where the authors established various Rellich-type identities, which in turn imply sharp Rellich-type improvements on the hyperbolic space.

The third result of the section is a simple application of Theorem 4.2.

Theorem 4.14 (see [39]). *Let (M, g) be an n-dimensional Cartan–Hadamard manifold $(n \geq 8)$, $\Omega \subseteq M$ be a domain, $x_0 \in M$ be fixed and define $\rho = d_{g,x_0}$. Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega |\Delta_g u|^2 \, \mathrm{d}v_g \geq \frac{n^2}{4} \int_\Omega \frac{|\nabla_g u|^2}{\rho^2} \, \mathrm{d}v_g.$$

Proof. We apply Theorem 4.2 with the choices

$$L(t) = \frac{n-1}{t}, \quad G(t) = \frac{n(n-4)}{4t^2}, \quad H(t) = \frac{n-4}{2t}, \quad \text{and} \quad W(t) = \frac{n(n-8)}{4},$$

for all $t > 0$, which provides the proof. □

Remark 4.15. Note that Theorem 4.14 is expected to hold for every $n \geq 5$; however, the technical condition $n \geq 8$ is required to guarantee the applicability of Theorem 4.2 ($W > 0$ whenever $n \geq 9$, and if $n = 8$, then $W = 0$, in which case the proof of Theorem 4.2 is obvious). A similar restriction also appeared in the Finsler context when proving quantitative Rellich inequalities; see Kristály and Repovš [73], where another approach was applied.

Chapter 5

Rellich inequalities on space forms

In this chapter, using the idea of Riccati pairs, we provide an alternative approach for proving Rellich inequalities on model space forms. We notice that our method can be extended to Riemannian manifolds with bounded sectional curvature; however, we use the former setting for the simplicity of presentation. We also note that numerous relevant applications are also formulated in this geometrical setting. The chapter is structured as follows.

In Section 5.1 we present our abstract approach. First, in Section 5.1.1 we provide a simplified definition of Riccati pairs (see Definition 5.1), which is tailored to prove Hardy inequalities on space forms (as anticipated in Remark 3.6). Next, we introduce dual Riccati pairs (see Definition 5.2), which can be used to prove Rellich inequalities in the same setting. Finally, we establish three general functional inequalities.

The first inequality connects $\Delta_\kappa u$ and $\nabla_\kappa^{\mathrm{rad}} u$, while the second inequality connects $\Delta_\kappa u$ and $\nabla_\kappa u$; see Theorems 5.3/(i) & 5.3/(ii), respectively. Both inequalities impose an extra condition on the parameter functions, which turns out to be less restrictive in the first case. This motivates the third inequality, which connects $\nabla_\kappa^{\mathrm{rad}} u$ and u (see Theorem 5.5), since its combination with the first inequality connects $\Delta_\kappa u$ and u. Obviously, this can also be done by combining the second inequality with Theorem 3.5, under more strict conditions. For this reason, when presenting applications, we follow the former line of thought. It is also worth mentioning that, if we restrict ourselves to space forms, Theorem 5.5 is stronger than Theorem 3.5, which follows from a simple application of the Cauchy–Schwarz inequality (2.10).

In Section 5.2 we provide applications in the Euclidean setting, presenting both the strengths and the limitations of the method. First, by combining Theorem 5.3/(i) and Theorem 5.5 we provide a general functional inequality formulated in terms of Bessel potentials (see Theorem 5.10). Next, we test two inequalities against our result: The first inequality meets the imposed extra condition (see Corollary 5.11), while the second does not (see Remark 5.12).

In Section 5.3 we provide applications on hyperbolic spaces. First, in Theorem 5.13 we establish a radial inequality, interpolating between a Rellich-type and a spectral gap estimate-type inequality. Next, in Theorem 5.16 we present lower-order counterparts of Theorem 5.13. Finally, in Theorem 5.18 we combine the previous two results, providing a sophisticated non-radial inequality on hyperbolic spaces connecting $\Delta_\kappa u$ and u.

5.1 General functional inequalities

In this section, we present an alternative, abstract approach for proving Rellich-type inequalities on model space forms, using the idea of Riccati pairs from Chapter 3. The choice of the geometrical setting, however, motivates a simplified definition of the original notion that can be used to provide Hardy inequalities. Furthermore, it turns out that Rellich inequalities require a corresponding dual notion. Both concepts are presented in the next section.

5.1.1 Simplified Riccati pairs and dual Riccati pairs

Let $\kappa \leq 0$ and recall from relation (2.7) the function $L_\kappa \colon (0,\infty) \to (0,\infty)$,

$$L_\kappa(t) = (n-1)\mathbf{ct}_\kappa(t).$$

A simplified definition of Riccati pairs refactored for space forms is as follows (for the original version, see Definition 3.3).

Definition 5.1 (see [57]). Let $\kappa \leq 0$ and $\Omega \subseteq \mathbf{M}^n_\kappa$ be an open domain. Fix $x_0 \in \Omega$ and let $\rho = d_{\kappa,x_0}$ the distance from x_0. Suppose that $w, W \colon (0, \sup_\Omega \rho) \to [0,\infty)$ are smooth functions. The couple (L_κ, W) is a (ρ, w)-*simplified Riccati pair* on $(0, \sup_\Omega \rho)$ if there exists a smooth function $G \colon (0, \sup_\Omega \rho) \to \mathbb{R}$ such that the following ODI holds:

$$G'(t) + \left(L_\kappa(t) + \frac{w'(t)}{w(t)}\right) G(t) - G(t)^2 \geq W(t), \qquad \forall t \in (0, \sup_\Omega \rho). \tag{5.1}$$

A function G satisfying (5.1) is said to be *admissible* for (L_κ, W).

The above inequality is a driving force for the Hardy-type functional inequalities. However, as we shall see soon, the Rellich-type inequalities are more compatible with its dual version, which can be stated as follows.

Definition 5.2 (see [57]). Let $\kappa \leq 0$ and $\Omega \subseteq \mathbf{M}^n_\kappa$ be an open domain. Fix $x_0 \in \Omega$ and let $\rho = d_{\kappa,x_0}$ the distance from x_0. Suppose that $v, V \colon (0, \sup_\Omega \rho) \to [0,\infty)$ are smooth functions. The couple (L_κ, V) is a (ρ, v)-*dual Riccati pair* on $(0, \sup_\Omega \rho)$ if there exists a smooth function $H \colon (0, \sup_\Omega \rho) \to \mathbb{R}$ such that the following ODI holds:

$$-H'(t) + \left(L_\kappa(t) - \frac{v'(t)}{v(t)}\right) H(t) - H(t)^2 \geq V(t), \qquad \forall t \in (0, \sup_\Omega \rho). \tag{5.2}$$

A function H satisfying (5.2) is said to be *dual admissible* for (L_κ, V).

We notice that the latter two concepts are in fact equivalent. Indeed, the following changes of functions provide the transitions between the two differential inequalities:

$$H(t) = L_\kappa(t) - G(t), \quad v(t) = w(t), \quad \text{and} \quad V(t) = W(t) - \frac{v'(t)}{v(t)} L_\kappa(t) - L'_\kappa(t).$$

The motivation behind the introduction of the latter concept is to highlight the true driving force of Rellich-type inequalities and to improve the clarity of the presentation. The first result of the section is a general functional inequality, which can be stated as follows.

Theorem 5.3 (see [57]). *Let $\kappa \leq 0$ and $\Omega \subseteq \mathbf{M}^n_\kappa$ be an open domain. Fix $x_0 \in \Omega$ and define $\rho = d_{\kappa,x_0}$. Suppose that (L_κ, V) is a dual Riccati pair on $(0, \sup_\Omega \rho)$ and H is dual admissible for (L_κ, V). Then the following statements hold.*

(i) *For every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega v(\rho)|\Delta_\kappa u|^2 \,\mathrm{d}x_\kappa \geq \int_\Omega v(\rho)V(\rho)|\nabla_\kappa^{\mathrm{rad}} u|^2 \,\mathrm{d}x_\kappa, \tag{5.3}$$

provided that $E_1(t) = (v(t)H(t))' + v(t)H(t)(L_\kappa(t) - 2\mathbf{ct}_\kappa(t)) \geq 0$, for all $t > 0$.

(ii) *For every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega v(\rho)|\Delta_\kappa u|^2 \,\mathrm{d}x_\kappa \geq \int_\Omega v(\rho)V(\rho)|\nabla_\kappa u|^2 \,\mathrm{d}x_\kappa, \tag{5.4}$$

provided that $E_2(t) = 2(v(t)H(t))' + v(t)H(t)(H(t) - 2\mathbf{ct}_\kappa(t)) \geq 0$, for all $t > 0$.

Proof. It is well-known that

$$|\xi|^2 \geq 2\xi\eta - |\eta|^2, \tag{5.5}$$

for every $\xi, \eta \in \mathbb{R}$. Choose

$$\xi = \Delta_\kappa u \quad \text{and} \quad \eta = H(\rho)\langle \nabla_\kappa u, \nabla_\kappa \rho\rangle$$

to obtain

$$|\Delta_\kappa u|^2 \geq 2H(\rho)\Delta_\kappa u \langle \nabla_\kappa u, \nabla_\kappa \rho\rangle - H(\rho)^2\langle \nabla_\kappa u, \nabla_\kappa \rho\rangle^2.$$

Multiplying both sides by $v(\rho) \geq 0$ and integrating over Ω yield

$$\int_\Omega v(\rho)|\Delta_\kappa u|^2 \,\mathrm{d}x_\kappa \geq 2\int_\Omega v(\rho)H(\rho)\Delta_\kappa u\langle \nabla_\kappa u, \nabla_\kappa \rho\rangle \,\mathrm{d}x_\kappa - \int_\Omega v(\rho)H(\rho)^2\langle \nabla_\kappa u, \nabla_\kappa \rho\rangle^2 \,\mathrm{d}x_\kappa.$$

First, applying an integration by parts (2.3) to the second term implies

$$\begin{aligned}
I_1 &\stackrel{\text{def}}{=} 2\int_\Omega v(\rho)H(\rho)\Delta_\kappa u\langle \nabla_\kappa u, \nabla_\kappa \rho\rangle \,\mathrm{d}x_\kappa = -2\int_\Omega \nabla_\kappa \left(v(\rho)H(\rho)\langle \nabla_\kappa u, \nabla_\kappa \rho\rangle\right) \nabla_\kappa u \,\mathrm{d}x_\kappa \\
&= -2\int_\Omega (v(\rho)H(\rho))'\langle \nabla_\kappa u, \nabla_\kappa \rho\rangle^2 \,\mathrm{d}x_\kappa - 2\int_\Omega v(\rho)H(\rho)\,\mathrm{Hess}_\kappa(u)(\nabla_\kappa u, \nabla_\kappa \rho)\,\mathrm{d}x_\kappa \\
&\quad - 2\int_\Omega v(\rho)H(\rho)\,\mathrm{Hess}_\kappa(\rho)(\nabla_\kappa u, \nabla_\kappa u)\,\mathrm{d}x_\kappa.
\end{aligned}$$

Next, another integration by parts and the expression (2.12) concerning $\Delta_\kappa \rho$ yield

$$\begin{aligned}
I_2 &\stackrel{\text{def}}{=} -2\int_\Omega v(\rho)H(\rho)\,\mathrm{Hess}_\kappa(u)(\nabla_\kappa u, \nabla_\kappa \rho)\,\mathrm{d}x_\kappa \\
&= -\int_\Omega v(\rho)H(\rho)\langle \nabla_\kappa |\nabla_\kappa u|^2, \nabla_\kappa \rho\rangle \,\mathrm{d}x_\kappa \\
&= \int_\Omega \left((v(\rho)H(\rho))' + v(\rho)H(\rho)L_\kappa(\rho)\right)|\nabla_\kappa u|^2 \,\mathrm{d}x_\kappa.
\end{aligned}$$

Next, expression (2.13) concerning $\mathrm{Hess}_\kappa(\rho)$, for $V = \nabla_\kappa u$ implies

$$\begin{aligned} I_3 &\overset{\text{def}}{=} -2\int_\Omega v(\rho)H(\rho)\,\mathrm{Hess}_\kappa(\rho)(\nabla_\kappa u, \nabla_\kappa u)\,\mathrm{d}x_\kappa \\ &= 2\int_\Omega v(\rho)H(\rho)\mathbf{ct}_\kappa(\rho)\left(\langle\nabla_\kappa u, \nabla_\kappa\rho\rangle^2 - |\nabla_\kappa u|^2\right)\,\mathrm{d}x_\kappa. \end{aligned}$$

Finally, by the above computation we obtain that

$$\begin{aligned} I_4 &\overset{\text{def}}{=} \int_\Omega v(\rho)|\Delta_\kappa u|^2\,\mathrm{d}x_\kappa \\ &\geq \int_\Omega \left(-2(v(\rho)H(\rho))' + 2v(\rho)H(\rho)\mathbf{ct}_\kappa(\rho) - v(\rho)H(\rho)^2\right)|\nabla_\kappa^{\mathrm{rad}} u|^2\,\mathrm{d}x_\kappa \\ &\quad + \int_\Omega \left((v(\rho)H(\rho))' + v(\rho)H(\rho)L_\kappa(\rho) - 2v(\rho)H(\rho)\mathbf{ct}_\kappa(\rho)\right)|\nabla_\kappa u|^2\,\mathrm{d}x_\kappa. \end{aligned} \tag{5.6}$$

If $E_1 \geq 0$, then inequality (5.6) and the Cauchy–Schwarz inequality (2.10) imply

$$\int_\Omega v(\rho)|\Delta_\kappa u|^2\,\mathrm{d}x_\kappa \geq \int_\Omega \left(-(v(\rho)H(\rho))' + v(\rho)H(\rho)L_\kappa(\rho) - v(\rho)H(\rho)^2\right)|\nabla_\kappa^{\mathrm{rad}} u|^2\,\mathrm{d}x_\kappa,$$

whence the dual Riccati ODI (5.2) yields the desired inequality (5.3).

Similarly, if $E_2 \geq 0$, then again inequalities (5.6), (2.10) and (5.2) imply the desired inequality (5.4), concluding the proof. □

Remark 5.4. We note that the above proof is valid for spaces with bounded sectional curvature. In that case, one needs to apply both a Laplace and a Hessian comparison, which have 'opposite' direction to each other, resulting the boundedness condition. We opted to present our results in model space forms for simplicity and better readability.

The second result of the section is tailored to complement Theorem 5.3/(i). It can be stated as follows.

Theorem 5.5 (see [57]). *Let $\kappa \leq 0$ and $\Omega \subseteq \mathbf{M}_\kappa^n$ be an open domain. Fix $x_0 \in \Omega$ and define $\rho = d_{\kappa,x_0}$. Suppose that (L_κ, V) is a simplified Riccati pair on $(0, \sup_\Omega \rho)$ and H is admissible for (L_κ, V). Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega w(\rho)|\nabla_\kappa^{\mathrm{rad}} u|^2\,\mathrm{d}x_\kappa \geq \int_\Omega w(\rho)W(\rho)u^2\,\mathrm{d}x_\kappa. \tag{5.7}$$

Proof. Choose $\xi = \langle\nabla_\kappa u, \nabla_\kappa\rho\rangle$ and $\eta = -G(\rho)u$ in the convexity inequality (5.5) to obtain

$$\langle\nabla_\kappa u, \nabla_\kappa\rho\rangle^2 \geq -2G(\rho)u\langle\nabla_\kappa u, \nabla_\kappa\rho\rangle - G(\rho)^2u^2.$$

Multiplying both sides by $w(\rho) \geq 0$ and integrating over Ω yield

$$\int_\Omega w(\rho)\langle\nabla_\kappa u, \nabla_\kappa\rho\rangle^2\,\mathrm{d}x_\kappa \geq -\int_\Omega 2w(\rho)G(\rho)u\langle\nabla_\kappa u, \nabla_\kappa\rho\rangle\,\mathrm{d}x_\kappa - \int_\Omega w(\rho)G(\rho)^2u^2\,\mathrm{d}x_\kappa.$$

Integration by parts (2.3) implies

$$-\int_\Omega 2w(\rho)G(\rho)u\langle\nabla_\kappa u, \nabla_\kappa\rho\rangle\,\mathrm{d}x_\kappa = \int_\Omega (w(\rho)G'(\rho) + w'(\rho)G(\rho) + w(\rho)G(\rho)L_\kappa(\rho))u^2\,\mathrm{d}x_\kappa,$$

whence the Riccati ODI (5.1) finishes the proof of the desired inequality (5.7). □

Remark 5.6. The latter proof is analogous to the proof of Theorem 3.1/(i) & 3.5, where similar steps with choices $\xi = \nabla_\kappa u$ and $\eta = -G(\rho)u\nabla_\kappa\rho$ imply the inequality:

$$\int_\Omega w(\rho)|\nabla_\kappa u|^2\,\mathrm{d}x_\kappa \geq \int_\Omega w(\rho)W(\rho)u^2\,\mathrm{d}x_\kappa, \qquad \forall u \in C_0^\infty(\Omega). \tag{5.8}$$

According to the Cauchy–Schwarz inequality (2.10), the inequality stated in Theorem 5.5 is stronger than the inequality (5.8).

Theorem 5.3/(i) can be used efficiently in combination with Theorem 5.5 to prove Rellich-type inequalities. For illustrative purposes, we sketch an alternative proof of the classical Rellich inequality (4.18) in $\mathbb{R}^n$.

Example 5.7 (see [57])**.** If $\Omega \subseteq \mathbb{R}^n$, with $n \geq 5$, then for every $u \in C_0^\infty(\Omega)$ one has

$$\int_\Omega |\Delta u|^2\,\mathrm{d}x \geq \frac{n^2}{4}\int_\Omega \frac{|\nabla^{rad}u|^2}{|x|^2}\,\mathrm{d}x \geq \frac{n^2(n-4)^2}{16}\int_\Omega \frac{u^2}{|x|^4}\,\mathrm{d}x. \tag{5.9}$$

Proof. On the one hand, in Theorem 5.3, let us choose

$$v(t) \equiv 1, \quad H(t) = \frac{n}{2t}, \quad \text{and} \quad V(t) = \frac{n^2}{4t^2}, \qquad \forall t > 0.$$

For these choices, the dual Riccati inequality (5.2) is verified with equality; moreover,

$$E_1(t) = \frac{n(n-4)}{2t^2} \quad \text{and} \quad E_2(t) = \frac{n(n-8)}{4t^2}, \qquad \forall t > 0,$$

hence, for every $u \in C_0^\infty(\Omega)$ one has

$$\int_\Omega |\Delta u|^2\,\mathrm{d}x \geq \frac{n^2}{4}\int_\Omega \frac{|\nabla^{rad}u|^2}{|x|^2}\,\mathrm{d}x, \qquad (\text{if } n \geq 5), \tag{5.10}$$

$$\int_\Omega |\Delta u|^2\,\mathrm{d}x \geq \frac{n^2}{4}\int_\Omega \frac{|\nabla u|^2}{|x|^2}\,\mathrm{d}x, \qquad (\text{if } n \geq 8). \tag{5.11}$$

On the other hand, in Theorem 5.5 choose

$$w(t) = \frac{n^2}{4t^2}, \quad G(t) = \frac{n-4}{2t}, \quad \text{and} \quad W(t) = \frac{n^2(n-4)^2}{16t^4}, \qquad \forall t > 0,$$

and combine the result with the inequality (5.10) to obtain (5.9). □

We conclude this section with additional comments.

Remark 5.8. (a) As one can see, our results provide an efficient framework to prove Rellich-type inequalities connecting $|\Delta u|^2$ and u^2 by combining Theorem 5.3/(i) and Theorem 5.5. We notice that this can also be done by combining Theorem 5.3/(ii) and Theorem 3.5 but at the price of some additional conditions, typically involving restrictive dimension constraints.

(b) When presenting applications, we restrict ourselves to the first approach. We note, however, that using Theorem 5.3/(ii), each of the obtained inequalities admits a non-radial counterpart; in order to avoid technicalities we omit them from the presentation.

(c) We also note that our results use a simpler approach for Rellich inequalities than other results from the literature, making them easy-to-use alternatives. In the next sections, some exact comparisons are made.

5.2 Applications I: Inequalities on Euclidean spaces

In the Euclidean setting, a number of well-known Rellich inequalities are discussed by Ghoussoub and Moradifam [50] in terms of Bessel potentials and/or Bessel pairs. After recalling the related concepts, we discuss two novel inequalities among them, which highlight the strengths and the limitations of our method. For the original proof of the selected inequalities (5.14) & (5.15), see Adimurthi, Grossi, and Santra [2].

For better readability, we repeat the definition of Bessel potentials and Bessel pairs introduced by Ghoussoub and Moradifam [50]; see Definitions 2.9 & 2.11, respectively. A function $Z > 0$ is a *Bessel potential* on $(0, R)$ if there exist a constant $c > 0$ and a function $z > 0$ such that following ODE holds:

$$z''(t) + \frac{z'(t)}{t} + c \cdot Z(t) \cdot z(t) = 0, \qquad \forall t \in (0, R). \tag{5.12}$$

The couple (X, Y) of positive functions is a *Bessel pair* on $(0, R)$ if there exist a constant $C > 0$ and a function $y > 0$ such that following ODE holds:

$$y''(t) + \left(\frac{n-1}{t} + \frac{X'(t)}{X(t)}\right) y'(t) + \frac{C \cdot Y(t)}{X(t)} y(t) = 0, \qquad \forall t \in (0, R). \tag{5.13}$$

The relation between the latter two concepts is presented in Proposition 2.12. The next proposition should clarify how they are related to simplified Riccati pairs and dual Riccati pairs.

Proposition 5.9 (see [57]). *Concerning Bessel potentials* (5.12)*, Bessel pairs* (5.13)*, simplified Riccati pairs* (5.1) *and dual Riccati pairs* (5.2)*, the following relations hold:*

(i) If equation (5.12) *holds on* $(0, R)$*, then relation* (5.1) *holds as well with equality, for the choices:*

$$G(t) = -\frac{z'(t)}{z(t)} + \frac{n-2}{2t}, \quad L_\kappa(t) = \frac{n-1}{t}, \quad w(t) \equiv 1, \quad \textit{and}$$

$$W(t) = \frac{(n-2)^2}{4t^2} + c \cdot Z(t).$$

(ii) If equation (5.12) *holds on* $(0, R)$*, then relation* (5.2) *holds as well with equality, for the choices:*

$$H(t) = \frac{n}{2t} + \frac{z'(t)}{z(t)}, \quad L_\kappa(t) = \frac{n-1}{t}, \quad v(t) \equiv 1, \quad \textit{and} \quad V(t) = \frac{n^2}{4t^2} + c \cdot Z(t).$$

(iii) If equation (5.13) *holds on* $(0, R)$*, then relation* (5.1) *holds as well with equality, for the choices:*

$$G(t) = -\frac{y'(t)}{y(t)}, \quad L_\kappa(t) = \frac{n-1}{t}, \quad w(t) = X(t), \quad \textit{and} \quad W(t) = \frac{C \cdot Y(t)}{X(t)}.$$

Using Theorems 5.3/(i) & 5.5 we obtain the following general result.

Theorem 5.10 (see [57]). *Let $n \geq 5$ and $B \subseteq \mathbb{R}^n$ be a ball centered at the origin with radius $R > 0$. Suppose that Z is a Bessel potential on $(0, R)$ with solution z and best constant c, such that Condition (**Z**) holds. Define $H(t) = \frac{n}{2t} + \frac{z'(t)}{z(t)}$. If $E_1(t) = H'(t) + H(t) \cdot \frac{n-3}{t} \geq 0$ for all $t \in (0, R)$, then for every $u \in C_0^\infty(B)$ one has*

$$\int_B |\Delta u|^2 \, \mathrm{d}x \geq \frac{n^2(n-4)^2}{16} \int_B \frac{u^2}{|x|^4} \, \mathrm{d}x + c \left(\frac{n^2}{4} + \frac{(n-\lambda-2)^2}{4} \right) \int_B \frac{Z(|x|)u^2}{|x|^2} \, \mathrm{d}x.$$

Proof. Proposition 5.9/(ii) implies that the dual Riccati ODI (5.2) is verified for

$$H(t) = \frac{n}{2t} + \frac{z'(t)}{z(t)}, \quad L_\kappa(t) = \frac{n-1}{t}, \quad v(t) \equiv 1, \quad \text{and} \quad V(t) = \frac{n^2}{4t^2} + c \cdot Z(t).$$

On the one hand, Theorem 5.3/(i) implies

$$\int_B |\Delta u|^2 \, \mathrm{d}x \geq \frac{n^2}{4} \int_B \frac{|\nabla^{\mathrm{rad}} u|^2}{|x|^2} \, \mathrm{d}x + c \int_B Z(t) |\nabla^{\mathrm{rad}} u|^2 \, \mathrm{d}x.$$

On the other hand, using Proposition 2.13 & 5.9/(iii) and Theorem 5.5 we obtain

$$\int_B \frac{|\nabla^{\mathrm{rad}} u|^2}{|x|^2} \, \mathrm{d}x \geq \frac{(n-4)^2}{4} \int_B \frac{u^2}{|x|^4} \, \mathrm{d}x + c \int_B \frac{Z(|x|)u^2}{|x|^2} \, \mathrm{d}x,$$
$$\int_B Z(|x|) |\nabla^{\mathrm{rad}} u|^2 \, \mathrm{d}x \geq \frac{(n-\lambda-2)^2}{4} \int_B \frac{Z(|x|)u^2}{|x|^2} \, \mathrm{d}x.$$

Combining these results yields the desired inequality. □

We note that the proof of the above result is analogous to [50, Theorem 3.6]. Using the notations of (2.46), i.e., for any function h let $h_{[0]}(t) = t$, $h_{[1]}(t) = h(t)$, and $h_{[i]}(t) = h(h_{[i-1]}(t))$, for all $i \geq 2$; a corollary of Theorem 5.10 can be stated as follows.

Corollary 5.11 (see [57]). *Let $n \geq 5$ and $B \subseteq \mathbb{R}^n$ be a ball centered at the origin with radius $R > 0$. If $k \geq 1$ and $r = R \cdot \exp_{[k-1]}(e)$, then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_B |\Delta u|^2 \, \mathrm{d}x \geq \frac{n^2(n-4)^2}{16} \int_B \frac{u^2}{|x|^4} \, \mathrm{d}x$$
$$+ \left(1 + \frac{n(n-4)}{8}\right) \sum_{j=1}^{k} \int_B \frac{u^2}{|x|^4} \left(\prod_{i=1}^{j} \log_{[i]} \left(\frac{r}{|x|} \right) \right)^{-2} \mathrm{d}x. \tag{5.14}$$

Proof. By Example 2.10/(b) the functions

$$Z_{k,r}(t) = \sum_{j=1}^{k} \frac{1}{t^2} \left(\prod_{i=1}^{j} \log_{[i]} \left(\frac{r}{t} \right) \right)^{-2} \quad \text{and} \quad z_{k,r}(t) = \left(\prod_{i=1}^{k} \log_{[i]} \left(\frac{r}{t} \right) \right)^{\frac{1}{2}}$$

satisfy (5.12) for every $t \in (0, R)$, with the best constant $c = \frac{1}{4}$; moreover, Condition (**Z**) holds for $\lambda = 2$.

Once we show that for all $t \in (0, R)$ one has

$$E_1(t) = H'(t) + H(t) \cdot \frac{n-3}{t} > 0, \quad \text{where} \quad H(t) = \frac{n}{2t} + \frac{z'_{k,r}(t)}{z_{k,r}(t)},$$

we can apply Theorem 5.10, which provides the desired inequality. To see this, using relation (5.12) it is enough to show for all $t \in (0, R)$ that

$$-q_{k,r}(t)^2 + (n-4)q_{k,r}(t) + \frac{n(n-4)}{2} - \frac{t^2}{4} Z_{k,r}(t) \geq 0, \quad \text{where} \quad q_{k,r}(t) = \frac{t \cdot z'_{k,r}(t)}{z_{k,r}(t)}.$$

To see this, observe that for all $t \in (0, R)$ and $i \in \{1, 2, \ldots, k-1\}$ one has

$$\log_{[k]}\left(\frac{r}{t}\right) \geq 1 \quad \text{and} \quad \log_{[i]}\left(\frac{r}{t}\right) \geq \exp_{[k-1-i]}(e) \geq e \geq 2.$$

For the products, the following estimates hold for all $t \in (0, R)$ and $j \in \{1, 2, \ldots, k-1\}$:

$$\prod_{i=1}^{k} \log_{[i]}\left(\frac{r}{t}\right) \geq 2^{k-1} \quad \text{and} \quad \prod_{i=1}^{j} \log_{[i]}\left(\frac{r}{t}\right) \geq 2^j.$$

On the one hand, by simple computation, we obtain for all $t \in (0, R)$ that

$$q_{k,r}(t) = -\frac{1}{2}\sum_{j=1}^{k}\left(\prod_{i=1}^{j} \log_{[i]}\left(\frac{r}{t}\right)\right)^{-1} \geq -\frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} \cdots + \frac{1}{2^{k-1}} + \frac{1}{2^{k-1}}\right) > -1$$

and $q_{k,r}(t) < 0$. On the other hand, for all $t \in (0, R)$ one has

$$-\frac{t^2 Z_{k,r}(t)}{4} \geq -\frac{1}{4}\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \cdots + \frac{1}{2^{2(k-1)}} + \frac{1}{2^{2(k-1)}}\right) \geq -\frac{1}{2}.$$

By the above computations, it is enough to show that

$$-q^2 + (n-4)q + \frac{n^2 - 4n - 1}{2} \geq 0, \qquad \forall q \in (-1, 0).$$

The zeros of the latter polynomial are

$$q_{\pm} = \frac{n-4}{2} \pm \frac{1}{2}\sqrt{3n^2 - 16n + 14},$$

thus, a basic computation implies $q_- \leq -1 < 0 < q_+$ for every $n \geq 5$, which concludes the proof. $\square$

The limitations of Theorem 5.10 can be illustrated using Example 2.10/(c) as follows.

Remark 5.12. Let $\ell(t) = \frac{1}{1-\log(t)}$, $k \geq 1$, $R > 0$ and consider the inequality

$$\int_\Omega |\Delta u|^2 \, \mathrm{d}x \geq \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} \, \mathrm{d}x + \left(1 + \frac{n(n-4)}{8}\right) \sum_{j=1}^{k} \int_\Omega \frac{u^2}{|x|^4} \prod_{i=1}^{j} \ell_{[i]}^2 \left(\frac{|x|}{R}\right) \mathrm{d}x. \tag{5.15}$$

The inequality (5.15) is generated by the Bessel potential with parameters

$$\widetilde{Z}_{k,R}(t) = \sum_{j=1}^{k} \frac{1}{t^2} \prod_{i=1}^{j} \ell_{[i]}^2 \left(\frac{t}{R}\right), \quad \widetilde{z}_{k,R}(t) = \left(\prod_{i=1}^{k} \ell_{[i]}\left(\frac{t}{R}\right)\right)^{-\frac{1}{2}}, \qquad \forall t \in (0, R),$$

and $c = \frac{1}{4}$. Moreover, Condition (**Z**) holds for $\lambda = 2$. Observe that

$$\lim_{t \to R} E_1(t) = \lim_{t \to R} \left(H'(t) + H(t) \cdot \frac{n-3}{t}\right) = \frac{n-k}{2}, \quad \text{where} \quad H(t) = \frac{n}{2t} + \frac{\widetilde{z}'_{k,r}(t)}{\widetilde{z}_{k,r}(t)}.$$

The above relation shows that if $k > n$, the positivity condition does not hold, and therefore, Theorem 5.10 cannot be applied.

5.3 Applications II: Inequalities on hyperbolic spaces

In this section, we show that our results can be used to provide Rellich inequalities on hyperbolic spaces as well. To illustrate this, we provide an improved Rellich inequality, which follows from Theorem 5.3/(i) & 5.5 for specific choices of parameter functions. We also note that other alternative choices lead to several different inequalities, yielding various possibilities.

The first result in this setting is the following interpolation inequality.

Theorem 5.13 (see [57]). *Let $\kappa < 0$, $n \geq 5$, and $\Omega \subseteq \mathbb{H}_\kappa^n$ be an open domain. Fix $x_0 \in \Omega$ and denote by $\rho = d_{\kappa,x_0}$ the Riemannian distance from x_0. Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\begin{aligned}\int_\Omega |\Delta_\kappa u|^2 \, \mathrm{d}x_\kappa \geq |\kappa|\lambda \int_\Omega |\nabla_\kappa^{\mathrm{rad}} u|^2 \, \mathrm{d}x_\kappa + h_n^2(\lambda) \int_\Omega \frac{|\nabla_\kappa^{\mathrm{rad}} u|^2}{\rho^2} \, \mathrm{d}x_\kappa \\ + |\kappa| \left(\frac{n^2}{4} - h_n^2(\lambda)\right) \int_\Omega \frac{|\nabla_\kappa^{\mathrm{rad}} u|^2}{\sinh^2(\sqrt{-\kappa}\rho)} \, \mathrm{d}x_\kappa \\ + \gamma_n(\lambda) h_n(\lambda) \int_\Omega \frac{(\rho \, \mathbf{ct}_\kappa(\rho) - 1)}{\rho^2} |\nabla_\kappa^{\mathrm{rad}} u|^2 \, \mathrm{d}x_\kappa,\end{aligned} \tag{5.16}$$

where $0 \leq \lambda \leq \frac{(n-1)^2}{4}$, $\gamma_n(\lambda) = \sqrt{(n-1)^2 - 4\lambda}$ and $h_n(\lambda) = \frac{\gamma_n(\lambda)+1}{2}$.

Proof. In Theorem 5.3/(i) let us choose $v(t) \equiv 1$, as well as

$$H(t) = \left(\frac{n}{2} - h_n(\lambda)\right) \mathbf{ct}_\kappa(t) + \frac{h_n(\lambda)}{t},$$
$$V(t) = |\kappa|\lambda + \frac{h_n^2(\lambda)}{t^2} + |\kappa| \left(\frac{n^2}{4} - h_n^2(\lambda)\right) \frac{1}{\sinh^2(\sqrt{-\kappa}t)} + \gamma_n(\lambda)h_n(\lambda)\frac{t\,\mathbf{ct}_\kappa(t) - 1}{t^2},$$

for all $t > 0$.

Observe that the dual Riccati ODI (5.2) is verified with equality; moreover,

$$E_1(t) = |\kappa| \left(\frac{n}{2} - h_n(\lambda)\right) + \frac{h_n(\lambda)((n-3)t\,\mathbf{ct}_\kappa(t) - 1)}{t^2} + |\kappa|(n-4)\left(\frac{n}{2} - h_n(\lambda)\right)\coth^2(\sqrt{-\kappa}t)$$

is positive for all $t > 0$, since $n \geq 5$, $\frac{1}{2} \leq h_n(\lambda) \leq \frac{n}{2}$, and $t\,\mathbf{ct}_\kappa(t) > 1$, $\forall t > 0$. □

Corollary 5.14 (see [57]). *Choose $\lambda = 0$ in (5.16) to obtain for every $u \in C_0^\infty(\Omega)$ that*

$$\int_\Omega |\Delta_\kappa u|^2 \,\mathrm{d}x_\kappa \geq \frac{n^2}{4}\int_\Omega \frac{|\nabla_\kappa^{\mathrm{rad}} u|^2}{\rho^2}\,\mathrm{d}x_\kappa + \frac{n(n-1)}{2}\int_\Omega \frac{(\rho\,\mathbf{ct}_\kappa(\rho) - 1)}{\rho^2}|\nabla_\kappa^{\mathrm{rad}} u|^2\,\mathrm{d}x_\kappa.$$

Corollary 5.15 (see [57]). *Choose $\lambda = \frac{(n-1)^2}{4}$ in (5.16) to obtain for every $u \in C_0^\infty(\Omega)$ that*

$$\int_\Omega |\Delta_\kappa u|^2 \,\mathrm{d}x_\kappa \geq \frac{(n-1)^2|\kappa|}{4}\int_\Omega |\nabla_\kappa^{\mathrm{rad}} u|^2\,\mathrm{d}x_\kappa + \frac{1}{4}\int_\Omega \frac{|\nabla_\kappa^{\mathrm{rad}} u|^2}{\rho^2}\,\mathrm{d}x_\kappa + \frac{(n^2-1)|\kappa|}{4}\int_\Omega \frac{|\nabla_\kappa^{\mathrm{rad}} u|^2}{\sinh^2(\sqrt{-\kappa}\rho)}\,\mathrm{d}x_\kappa.$$

Lower-order counterparts of Corollary 5.15 can be stated as follows.

Theorem 5.16 (see [57]). *Let $\kappa < 0$, $n \geq 5$, and $\Omega \subseteq \mathbb{H}_\kappa^n$ be an open domain. Fix $x_0 \in \Omega$ and denote by $\rho = d_{\kappa,x_0}$ the Riemannian distance from x_0. Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega |\nabla_\kappa^{\mathrm{rad}} u|^2\,\mathrm{d}x_\kappa \geq \frac{(n-1)^2|\kappa|}{4}\int_\Omega u^2\,\mathrm{d}x_\kappa + \frac{1}{4}\int_\Omega \frac{u^2}{\rho^2}\,\mathrm{d}x_\kappa + \frac{(n-1)(n-3)|\kappa|}{4}\int_\Omega \frac{u^2}{\sinh^2(\sqrt{-\kappa}\rho)}\,\mathrm{d}x_\kappa, \tag{5.17}$$

$$\int_\Omega \frac{|\nabla_\kappa^{\mathrm{rad}} u|^2}{\rho^2}\,\mathrm{d}x_\kappa \geq \frac{9}{4}\int_\Omega \frac{u^2}{\rho^4}\,\mathrm{d}x_\kappa - (n-1)\int_\Omega \frac{\mathbf{ct}_\kappa(\rho)}{\rho^3}\,\mathrm{d}x_\kappa + \frac{(n-1)^2|\kappa|}{4}\int_\Omega \frac{u^2}{\rho^2}\,\mathrm{d}x_\kappa + \frac{(n-1)(n-3)|\kappa|}{4}\int_\Omega \frac{u^2}{\rho^2\sinh^2(\sqrt{-\kappa}\rho)}\,\mathrm{d}x, \tag{5.18}$$

$$\int_\Omega \frac{|\nabla_\kappa^{\mathrm{rad}} u|^2}{\sinh^2(\sqrt{-\kappa}\rho)}\,\mathrm{d}x_\kappa \geq \frac{1}{4}\int_\Omega \frac{u^2}{t^2\sinh^2(\sqrt{-\kappa}\rho)}\,\mathrm{d}x_\kappa + \frac{(n-3)^2|\kappa|}{4}\int_\Omega \frac{u^2}{\sinh^2(\sqrt{-\kappa}\rho)}\,\mathrm{d}x_\kappa + \frac{(n-3)(n-5)|\kappa|}{4}\int_\Omega \frac{u^2}{\sinh^4(\sqrt{-\kappa}\rho)}\,\mathrm{d}x_\kappa. \tag{5.19}$$

Proof. All these inequalities follow from Theorem 5.5. In each case, the choice of G is

$$G_1(t) = \frac{n-1}{2}\mathbf{ct}_\kappa(t) - \frac{1}{2t}, \quad G_2(t) = \frac{n-1}{2}\mathbf{ct}_\kappa(t) - \frac{3}{2t}, \quad \text{and } G_3(t) = \frac{n-3}{2}\mathbf{ct}_\kappa(t) - \frac{1}{2t},$$

respectively, while

$$L_\kappa(t) = (n-1)\mathbf{ct}_\kappa(t),$$

and the functions w and W must be chosen corresponding to the left- and right-hand side of the given inequality. □

Remark 5.17. We note that non-radial versions of the latter three inequalities can be obtained using Theorem 3.5 by choosing the same parameter functions; see also Section 3.2.4 and the results therein.

We conclude the section with the following result, which combines Theorem 5.16 and Corollary 5.15. This last result can be stated as follows.

Theorem 5.18 (see [57]). *Let $\kappa < 0$, $n \geq 5$, and $\Omega \subseteq \mathbb{H}^n_\kappa$ be an open domain. Fix $x_0 \in \Omega$ and denote $\rho = d_{\kappa,x_0}$ the Riemannian distance from x_0. Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\begin{aligned}
\int_\Omega |\Delta_\kappa u|^2 \,\mathrm{d}x_\kappa \geq{}& \frac{(n-1)^4|\kappa|}{16}\int_\Omega u^2 \,\mathrm{d}x_\kappa + \frac{(n-1)^2|\kappa|}{8}\int_\Omega \frac{u^2}{\rho^2}\,\mathrm{d}x_\kappa \\
&+ \frac{(n-1)^2|\kappa|}{8}\int_\Omega \frac{u^2}{\rho^2\sinh^2(\sqrt{-\kappa}\rho)}\,\mathrm{d}x_\kappa \\
&+ \frac{(n-1)(n-3)(n^2-2n-1)\kappa^2}{8}\int_\Omega \frac{u^2}{\sinh^2(\sqrt{-\kappa}\rho)}\,\mathrm{d}x_\kappa \\
&- \frac{(n-1)\kappa}{4}\int_\Omega \frac{\mathbf{ct}_\kappa(\rho)|u^2|}{t^3}\,\mathrm{d}x_\kappa \\
&+ \frac{(n^2-1)(n-3)(n-5)\kappa^2}{16}\int_\Omega \frac{u^2}{\sinh^4(\sqrt{-\kappa}\rho)}\,\mathrm{d}x_\kappa + \frac{9}{16}\int_\Omega \frac{u^2}{\rho^4}\,\mathrm{d}x_\kappa.
\end{aligned}$$

Proof. Combining inequalities (5.17), (5.18) and (5.19) as well as inequality (5.16), and a simple computation yield the proof. □

Chapter 6

Hardy inequalities on Finsler manifolds

In this chapter, we revisit Riccati pairs and Hardy-type inequalities, but now form a Finslerian point of view. For a brief summary of relevant concepts of Finsler geometry, see Section 2.2.

The natural habitat for the Finslerian Hardy-type inequalities is a *Finsler metric measure manifold* (M, F, m) (or FMMM for short), which is a differentiable manifold M equipped with a smooth, positive-homogeneous metric F on the tangent bundle TM, and a smooth positive measure m. When formulating Hardy inequalities in this setting, the expression $|\nabla u(x)|$ is replaced by either $\max\{F^*(\pm Du)\}$ or $F^*(Du)$, where F^* is the dual metric of F, and Du denotes the *derivative* of u (see relations (2.16) & (2.23)). The quantity $|x|$ is typically replaced by a *distance function* $\rho = d(x_0, \cdot)$ from a point $x_0 \in M$. It turns out that three quantities control the behavior of Hardy inequalities:

- The *flag curvature* $\mathbf{K}$ (see relation (2.26)), which is the natural Finslerian extension of the sectional curvature of Riemannian manifolds.
- The S*-curvature* $\mathbf{S}$ (see relation (2.27)), which is a pure Finslerian quantity that measures the rate of change of *distortion* along geodesics. For convenience, one can also use the *reduced* S*-curvature* $\overline{\mathbf{S}} = \mathbf{S}/F$.
- The *reversibility* $\lambda_F(M) = \sup_{(x,y)\in TM} \mathbf{rev}(x, y)$, where $\mathbf{rev}(x, y) = \frac{F(x,-y)}{F(x,y)}$. Note that $1 \le \lambda_F(M) \le +\infty$. If $\lambda_F(M) = 1$, then the manifold is said to be *reversible*. In particular, every Riemannian metric is reversible.

Focusing on spaces with non-positive curvature ($\mathbf{K} \le 0$), a fundamental inequality is the Finslerian version of the classical Hardy inequality, which is due to Huang, Kristály, and Zhao [56, Theorem 1.4]. It states that if $\mathbf{K} \le 0$ and $\mathbf{S} \le 0$, then one has

$$\int_M \max\{F^*(\pm Du)\}^2 \, \mathsf{dm} \ge \left(\frac{n-2}{2}\right)^2 \int_M \frac{u^2}{\rho^2} \, \mathsf{dm}, \quad \forall u \in C_0^\infty(M); \tag{6.1}$$

in addition, if F is reversible, then the constant $\left(\frac{n-2}{2}\right)^2$ is *sharp*. On the other hand, since $\lambda_F(M) F^*(Du) \ge \max\{F^*(\pm Du)\}$, we also have

$$\lambda_F(M)^2 \int_M F^*(Du)^2 \, \mathsf{dm} \ge \left(\frac{n-2}{2}\right)^2 \int_M \frac{u^2}{\rho^2} \, \mathsf{dm}, \quad \forall u \in C_0^\infty(M). \tag{6.2}$$

Note that if $\lambda_F(M) = 1$, then inequalities (6.1) & (6.2) coincide. If $\lambda_F(M) = +\infty$, then left hand side of the latter inequality blows up. In this case, the authors in [56, Example 6.2] showed that concerning the *Funk metric* with $\mathbf{K} = -\frac{1}{2}$, $\overline{\mathbf{S}} = \frac{n+1}{2}$, and $\lambda_F = +\infty$ one has

$$\inf_{u \in C_0^\infty(M)} \frac{\int_M F^{*2}(Du)\,\mathsf{dm}}{\int_M \frac{u^2}{\rho^2}\,\mathsf{dm}} = 0,$$

which aligns (in some sense) with inequality (6.2). We also refer to Kristály, Li, and, Zhao [75, Theorem 1.1] for additional results describing the failure of various functional inequalities. In the remaining case when $1 < \lambda_F(M) < \infty$, to the best of our knowledge, there is no evidence on the sharpness of (6.1) & (6.2). In fact, we are not aware of any particular examples where the sharpness persists.

Concerning spaces with strong negative curvature, ($\mathbf{K} \leq -\kappa^2 < 0$) a fundamental inequality is the celebrated spectral gap estimate of McKean [79]: if a Riemannian manifold has strong negative sectional curvature, then there exist a positive, domain independent lower bound for the first eigenvalue of the Laplacian (which is also the fundamental tone of the fixed membrane). The Finslerian version of this result is due to Yin and He (see [108, Theorem 3.5]), which can be stated as follows (for $p = 2$): if $\mathbf{K} \leq -\kappa^2$ and $\sup_{TM} |\overline{\mathbf{S}}| < (n-1)\kappa$ for some $\kappa > 0$, and $\lambda_F(M) < \infty$, then

$$\inf_{u \in W_0^{1,2}(M)} \frac{\int_M F^*(Du)^2\,\mathsf{dm}}{\int_M |u|^2\,\mathsf{dm}} \geq \left(\frac{(n-1)\kappa - \sup_{TM} |\overline{\mathbf{S}}|}{2\lambda} \right)^2. \tag{6.3}$$

The goal of tha chapter is twofold. Fist, we provide a Finslerian extension of the method of Riccati pairs, which can yield inequalities of type (6.1), (6.2) & (6.3). Next, we address the issue of sharpness, and provide some special Finsler manifolds on which the obtained inequalities turn out to be sharp. Our constructions also suggest that it is enough to consider the relevant curvature assumptions along geodesics starting from a point. The plan of the chapter is as follows.

In Section 6.1 we present out Finslerian extension of Riccati pairs, for the main result, see Theorem 6.1. In Section 6.1.1 we provide some applications: In Theorem 6.3 we prove a weighted version of Hardy inequalities (6.1) & (6.2). In Theorem 6.4, we provide a McKean-type spectral gap estimate in the Finsler setting.

In Section 6.2 we address the issue of sharpness in case of non-positive flag curvature. First, in Section 6.2.1 we construct a family of Finsler spaces on which we investigate the sharpness (see Theorem 6.5). Next, in Section 6.2.2 we establish some sufficient conditions for the sharpness of a Hardy-type inequality on the aforementioned spaces (see Theorem 6.7). Finally, in Section 6.2.3 we provide an application: We prove the sharpness of (the weighted version of) inequalities (6.1) & (6.2) on the constructed spaces (see Theorem 6.8 and Corollary 6.9).

In Section 6.3 we proceed similarly in case of the strong negative flag curvature. First, in Section 6.3.1 we construct a family of Finsler spaces on which we investigate the sharpness (see Theorem 6.10). Next, in Section 6.3.2 we establish some sufficient conditions for the sharpness of a Hardy-type inequality on the constructed spaces (see Theorem 6.12). Finally, in Section 6.3.3 we provide an application: We prove the sharpness of a McKean-type estimate (see Theorem 6.13).

6.1 A Finslerian extension of Riccati pairs

In this section, we present a Finslerian extension of the approach of Riccati pairs presented in Section 3.2. The key ideas of the proof are similar, however, some extra workarounds are needed due to the specificities of the Finsler geometry. Our extension can be stated as follows.

Theorem 6.1 (see [58]). *Let (M, F, m) be a forward complete, non-compact FMMM, with dimension $n \geq 2$ and reversibility $\lambda \in (1, \infty)$. Let $\Omega \subseteq M$ be a domain, $p > 1$, $x_0 \in M$ and $\rho = d(x_0, \cdot)$ the distance from x_0. Suppose that the couple (L, W) is a (p, ρ, w)-*Riccati pair *in $(0, \sup_\Omega \rho)$ that is*

(c1') *$L, W, w\colon (0, \sup_\Omega \rho) \to (0, \infty)$ such that L, W are continuous and w is C^1;*

(c2') *$\Delta_F \rho \geq L(\rho)$ in the distributional sense in Ω;*

(c3') *there exists a C^1 function $G\colon (0, \sup_\Omega \rho) \to (0, \infty)$ such that the following Riccati ODI holds:*

$$(G(t)w(t))' + G(t)w(t)L(t) - (p-1)G(t)^{p'}w(t) \geq W(t)w(t), \tag{6.4}$$

where $t \in (0, \sup_\Omega \rho)$ and $p' = \frac{p}{p-1}$. Then for every $u \in C_0^\infty(\Omega)$ one has

$$\lambda^p \int_\Omega w(\rho) F^*(Du)^p \,\mathrm{dm} \geq \int_\Omega w(\rho) \max\{F^*(\pm Du)\}^p \,\mathrm{dm} \geq \int_\Omega w(\rho) W(\rho) |u|^p \,\mathrm{dm}. \tag{6.5}$$

Proof. The convexity of the map $\xi \mapsto F^*(\xi)^p$ implies

$$F^*(\xi)^p \geq F^*(\eta)^p + \left\langle \xi - \eta, \frac{\partial}{\partial \eta} F^*(\eta)^p \right\rangle, \qquad \forall \xi, \eta \in T^*\Omega. \tag{6.6}$$

By using the chain rule and the properties of Legendre transforms (see relations (2.20) & (2.21)), we obtain

$$F^*(\xi)^p \geq pF^*(\eta)^{p-2}\langle \xi, \ell^*(\eta)\rangle - (p-1)F^*(\eta)^p, \qquad \forall \xi, \eta \in T^*\Omega. \tag{6.7}$$

Introduce the notation

$$\operatorname{sgn}(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0, \end{cases}$$

and let us make in inequality (6.7) the following choices:

$$\xi = -\operatorname{sgn}(u) Du \quad \text{and} \quad \eta = \operatorname{sgn}(u) G(\rho)^{\frac{1}{p-1}} u D\rho. \tag{6.8}$$

On the one hand, the positive homogeneity of F^* implies

$$pF^*(\eta)^{p-2} = pG(\rho)^{\frac{p-2}{p-1}} F^*(D\rho)^{p-2} |u|^{p-2},$$
$$(p-1)F^*(\eta)^p = (p-1)G(\rho)^{p'} F^*(D\rho)^p |u|^p.$$

On the other hand, since ℓ^* is positive homogeneous and the canonical pairing is bilinear, we have

$$\langle \xi, \ell^*(\eta)\rangle = -G(\rho)^{\frac{1}{p-1}} u \langle Du, \ell^*(D\rho)\rangle.$$

Thus, the chain rule yields

$$pF^*(\eta)^{p-2}\langle \xi, \ell^*(\eta)\rangle = -G(\rho)F^*(D\rho)^{p-2}\langle D(|u|^p), \ell^*(D\rho)\rangle.$$

By the above computations, it follows that

$$F^*(-\operatorname{sgn}(u)Du)^p \geq -G(\rho)F^*(D\rho)^{p-2}\langle D(|u|^p), \ell^*(D\rho)\rangle - (p-1)G(\rho)^{p'}F^*(D\rho)^p|u|^p.$$

Multiplying both sides by $w(\rho) > 0$ and integrating over Ω yields

$$\begin{aligned} \int_\Omega w(\rho)F^*(-\operatorname{sgn}(u)Du)^p \,\mathsf{dm} \geq &- \int_\Omega w(\rho)G(\rho)F^*(D\rho)^{p-2}\langle D(|u|^p), \ell^*(D\rho)\rangle \,\mathsf{dm} \\ &- (p-1)\int_\Omega w(\rho)G(\rho)^{p'}F^*(D\rho)^p|u|^p \,\mathsf{dm}. \end{aligned} \tag{6.9}$$

First, by the eikonal equation (2.18), both the factors $F^*(D\rho)^{p-2}$ and $F^*(D\rho)^p$ can be omitted. Second, the integration by parts formula (2.25) for $u_1 = w(\rho)G(\rho)|u|^p$ and $u_2 = \rho$ implies

$$\begin{aligned} \int_\Omega (w(\rho)G(\rho)\Delta_F\rho)|u|^p \,\mathsf{dm} = &- \int_\Omega (w(\rho)G(\rho))'\langle D\rho, \nabla\rho\rangle|u|^p \,\mathsf{dm} \\ &- \int_\Omega w(\rho)G(\rho)\langle D(|u|^p), \nabla\rho\rangle \,\mathsf{dm}. \end{aligned}$$

Next, since $\nabla\rho = \ell^*(D\rho)$, by relation (2.19) and the eikonal equation, the factor $\langle D\rho, \nabla\rho\rangle$ can also be omitted. Thus, inequality (6.9) can be rewritten as

$$\begin{aligned} \int_\Omega w(\rho)F^*(-\operatorname{sgn}(u)Du)^p \,\mathsf{dm} \geq &\int_\Omega \left((w(\rho)G(\rho))' + w(\rho)G(\rho)\Delta_F\rho\right)|u|^p \,\mathsf{dm} \\ &- \int_\Omega (p-1)w(\rho)G(\rho)^{p'}|u|^p \,\mathsf{dm}. \end{aligned}$$

Next, since w and G are positive, condition **(c2')** implies

$$\begin{aligned} \int_\Omega w(\rho)F^*(-\operatorname{sgn}(u)Du)^p \,\mathsf{dm} \geq &\int_\Omega \left((w(\rho)G(\rho))' + w(\rho)G(\rho)L(\rho)\right)|u|^p \,\mathsf{dm}, \\ &- \int_\Omega (p-1)w(\rho)G(\rho)^{p'}|u|^p \,\mathsf{dm}, \end{aligned}$$

thus, applying the Riccati ODI (6.4) from condition **(c3')** yields

$$\int_\Omega w(\rho)F^*(-\operatorname{sgn}(u)Du)^p \,\mathsf{dm} \geq \int_\Omega w(\rho)W(\rho)\,\mathsf{dm}.$$

Finally, since $\lambda_F(M) = \lambda_{F^*}(M) = \lambda$ (see relation (2.17)), one has

$$\lambda F^*(Du) \geq \max\{F^*(\pm Du)\} \geq F^*(-\operatorname{sgn}(u)Du), \tag{6.10}$$

which combined with the positivity of w finishes the proof. □

Several comments are in order.

Remark 6.2. (a) The above theorem extends the combination of Theorems 3.1 & 3.5. For a more simple presentation we imposed more simpler but stricter conditions on the parameters: We assumed smoothness instead of local integrability, eliminated an extra parameter, considered only distances from a point, and so on. However, if the applications require, one can easily relax these conditions, without affecting the proof.

(b) Conceptually, Theorem 6.1 has the same benefit as its Riemannian counterpart: It reduces the proof of a Hardy inequality from the relation (6.5) to the solvability of the Riccati ODI (6.4). In addition, the Riemannian applications presented in Section 3.2 gain natural Finslerian extensions: One only needs to formally replace $|\nabla u|$ by either $\max\{F^*(\pm Du)\}$ or $\lambda F^*(Du)$ in the corresponding inequalities. In addition, by using a similar scaling argument as in the proof Theorem 3.1/(ii) one can deal with uncertainty principles (such as the Heisenberg–Pauli–Weyl uncertainty principle and the Hydrogen uncertainty principle) and Caffarelli–Kohn–Nirenberg inequalities. The details are left to the interested reader.

(c) Theorem 6.1 highlights the effect of $\mathbf{K}$, $\overline{\mathbf{S}}$, and **rev** on Hardy inequalities. First, condition **(c2')** is ensured by the Laplace comparison theorem (see Theorem 2.3), the exact form of L depends on the upper bound of the flag curvature $\mathbf{K}$ and the reduced S-curvature $\overline{\mathbf{S}}$ along the geodesics starting from x_0. The effect of reversibility is made precise in relation (6.10).

(d) Theorem 6.1 also gives some hints about the sharpness of Hardy-type inequalities: one can expect the inequalities of (6.5) to be *close* to the equality if the inequality of condition **(c2')**, the inequalities from (6.10), and the convexity inequality (6.6) are close to equality. The first two of them are related to the geometry of the ambient space (see also Theorem 2.3), while the third provides information concerning u.

Define the *limit function* $u^\star$ of the Hardy inequality (3.5), as a radially symmetric function $u^\star = v(\rho)$ for some $v\colon (0, \sup_\Omega \rho) \to \mathbb{R}$, for which $\xi = \eta$ in relation (6.8), and thus inequality (6.6) becomes an equality. Then we obtain

$$-(\log(v(t)))' = -\frac{v'(t)}{v(t)} = G(t)^{\frac{1}{p-1}}, \quad \forall t \in (0, \sup_\Omega \rho).$$

For $u = u^\star$ the integrals of the inequality (3.5) do not converge (there exists no extremal function), but considering its approximations/truncations may yield sharpness. To get close to the equality in **(c2')** & (6.10), in Sections 6.2.1 & 6.3.1, we construct some suitable spaces for these approximations. We note, however, that a general proof of sharpness (if exists) would not require such constructions.

6.1.1 Applications

In this section, by applying Theorem 6.1, we prove a Finslerian version of a weighted Hardy inequality and a McKean-type spectral gap estimate. We note that our choice of parameters agree with the corresponding Riemannian counterparts from Section 3.2. The first application can be stated as follows.

Theorem 6.3 (see [58]). *Let (M, F, m) be a forward complete, non-compact FMMM, with dimension $n \geq 3$, and reversibility $\lambda < \infty$. Let $\Omega \subseteq M$ be a domain, $x_0 \in M$ and $\rho = d(x_0, \cdot)$ the distance from x_0. Suppose that $\mathbf{K} \leq 0$, $\overline{\mathbf{S}} \leq 0$ along $\nabla\rho$. Let $\alpha, p \in \mathbb{R}$ such that $1 < p < n + \alpha$. Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\lambda^p \int_\Omega \rho^\alpha \cdot F^*(Du)^p \,\mathrm{dm} \geq \int_\Omega \rho^\alpha \cdot \max\{F^*(\pm Du)\}^p \,\mathrm{dm} \geq c_{n,p,\alpha} \int_\Omega \rho^\alpha \frac{|u|^p}{\rho^p} \,\mathrm{dm}, \tag{6.11}$$

where $c_{n,p,\alpha} = \left(\frac{n+\alpha-p}{p}\right)^p$.

Proof. In Theorem 6.1 let us choose $\Omega = M \setminus \{x_0\}$, as well as,

$$w(t) = t^\alpha, \quad L(t) = \frac{n-1}{t}, \quad W(t) = c_{n,p,\alpha} \cdot \frac{1}{t^p}, \quad \text{and} \quad G(t) = (c_{n,p,\alpha})^{\frac{p-1}{p}} \cdot \frac{1}{t^{p-1}},$$

for every $t > 0$. Obviously, the above functions satisfy **(c1')**. The Laplace comparison (see Theorem 2.3) implies that **(c2')** holds as well. By a simple computation, the Riccati ODI of condition **(c3')** is verified with equality. Thus, Theorem 6.1 implies that inequality (6.11) holds for every $u \in C_0^\infty(M \setminus \{x_0\})$. Finally, according to Remark 3.14, since the p-capacity of $\rho^{-1}(0) \subset M$ is zero, thus, inequality (6.11) holds for every $u \in C_0^\infty(M)$ as well. □

We notice that Zhao in [109, Theorem 1.1] showed that if $\mathbf{K} \leq 0$ (globally) and $\mathbf{S} \geq 0$ along $\nabla \overleftarrow{\rho}$, where $\overleftarrow{\rho} = d(\cdot, x_0)$, then one has

$$\int_\Omega \overleftarrow{\rho}^\alpha \cdot \max\{F^*(\pm Du)\}^2 \,\mathrm{dm} \geq c_{n,p,\alpha} \int_\Omega \overleftarrow{\rho}^\alpha \frac{|u|^p}{\overleftarrow{\rho}^p} \,\mathrm{dm}, \quad \forall u \in C_0^\infty(M);$$

additionally, if F is reversible, then the constant $c_{n,p,\alpha}$ is sharp. Since, $\mathbf{S} \geq 0$ along $\nabla \overleftarrow{\rho}$ if and only if $\mathbf{S} \leq 0$ along $\nabla\rho$, this result perfectly aligns with inequality (6.11).

The second application can be stated as follows.

Theorem 6.4 (see [58]). *Let (M, F, m) be a forward complete, non-compact FMMM, with dimension $n \geq 3$, and reversibility $\lambda < \infty$. Let $\Omega \subseteq M$ be a domain, $x_0 \in M$ and $\rho = d(x_0, \cdot)$ the distance from x_0. Suppose that $\mathbf{K} \leq -\kappa^2$, $\overline{\mathbf{S}} \leq (n-1)h$ along $\nabla\rho$, for some $\kappa > h \geq 0$, and let $p > 1$. Then for every $u \in C_0^\infty(\Omega)$ one has*

$$\lambda^p \int_\Omega F^*(Du)^p \,\mathrm{dm} \geq \int_\Omega \max\{F^*(\pm Du)\}^p \,\mathrm{dm} \geq c_{n,p,\kappa,h} \int_\Omega |u|^p \,\mathrm{dm}, \tag{6.12}$$

where $c_{n,p,\kappa,h} = \left(\frac{(n-1)(\kappa-h)}{p}\right)^p$.

Proof. In Theorem 6.1 let us choose $\Omega = M$, as well as

$$w(t) \equiv 1, \quad L(t) \equiv (n-1)(\kappa - h), \quad W(t) \equiv c_{n,p,\kappa,h}, \quad \text{and} \quad G(t) \equiv (c_{n,p,\kappa,h})^{\frac{p-1}{p}}.$$

These functions satisfy **(c1')**, by the Laplace comparison theorem (see Theorem 2.3) one has

$$\Delta_F(\rho) \geq (n-1)(\kappa \coth(\kappa\rho) - h) \geq (n-1)(\kappa - h) = L,$$

hence **(c2')** holds as well. The Riccati ODI from **(c3')** is verified with inequality, thus, Theorem 6.1 proves inequality (6.12). □

6.2 Sharp Hardy inequalities – non-positive curvature

In this section, we present our sharpness results concerning Hardy-type inequalities assuming non-positive flag curvature. First, in Section 6.2.1, we present a family of Finsler spaces on which our sharpness results are established (see Theorem 6.5). Next, In Section 6.2.2, we provide some sufficient conditions ensuring the sharpness of Hardy inequalities obtained by Riccati pairs, on the aforementioned spaces (see Theorem 6.7). Finally, in Section 6.2.3 we present an application: In Theorem 6.8 we prove a sharpness result concerning a weighted Hardy inequality, as a consequence we also obtain the sharpness of the classical Finlserian Hardy inequality (see Corollary 6.9).

In the sequel, we use various properties of projectively flat Randers spaces; we refer to Section 2.2.2 for a brief summary on these special manifolds.

6.2.1 Construction

According to the proof of Theorem 6.1 (see also Remark 6.2/(d)) an *ideal* space for discussing the sharpness of Hardy-type inequalities has a radially symmetric metric F and constant $\mathbf{K}$, $\overline{\mathbf{S}}$, and $\mathbf{rev}$ along geodesics starting for a point x_0. Unfortunately, such spaces are not available in the Finsler setting, however, we can make the following observation.

Consider the triple $(\mathbb{R}^n, F, \mathsf{m})$, where $F\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ is defined by

$$F(x,y) = |y| - \frac{\langle x, y\rangle \theta}{|x|}, \quad \text{if } x \neq O \quad \text{and} \quad F(O,y) = |y|, \tag{6.13}$$

for some $\theta \in (0,1)$ and m stands for the Busemann–Hausdorff measure induced by F. Here, $O \in \mathbb{R}^n$ is the origin, $|\cdot|$ and $\langle \cdot, \cdot \rangle$ denote the norm and the inner product in $\mathbb{R}^n$. The function F is *not* a Finsler metric, since it is not continuous at the origin. However, it has various remarkable properties (away from O).

On the one hand, since relation (2.29) holds, Theorem 2.4 implies projectively flatness, i.e. all the geodesics are straight (Euclidean) lines. Denote $\rho = d(O,\cdot)$ the Finslerian distance from the origin. Since the geodesics starting from the origin are precisely the integral curves of $\nabla\rho$, if follows that x is parallel with $\nabla\rho$, as vectors in $\mathbb{R}^n$. Hence, using Proposition 2.5 and the homogeneity of the flag curvature and the reduced S-curvature, we obtain

$$\mathbf{K}(x, \nabla\rho(x)) = \mathbf{K}(x,x) = 0 \quad \text{and} \quad \overline{\mathbf{S}}(x, \nabla\rho(x)) = \overline{\mathbf{S}}(x,x) = 0, \tag{6.14}$$

for every $x \in \mathbb{R}^n$ with $x \neq O$; which precisely means that $\mathbf{K} = 0$, $\overline{\mathbf{S}} = 0$ along $\nabla\rho$.

On the other hand, by an easy computation we obtain for every $x, y \in \mathbb{R}^n$ that

$$\mathbf{rev}(x,y) \leq \mathbf{rev}(x,x) = \frac{1+\theta}{1-\theta} \stackrel{\text{def}}{=} \lambda, \quad \text{if } x \neq O, \quad \text{and} \quad \mathbf{rev}(O,y) = 1, \tag{6.15}$$

hence $\lambda_F(\mathbb{R}^n) = \lambda \in (1,\infty)$. If $u = v(\rho)$ is a radially symmetric function with $v' < 0$, then one has equality in (6.10); the latter provides the difference between the two common forms of Hardy inequalities in the Finsler setting (compare (6.1) & (6.2)).

To remedy the discontinuity at the origin, one could simply puncture the space (considering only $\mathbb{R}^n \setminus \{O\}$), which results in loss of computational convenience: By construction, the distance from O is easier to compute than the distance from any other base point. An other option (what is chosen here) is to 'smooth out' the jump at the origin, which can be done by a careful approximation of F, partially retaining the aforementioned favorable properties. We have the following result.

Theorem 6.5 (see [58]). *Let $\lambda \in (1,\infty)$ and $\theta = \frac{\lambda-1}{\lambda+1}$. Construct the family $\mathcal{F}_{0,0,\lambda} = \{(\mathbb{R}^n, F_\varepsilon, \mathsf{m}_\varepsilon)\}_{\varepsilon>0}$ of FMMMs, where $F_\varepsilon \colon T\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ is defined by*

$$F_\varepsilon(x,y) = |y| - \frac{\langle x,y\rangle(|x|+2\varepsilon)\theta}{(|x|+\varepsilon)^2}, \tag{6.16}$$

and m_ε is the Busemann–Hausdorff measure induced by F_ε. The following statements hold:

(i) For every $\varepsilon > 0$, the triple $(\mathbb{R}^n, F_\varepsilon, \mathsf{m}_\varepsilon)$ is a Finsler metric measure manifold. In addition, F_ε is a projectively flat Randers metric, with reversibility $\lambda_{F_\varepsilon}(\mathbb{R}^n) = \lambda$.

(ii) For every $x \in \mathbb{R}^n$ with $x \neq 0$ one has

$$\mathbf{K}_\varepsilon(x,x) \to 0, \quad \overline{\mathbf{S}}_\varepsilon(x,x) \to 0, \quad \textit{and} \quad \mathbf{rev}_\varepsilon(x,x) \to \lambda, \quad \textit{as} \quad \varepsilon \to 0.$$

(iii) For every $\varepsilon > 0$ and $x \in \mathbb{R}^n$, one has

$$\mathbf{K}_\varepsilon(x,x) \le 0, \quad \overline{\mathbf{S}}_\varepsilon(x,x) \le 0, \quad \textit{and} \quad \mathbf{rev}_\varepsilon(x,x) \le \lambda.$$

Proof. Denote $r = |x|$. Using the familiar notation of Randers metrics presented in Section 2.2.2, we can write

$$F_\varepsilon(x,y) = \sqrt{a_{ij}(x)y^iy^j} + b_i(x)y^i, \quad \text{where} \quad a_{ij}(x) = \delta_{ij} \quad \text{and} \quad b_i(x) = -\frac{x^i(r+2\varepsilon)\theta}{(r+\varepsilon)^2},$$

while δ_{ij} is the Kronecker delta (see (2.32)). Since $a^{ij}(x) = \{a_{ij}(x)\}^{-1} = \delta_{ij}$, we have

$$\|b(x)\|_a = \sqrt{a^{ij}(x)b_i(x)b_j(x)} = \frac{r(r+2\varepsilon)\theta}{(r+\varepsilon)^2} < \theta < 1.$$

Obviously F_ε is smooth, hence it is a Randers metric on $\mathbb{R}^n$. In addition, according to Remark 2.6, the metric F_ε is projectively flat. Indeed, the Euclidean metric is projectively flat, and one has

$$b_i\,\mathrm{d}x^i = D\beta(x), \quad \text{where} \quad \beta(x) = -\left(r + \frac{\varepsilon^2}{r+\varepsilon}\right)\theta.$$

Let $O \in \mathbb{R}^n$ be the origin. For $\rho_\varepsilon(x) = d(O,x)$ and $\overleftarrow{\rho_\varepsilon}(x) = d(x,O)$ we obtain:

$$\rho_\varepsilon(x) = r \cdot \frac{r(1-\theta)+\varepsilon}{r+\varepsilon} \quad \text{and} \quad \overleftarrow{\rho_\varepsilon}(x) = r \cdot \frac{r(1+\theta)+\varepsilon}{r+\varepsilon}.$$

In addition, one has

$$D\rho_\varepsilon(x) = \left(\frac{1}{r} - \frac{(r+2\varepsilon)\theta}{(r+\varepsilon)^2}\right)\cdot x \quad \text{and} \quad \nabla\rho_\varepsilon(x) = \frac{(r+\varepsilon)^2}{r(r+\varepsilon)^2 - r^2(r+2\varepsilon)\theta}\cdot x. \quad (6.17)$$

According to relation (2.31), the density of the Busemann–Hausdorff measure m_ε induced by F_ε is

$$\sigma_{F_\varepsilon}(x) = \left(1 - \frac{r^2(r+2\varepsilon)^2\theta^2}{(r+\varepsilon)^4}\right)^{\frac{n+1}{2}}.$$

By Proposition 2.5, for every $\varepsilon > 0$ and $x \in \mathbb{R}^n$ we obtain

$$\mathbf{K}_{\rho_\varepsilon}(r) \stackrel{\text{def}}{=} \mathbf{K}_\varepsilon(x,x) = -\frac{3\varepsilon^2(r+\varepsilon)^4\theta(1-\theta)}{\left((r+\varepsilon)^2 - r(r+2\varepsilon)\theta\right)^4} \le 0,$$

$$\overline{\mathbf{S}}_{\rho_\varepsilon}(r) \stackrel{\text{def}}{=} \overline{\mathbf{S}}_\varepsilon(x,x) = -\frac{(n+1)\varepsilon^2(r+\varepsilon)\theta}{(r+\varepsilon)^4 - r^2\left(r+2\varepsilon\right)^2\theta^2} \le 0,$$

$$\mathbf{rev}_{\rho_\varepsilon}(r) \stackrel{\text{def}}{=} \mathbf{rev}_\varepsilon(x,x) = \frac{1+\|b(x)\|_a}{1-\|b(x)\|_a} = \frac{1+\frac{r(r+2\varepsilon)\theta}{(r+\varepsilon)^2}}{1-\frac{r(r+2\varepsilon)\theta}{(r+\varepsilon)^2}} \le \frac{1+\theta}{1-\theta} = \lambda.$$

Considering the limit cases of the above inequalities, we obtain

$$\lim_{r\to\infty}\mathbf{K}_{\rho_\varepsilon}(r) = 0, \quad \lim_{r\to\infty}\overline{\mathbf{S}}_{\rho_\varepsilon}(r) = 0, \quad \lim_{r\to\infty}\mathbf{rev}_{\rho_\varepsilon}(r) = \lambda, \quad \forall\varepsilon > 0,$$

and

$$\lim_{\varepsilon\to 0}\mathbf{K}_{\rho_\varepsilon}(r) = 0, \quad \lim_{\varepsilon\to 0}\overline{\mathbf{S}}_{\rho_\varepsilon}(r) = 0, \quad \lim_{\varepsilon\to 0}\mathbf{rev}_{\rho_\varepsilon}(r) = \lambda, \quad \forall r > 0.$$

Moreover, for every $\varepsilon > 0$ and $x, y \in \mathbb{R}^n$ one has

$$\mathbf{rev}_\varepsilon(x,y) = \frac{F_\varepsilon(x,-y)}{F_\varepsilon(x,y)} \le \frac{F_\varepsilon(x,-x)}{F_\varepsilon(x,x)} = \mathbf{rev}_\varepsilon(x,x) \le \lambda, \quad \text{if } x \neq O \qquad (6.18)$$

and $\mathbf{rev}_\varepsilon(O,y) = 1$, thus, one also has $\lambda_{F_\varepsilon}(\mathbb{R}^n) = \lambda$, for every $\varepsilon > 0$. □

We make the following of observations.

Remark 6.6. (a) The construction presented in Theorem 6.5 can be motivated as follows. First, Randers-type metrics are often considered as a first step towards Finsler geometry from Riemannian geometry, it is quite natural to consider them in this case as well. Second, every radially symmetric Randers metric with respect to some x_0, needs to have $\mathbf{rev}(x_0,\cdot) = 1$ (as in (6.18)), otherwise it would be discontinuous at x_0. Since our F in (6.13) has constant $\mathbf{rev} \neq 1$ along $\nabla\rho$, some smoothing/perturbation is unavoidable at x_0. Indeed, one can consider other perturbations instead of the one in (6.16). The latter is chosen due to its simplicity and computational convenience.

(b) One can consider a measure m_ε with density $\sigma_\varepsilon = \sigma_F \cdot e^{-s\rho_\varepsilon}$, to obtain

$$\sup_{x\in\mathbb{R}^n}\overline{\mathbf{S}}_\varepsilon(x,x) \le s \quad \text{and} \quad \lim_{\varepsilon\to 0}\overline{\mathbf{S}}_\varepsilon(x,x) = s, \quad \forall x \in \mathbb{R}^n\setminus\{x_0\}.$$

In this section we only need $s = 0$, however, this observation is exploited in Section 6.3.

(c) We note that for the reduced S-curvature, $\overline{\mathbf{S}} \leq 0$ holds also globally. Surprisingly, for the flag curvature, $\mathbf{K} \leq 0$ is *not* a global property. For example, one has

$$\mathbf{K}_\varepsilon(x, -x) = \frac{3\varepsilon^2(r+\varepsilon)^4\theta(1+\theta)}{\left((r+\varepsilon)^2 + r(r+2\varepsilon)\theta\right)^4} > 0, \quad \forall x \in \mathbb{R}^n.$$

Fortunately, this phenomena is permissible (see Theorem 6.3), it also suggests that the well-known global assumptions may be relaxed.

(d) Observe that $\max\{F_\varepsilon^*(\pm D\rho_\varepsilon)\} = F_\varepsilon^*(D\rho_\varepsilon) = 1$. Indeed, relation (2.18) implies $F_\varepsilon^*(D\rho_\varepsilon) = 1$, and one has

$$F_\varepsilon^*(-D\rho_\varepsilon) = \frac{1 - \frac{r(r+2\varepsilon)}{(r+\varepsilon)^2}\theta}{1 + \frac{r(r+2\varepsilon)}{(r+\varepsilon)^2}\theta} \stackrel{\text{def}}{=} h_\varepsilon(r), \quad \text{where} \quad \frac{1-\theta}{1+\theta} \leq h_\varepsilon(r) \leq 1.$$

Note that F_ε^* can be computed as in Shen [101, Example 3.1.1], while $D\rho_\varepsilon$ is given by relation (6.17).

6.2.2 Sufficient conditions for sharpness

In this section, we present sufficient conditions for the sharpness of Hardy inequalities provided by Riccati pairs on the spaces of $\mathcal{F}_{0,0,\lambda}$, introduced in Theorem 6.5.

Let $(\mathbb{R}^n, F_\varepsilon, \mathsf{m}_\varepsilon) \in \mathcal{F}_{0,0,\lambda}$ for some $\varepsilon > 0$ and $\lambda \in (1, \infty)$. Then for every $x \in \mathbb{R}^n$, one has

$$\mathbf{K}_\varepsilon(x, x) \leq 0, \quad \overline{\mathbf{S}}_\varepsilon(x, x) \leq 0, \quad \text{and} \quad \mathbf{rev}_\varepsilon(x, x) \leq \lambda = \lambda_{F_\varepsilon}(\mathbb{R}^n).$$

Denote $\rho = d(O, \cdot)$ the distance from the origin. The Laplace comparison theorem (see Theorem 2.3) implies

$$\Delta_{F_\varepsilon}(\rho_\varepsilon) \geq \frac{n-1}{\rho_\varepsilon} \stackrel{\text{def}}{=} L^{0,0}(\rho_\varepsilon).$$

Let $R \in (0, +\infty]$, $L(t) \leq L^{0,0}(t)$ for every $t \in (0, R)$, and define

$$\Omega_\varepsilon = \{x \in \mathbb{R}^n : \rho_\varepsilon(x) < R\}.$$

Suppose that (L, W) is a (p, ρ_ε, w)-*Riccati pair* in $(0, R)$, that is $W\colon (0, R) \to (0, \infty)$ is continuous, $w\colon (0, R) \to (0, \infty)$ is of class C^1, and there exists $G\colon (0, R) \to (0, \infty)$ of class C^1 such that

$$(G(t)w(t))' + G(t)w(t)L(t) - (p-1)G(t)^{p'}w(t) \geq W(t)w(t), \quad \forall t \in (0, R). \tag{6.19}$$

Then by Theorem 6.1, for every $u \in C_0^\infty(\Omega_\varepsilon)$ one has:

$$\lambda^p \int_{\Omega_\varepsilon} w(\rho_\varepsilon) F_\varepsilon^*(Du)^p \, \mathsf{dm}_\varepsilon \geq \int_{\Omega_\varepsilon} w(\rho_\varepsilon) W(\rho_\varepsilon) |u|^p \, \mathsf{dm}_\varepsilon. \tag{6.20}$$

As in Remark 6.2/(d), suppose that a function $G > 0$ verifies the Riccati ODI (6.19) with equality, and define $v\colon (0, R) \to \mathbb{R}$ by

$$-(\log v(t))' = -\frac{v'(t)}{v(t)} = G(t)^{\frac{1}{p-1}}. \tag{6.21}$$

Since G is smooth and positive, it follows that v is smooth, positive, and decreasing. Introduce the notation

$$\mathcal{I}(f;a,b) = \int_a^b f(t)t^{n-1}\,\mathrm{d}t, \tag{6.22}$$

for arbitrary continuous functions $f\colon [a,b] \to \mathbb{R}$, with $0 < a < b < R$.

We have the following result.

Theorem 6.7 (see [58]). *Fix $\delta_0 > 0$ and let $t_1 = t_1(\delta), t_2 = t_2(\delta), t_3 = t_3(\delta), t_4 = t_4(\delta)$ be continuous functions admitting a (possibly infinite) limit at infinity, such that for every $\delta > \delta_0$ one has $0 < t_1 < t_2 < t_3 < t_4 < R$. Suppose that the following limits hold:*

$$l_0(\delta) \overset{def}{=} \frac{v(t_2)^p \cdot \mathcal{I}(w;t_1,t_2)}{(t_2-t_1)^p \cdot \mathcal{I}(wWv^p;t_2,t_3)} + \frac{v(t_3)^p \cdot \mathcal{I}(w;t_3,t_4)}{(t_4-t_3)^p \cdot \mathcal{I}(wWv^p;t_2,t_3)} \to 0, \quad \textit{as} \quad \delta \to \infty,$$

$$l_1(\delta) \overset{def}{=} \max_{t\in[t_2,t_3]} \frac{G(t)^{p'}}{W(t)} \to 1, \quad \textit{as} \quad \delta \to \infty.$$

Then inequality (6.20) is sharp on $\mathcal{F}_{0,0,\lambda}$, in the sense that it no longer holds on each FMMM of $\mathcal{F}_{0,0,\lambda}$ if we multiply its right hand side by a constant $c > 1$.

Proof. Let $\varepsilon > 0$ and define the *limit function* $u_\varepsilon^\star\colon \Omega \to (0,\infty)$ of inequality (6.20) by

$$u_\varepsilon^\star(x) = v(\rho_\varepsilon(x)).$$

Denote $T = (t_1,t_2,t_3,t_4)$ and construct

$$u_{\varepsilon,T}^\star = v_T(\rho_\varepsilon),$$

where the function v_T and its derivative satisfy

$$v_T(t) = \begin{cases} 0, & \text{if } 0 < t \le t_1,\\ v(t_2)\frac{t-t_1}{t_2-t_1}, & \text{if } t_1 \le t \le t_2,\\ v(t), & \text{if } t_2 \le t \le t_3,\\ v(t_3)\frac{t_4-t}{t_4-t_3}, & \text{if } t_3 \le t \le t_4,\\ 0, & \text{if } t_4 \le t < R, \end{cases} \quad\text{and}\quad v_T'(t) = \begin{cases} 0, & \text{if } 0 < t < t_1,\\ \frac{v(t_2)}{t_2-t_1}, & \text{if } t_1 < t < t_2,\\ v'(t), & \text{if } t_2 < t < t_3,\\ -\frac{v(t_3)}{t_4-t_3}, & \text{if } t_3 < t < t_4,\\ 0, & \text{if } t_4 < t < R. \end{cases} \tag{6.23}$$

Note that v_T' is not defined in t_1, t_2, t_3 and t_4. In addition, both v_T and v_T' are supported in $[t_1,t_4]$.

To prove the sharpness of inequality (6.20), we show that under a suitable limiting procedure, one has

$$\frac{\lambda^p \displaystyle\int_{\Omega_\varepsilon} w(\rho_\varepsilon)F_\varepsilon^*(Du_{\varepsilon,T}^\star)^p\,\mathsf{dm}_\varepsilon}{\displaystyle\int_{\Omega_\varepsilon} w(\rho_\varepsilon)W(\rho_\varepsilon)(u_{\varepsilon,T}^\star)^p\,\mathsf{dm}_\varepsilon} \to 1.$$

As a first step, we use polar transforms to reduce the above integrals to one dimension. We also establish some fine estimates on the distance and the density of the measure.

Using the notations $r=|x|$ and $t=\rho_\varepsilon(x)$, it follows that

$$t=r\cdot\frac{r(1-\theta)+\varepsilon}{r+\varepsilon}\qquad\Longleftrightarrow\qquad r=\frac{t-\varepsilon+\sqrt{(t+\varepsilon)^2-4t\varepsilon\theta}}{2(1-\theta)}\stackrel{\text{def}}{=}\varphi_\varepsilon(t),$$

hence we obtain

$$\sigma_\varepsilon(r)r^{n-1}\,\mathrm{d}r=\sigma_\varepsilon(\varphi_\varepsilon(t))\varphi_\varepsilon(t)^{n-1}\varphi_\varepsilon'(t)\,\mathrm{d}t\stackrel{\text{def}}{=}\psi_\varepsilon(t)\,\mathrm{d}t. \tag{6.24}$$

Since for every $t\in(0,R)$ one has

$$t\le\varphi_\varepsilon(t)\le\frac{t}{1-\theta},\quad 1\le\varphi_\varepsilon'(t)\le\frac{1}{1-\theta},\quad\text{and}\quad \left(1-\theta^2\right)^{\frac{n+1}{2}}\le\sigma_\varepsilon(\varphi_\varepsilon(t))\le 1,$$

we obtain

$$ct^{n-1}\stackrel{\text{def}}{=}\left(1-\theta^2\right)^{\frac{n+1}{2}}t^{n-1}\le\psi_\varepsilon(t)\le\frac{t^{n-1}}{(1-\theta)^n}\stackrel{\text{def}}{=}Ct^{n-1}. \tag{6.25}$$

Note that c and C are independent of ε.

Since v is positive and decreasing, the positive homogeneity of F_ε^* and the relations from (6.23) imply

$$F_\varepsilon^*(Du_{\varepsilon,T}^\star)=F_\varepsilon^*(v_T'(\rho_\varepsilon)D\rho_\varepsilon)=\begin{cases}\frac{v(t_2)}{t_2-t_1}F_\varepsilon^*(D\rho_\varepsilon), & \text{if } \rho_\varepsilon\in(t_1,t_2),\\ |v'(\rho_\varepsilon)|F_\varepsilon^*(-D\rho_\varepsilon) & \text{if } \rho_\varepsilon\in(t_2,t_3),\\ \frac{v(t_3)}{t_4-t_3}F_\varepsilon^*(-D\rho_\varepsilon) & \text{if } \rho_\varepsilon\in(t_3,t_4),\\ 0, & \text{if } \rho_\varepsilon\in(0,t_1)\cup(t_4,R).\end{cases} \tag{6.26}$$

According to Remark 6.6/(d), one has

$$F_\varepsilon^*(D\rho_\varepsilon)=1,\quad\text{and}\quad F_\varepsilon^*(-D\rho_\varepsilon)=\frac{1-\frac{r(r+2\varepsilon)}{(r+\varepsilon)^2}\theta}{1+\frac{r(r+2\varepsilon)}{(r+\varepsilon)^2}\theta}\stackrel{\text{def}}{=}h_\varepsilon(r),\quad\text{where}\quad\frac{1-\theta}{1+\theta}\le h_\varepsilon(r)\le 1.$$

By a simple computation, we obtain that $h_\varepsilon(r)$ is decreasing in r, increasing in ε, and approaches to $\frac{1-\theta}{1+\theta}=\frac{1}{\lambda}$ as $\varepsilon\to 0$. Since $r=\varphi_\varepsilon(t)\ge t$, we also have

$$h_\varepsilon(\varphi_\varepsilon(t))\le h_\varepsilon(t)\le 1. \tag{6.27}$$

In the sequel, denote by ω the volume of the Euclidean unit ball of dimension n. On the one hand, relations (6.24) and (6.26) yield

$$\begin{aligned}I\stackrel{\text{def}}{=}\frac{1}{n\omega}\int_{\Omega_\varepsilon}w(\rho_\varepsilon)F_\varepsilon^*(Du_{\varepsilon,T}^\star)^p\,\mathsf{dm}_\varepsilon&=\frac{v(t_2)^p}{(t_2-t_1)^p}\int_{t_1}^{t_2}w(t)\psi_\varepsilon(t)\,\mathrm{d}t\\&+\int_{t_2}^{t_3}w(t)|v'(t)|^p h_\varepsilon^p(\varphi_\varepsilon(t))\psi_\varepsilon(t)\,\mathrm{d}t\\&+\frac{v(t_3)^p}{(t_4-t_3)^p}\int_{t_3}^{t_4}w(t)h_\varepsilon^p(\varphi_\varepsilon(t))\psi_\varepsilon(t)\,\mathrm{d}t.\end{aligned}$$

Applying relations (6.25) & (6.27), and using the definition of $\mathcal{I}$ from (6.22), it follows that

$$\begin{aligned}
I &\leq \frac{v(t_2)^p}{(t_2-t_1)^p}\int_{t_1}^{t_2} w(t)\psi_\varepsilon(t)\,\mathrm{d}t + \int_{t_2}^{t_3} w(t)|v'(t)|^p h_\varepsilon^p(t)\psi_\varepsilon(t)\,\mathrm{d}t \\
&\quad + \frac{v(t_3)^p}{(t_4-t_3)^p}\int_{t_3}^{t_4} w(t)h_\varepsilon^p(t)\psi_\varepsilon(t)\,\mathrm{d}t \\
&\leq \frac{Cv(t_2)^p}{(t_2-t_1)^p}\int_{t_1}^{t_2} w(t)t^{n-1}\,\mathrm{d}t + h_\varepsilon^p(t_2)\int_{t_2}^{t_3} w(t)|v'(t)|^p\psi_\varepsilon(t)\,\mathrm{d}t \\
&\quad + \frac{Ch_\varepsilon^p(t_3)v(t_3)^p}{(t_4-t_3)^p}\int_{t_3}^{t_4} w(t)t^{n-1}\,\mathrm{d}t \\
&= \frac{Cv(t_2)^p\cdot\mathcal{I}(w;t_1,t_2)}{(t_2-t_1)^p} + h_\varepsilon^p(t_2)\int_{t_2}^{t_3} w(t)|v'(t)|^p\psi_\varepsilon(t)\,\mathrm{d}t + \frac{Cv(t_3)^p\cdot\mathcal{I}(w;t_3,t_4)}{(t_4-t_3)^p}.
\end{aligned}$$

By relation (6.21), for every $t\in[t_2,t_3]$ one has

$$\frac{1}{W(t)}\left|\frac{v'(t)}{v(t)}\right|^p = \frac{1}{W(t)}\left(-\frac{v'(t)}{v(t)}\right)^p = \frac{G(t)^{p'}}{W(t)}.$$

This function is continuous, thus, it attains its maximum on $[t_2,t_3]$. Hence, we have

$$\begin{aligned}
\int_{t_2}^{t_3} w(t)|v'(t)|^p\psi_\varepsilon(t)\,\mathrm{d}t &= \int_{t_2}^{t_3}\frac{1}{W(t)}\left|\frac{v'(t)}{v(t)}\right|^p w(t)W(t)|v(t)|^p\psi_\varepsilon(t)\,\mathrm{d}t \\
&\leq \max_{t\in[t_2,t_3]}\frac{G(t)^{p'}}{W(t)}\cdot\int_{t_2}^{t_3} w(t)W(t)(v(t))^p\psi_\varepsilon(t)\,\mathrm{d}t.
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
J &\stackrel{\text{def}}{=} \frac{1}{n\omega}\int_{\Omega_\varepsilon} w(\rho_\varepsilon)W(\rho_\varepsilon)(u_{\varepsilon,T}^\star)^p\,\mathsf{dm}_\varepsilon = \int_0^R w(t)W(t)(v_T(t))^p\psi_\varepsilon(t)\,\mathrm{d}t \\
&\geq \int_{t_2}^{t_3} w(t)W(t)(v(t))^p\psi_\varepsilon(t)\,\mathrm{d}t \geq c\cdot\mathcal{I}(wWv^p;t_2,t_3).
\end{aligned} \tag{6.28}$$

Combining the above results, and using both estimates of (6.28), it follows that

$$\begin{aligned}
\frac{\lambda^p I}{J} &\leq \lambda^p h_\varepsilon^p(t_2)\cdot\max_{t\in[t_2,t_3]}\frac{G(t)^{p'}}{W(t)} \\
&\quad + \frac{\lambda^p C}{c}\left(\frac{v(t_2)^p\cdot\mathcal{I}(w;t_1,t_2)}{(t_2-t_1)^p\cdot\mathcal{I}(wWv^p;t_2,t_3)} + \frac{v(t_3)^p\cdot\mathcal{I}(w;t_3,t_4)}{(t_4-t_3)^p\cdot\mathcal{I}(wWv^p;t_2,t_3)}\right) \\
&= \lambda^p h_\varepsilon^p(t_2)l_1(\delta) + \frac{\lambda^p C}{c}l_0(\delta).
\end{aligned}$$

Observe that the above inequality holds for every $\varepsilon>0$ and $\delta>\delta_0$, in addition, δ, $T=(t_1,t_2,t_3,t_3)$, c, C, λ, l_0, and l_1 are independent of ε.

To prove the sharpness of (6.20), it is enough to choose for every $\delta>\delta_0$ an $\varepsilon=\varepsilon(\delta)$ such that $h_\varepsilon(t_2(\delta))\to\frac{1}{\lambda}$ as $\delta\to\infty$. Indeed, in this case, since $l_0\to 0$ and $l_1\to 1$ as $\delta\to\infty$ we obtain $\lambda^p I/J\to 1$ as $\delta\to\infty$.

By assumption there exists

$$\lim_{\delta\to\infty} t_2(\delta) \overset{\text{def}}{=} l \in [0, R].$$

We distinguish two cases: If $l > 0$, then choose $\varepsilon = 1/\delta$, to obtain

$$\lim_{\delta\to\infty} h_\varepsilon(t_2(\delta)) = \lim_{\delta\to\infty} \frac{1 - \frac{t_2(\delta)(t_2(\delta)+2/\delta)}{(t_2(\delta)+1/\delta)^2}\theta}{1 + \frac{t_2(\delta)(t_2(\delta)+2/\delta)}{(t_2(\delta)+1/\delta)^2}\theta} = \frac{1-\theta}{1+\theta} = \frac{1}{\lambda}.$$

If $l = 0$, then choosing $\varepsilon = t_2(\delta)^2$ yields

$$\lim_{\delta\to\infty} h_\varepsilon(t_2(\delta)) = \lim_{t_2\to 0} \frac{1 - \frac{t_2(t_2+2t_2^2)}{(t_2+t_2^2)^2}\theta}{1 + \frac{t_2(t_2+2t_2^2)}{(t_2+t_2^2)^2}\theta} = \lim_{t_2\to 0} \frac{1 - \frac{(1+2t_2)}{(1+t_2)^2}\theta}{1 + \frac{(1+2t_2)}{(1+t_2)^2}\theta} = \frac{1-\theta}{1+\theta} = \frac{1}{\lambda},$$

which concludes the proof. □

6.2.3 The sharpness of a weighted Hardy inequality

In this section, we present a sharpness result concerning a weighted Hardy inequality on $\mathcal{F}_{0,0,\lambda}$. Our result can be stated as follows.

Theorem 6.8 (see [58]). *Let $n \geq 3$, $\lambda \in (1, \infty)$, and $\alpha, p \in \mathbb{R}$ such that $1 < p < n+\alpha$. For every $(\mathbb{R}^n, F_\varepsilon, \mathsf{m}_\varepsilon) \in \mathcal{F}_{0,0,\lambda}$ and $u \in C_0^\infty(\mathbb{R}^n)$ one has*

$$\lambda^p \int_{\mathbb{R}^n} \rho^\alpha \cdot F_\varepsilon^*(Du)^p \,\mathrm{d}\mathsf{m}_\varepsilon \geq c_{n,p,\alpha} \int_{\mathbb{R}^n} \rho^\alpha \frac{u^p}{\rho_\varepsilon^p} \,\mathrm{d}\mathsf{m}_\varepsilon, \quad \textit{where } c_{n,p,\alpha} = \left(\frac{n+\alpha-p}{p}\right)^p. \tag{6.29}$$

Moreover, $c_{n,p,\alpha}$ is sharp *in $\mathcal{F}_{0,0,\lambda}$, in the sense that it is the greatest constant with this property.*

Proof. The proof of inequality (6.29) follows from Theorem 6.3. To prove the sharpness, we use Theorem 6.7. According to the relation (6.21), we obtain

$$v(t)^p = \frac{1}{t^{n+\alpha-p}}, \quad \forall t \in (0, \infty).$$

For every $\delta > 1$ define

$$t_1(\delta) = \delta/2, \quad t_2(\delta) = \delta, \quad t_3(\delta) = \delta^2, \quad \text{and} \quad t_4(\delta) = 2\delta^2.$$

By simple computations we obtain

$$\begin{aligned} \frac{v(t_2)^p \cdot \mathcal{I}(w; t_1, t_2)}{(t_2 - t_1)^p} &= \frac{2^{n+\alpha} - 1}{(n+\alpha)2^{n+\alpha-p}} = \text{const}, \\ \frac{v(t_3)^p \cdot \mathcal{I}(w; t_3, t_4)}{(t_4 - t_3)^p} &= \frac{2^{n+\alpha} - 1}{n+\alpha} = \text{const}, \\ \mathcal{I}(wWv^p; t_2, t_3) &= c_{n,p,\alpha} \log(\delta) \to \infty, \quad \text{as } \delta \to \infty. \end{aligned}$$

It follows that $l_0 \to 0$ as $\delta \to \infty$. Since $l_1 = \frac{G(t)^{p'}}{W(t)} = 1$, for every $t > 0$, thus, Theorem 6.7 yields the sharpness of the inequality (6.29). □

By choosing $\alpha = 0$ and $p = 2$ in Theorem 6.8 we obtain the following result.

Corollary 6.9 (see [58]). *Let $n \geq 3$, $\lambda \in (1, \infty)$. For every $(\mathbb{R}^n, F_\varepsilon, \mathsf{m}_\varepsilon) \in \mathcal{F}_{0,0,\lambda}$ and $u \in C_0^\infty(\mathbb{R}^n)$ one has*

$$\lambda^2 \int_{\mathbb{R}^n} F_\varepsilon^*(Du)^2 \,\mathrm{d}\mathsf{m}_\varepsilon \geq c_n \int_{\mathbb{R}^n} \frac{u^2}{\rho_\varepsilon^2} \,\mathrm{d}\mathsf{m}_\varepsilon, \quad \text{where} \quad c_n = \left(\frac{n-2}{2}\right)^2. \tag{6.30}$$

Moreover, c_n is sharp *in $\mathcal{F}_{0,0,\lambda}$, in the sense that it is the greatest constant with this property.*

6.3 Sharp Hardy inequalities – strong negative curvature

In this section, we present our sharpness results concerning Hardy-type inequalities, assuming strong negative curvature. We proceed similarly to the previous case. First, in Section 6.3.1 we present a second construction of spaces (see Theorem 6.10). Next, in Section 6.3.2 we provide sufficient conditions for sharpness of Hardy-type inequalities on them (see Theorem 6.12). Finally, in Section 6.3.3 we present an application: In Theorem 6.13 we prove a sharpness result concerning a McKean-type spectral gap estimate.

6.3.1 Construction

Consider the triple $(\mathbb{B}^n, F, \mathsf{m})$, where $\mathbb{B}^n$ denotes the Euclidean (open) unit ball, the function $F\colon \mathbb{B}^n \times \mathbb{R}^n \to [0, \infty)$ is defined by

$$F(x, y) = \frac{\sqrt{|y|^2 - |x|^2|y|^2 + \langle x, y\rangle^2}}{\kappa(1-\theta)(1-|x|^2)} - \frac{\theta\langle x, y\rangle}{\kappa(1-\theta)|x|(1-|x|^2)}, \tag{6.31}$$

for some $\theta \in (0, 1)$ and $\kappa > 0$; while m is a measure with density

$$\sigma = \exp(-(n-1)h\rho) \cdot \sigma_F,$$

for some $h \in \mathbb{R}$. As before, ρ is the distance from the origin and σ_F is the density of the Busemann–Hausdorff measure induced by F. Note that F is precisely a perturbation of the *Klein metric* on $\mathbb{B}^n$ (to see this, substitute $\theta = 0$ in (6.31)).

We observe that F is not a Finsler metric, since it is not continuous at the origin. However, for every $x \neq O$, one has

$$\mathbf{K}(x, x) = -\kappa^2, \quad \overline{\mathbf{S}}(x, x) = (n-1)h, \quad \text{and} \quad \mathbf{rev}(x, x) = \frac{1+\theta}{1-\theta} \stackrel{\text{def}}{=} \lambda = \lambda_F(\mathbb{B}^n).$$

The following result provides an approximation of the above structure by a family of Randers spaces.

Theorem 6.10 (see [58]). *Let $\lambda \in (1,\infty)$, $\theta = \frac{\lambda-1}{\lambda+1}$, $\kappa > 0$ and $h \in \mathbb{R}$. Construct the family $\mathcal{F}_{\kappa,h,\lambda} = \{(\mathbb{B}^n, F_\varepsilon, \mathsf{m}_\varepsilon)\}_{\varepsilon>0}$ of FMMMs, where $F_\varepsilon \colon T\mathbb{B}^n \simeq \mathbb{B}^n \times \mathbb{R}^n \to [0,\infty)$ is defined by*

$$
\begin{aligned}
F_\varepsilon(x,y) = {} & \frac{\sqrt{|y|^2 - |x|^2|y|^2 + \langle x,y\rangle^2}}{k_\varepsilon(1-\theta)(1-|x|^2)} \\
& - \frac{\theta\langle x,y\rangle}{k_\varepsilon(1-\theta)|x|(1-|x|^2)} \cdot \frac{\operatorname{arctanh}(|x|)\,(\operatorname{arctanh}(|x|)+2\varepsilon)}{(\operatorname{arctanh}(|x|)+\varepsilon)^2},
\end{aligned}
$$

k_ε is given by relation (6.34), m_ε is a measure with density $\sigma_\varepsilon = \exp(-(n-1)h\rho_\varepsilon)\cdot\sigma_{F_\varepsilon}$, and σ_{F_ε} denotes the density of the Busemann–Hausdorff measure induced by F_ε. The following statements hold:

(i) For every $\varepsilon > 0$, the triple $(\mathbb{B}^n, F_\varepsilon, \mathsf{m}_\varepsilon)$ is a Finsler metric measure manifold. In addition, F_ε is a projectively flat Randers metric, with reversibility $\lambda_{F_\varepsilon}(\mathbb{B}^n) = \lambda$.

(ii) For every $x \in \mathbb{B}^n$ with $x \neq 0$, one has

$$
\mathbf{K}_\varepsilon(x,x) \to -\kappa^2, \quad \overline{\mathbf{S}}_\varepsilon(x,x) \to (n-1)h, \quad \text{and} \quad \mathbf{rev}_\varepsilon(x,x) \to \lambda, \quad \text{as} \quad \varepsilon \to 0. \tag{6.32}
$$

(iii) For every $\varepsilon > 0$ and $x \in \mathbb{B}^n$, one has

$$
\mathbf{K}_\varepsilon(x,x) \le -\kappa^2, \quad \overline{\mathbf{S}}_\varepsilon(x,x) \le (n-1)h, \quad \text{and} \quad \mathbf{rev}_\varepsilon(x,x) \le \lambda. \tag{6.33}
$$

Proof. For simplicity of presentation, denote

$$
r = |x|, \quad \overline{r} = \operatorname{arctanh}(|x|), \quad \text{and} \quad \vartheta_\varepsilon = k_\varepsilon(1-\theta).
$$

Using the standard notation of Randers metrics (see Section 2.2.2) one has

$$
F_\varepsilon(x,y) = \sqrt{a_{ij}(x)y^iy^j} + b_i(x)y^i,
$$

where

$$
a_{ij}(x) = \frac{\delta_{ij}}{\vartheta_\varepsilon^2(1-r^2)} + \frac{x^ix^j}{\vartheta_\varepsilon^2(1-r^2)^2} \quad \text{and} \quad b_i(x) = -\frac{\theta\overline{r}(\overline{r}+2\varepsilon)x^i}{\vartheta_\varepsilon r(1-r^2)(\overline{r}+\varepsilon)^2}.
$$

Let $a^{ij}(x) = \{a_{ij}(x)\}^{-1}$, it follows that

$$
\|b\|_a(x) = \sqrt{a^{ij}(x)b_i(x)b_j(x)} = \frac{\theta\overline{r}(\overline{r}+2\varepsilon)}{(\overline{r}+\varepsilon)^2} < \theta < 1.
$$

Since F_ε is also smooth, it is a Randers metric on $\mathbb{R}^n$. According to Remark 2.6, the metric F_ε is projectively flat. Indeed, the Klein metric (obtained from F_ε for $\theta = 0$) is projectively flat, and

$$
b_i\,\mathrm{d}x^i = D\beta(x), \quad \text{where} \quad \beta(x) = -\frac{\theta}{\vartheta_\varepsilon}\left(\overline{r} + \frac{\varepsilon^2}{\overline{r}+\varepsilon}\right).
$$

Let $O \in \mathbb{R}^n$ denote the origin, $\rho_\varepsilon(x) = d(O,x)$ and $\overleftarrow{\rho_\varepsilon}(x) = d(x,O)$. Then

$$\rho_\varepsilon(x) = \overline{r} \cdot \frac{\overline{r}(1-\theta)+\varepsilon}{\vartheta_\varepsilon(\overline{r}+\varepsilon)} \quad \text{and} \quad \overleftarrow{\rho_\varepsilon}(x) = \overline{r} \cdot \frac{\overline{r}(1+\theta)+\varepsilon}{\vartheta_\varepsilon(\overline{r}+\varepsilon)}.$$

In addition, one has

$$D\rho_\varepsilon(x) = \left(\frac{(\overline{r}+\varepsilon)^2 - \overline{r}(\overline{r}+2\varepsilon)\theta}{\vartheta_\varepsilon r(1-r^2)(\overline{r}+\varepsilon)^2}\right) \cdot x \quad \text{and} \quad \nabla\rho_\varepsilon(x) = \frac{\vartheta_\varepsilon(1-r^2)(\overline{r}+\varepsilon)^2}{r((\overline{r}+\varepsilon)^2 - \overline{r}(\overline{r}+2\varepsilon)\theta)} \cdot x.$$

According to relation (2.31), the density of the Busemann–Hausdorff measure induced by F_ε is

$$\sigma_{F_\varepsilon} = \frac{1}{\vartheta_\varepsilon^n} \cdot \left(\frac{1}{1-r^2}\right)^{\frac{n+1}{2}} \cdot \left(1 - \frac{\overline{r}^2(\overline{r}+2\varepsilon)^2\theta^2}{(\overline{r}+\varepsilon)^4}\right)^{\frac{n+1}{2}} \stackrel{\text{def}}{=} \frac{1}{\vartheta_\varepsilon^n} \cdot \left(\frac{1}{1-\overline{r}^2}\right)^{\frac{n+1}{2}} \cdot f_\varepsilon^{\frac{n+1}{2}}.$$

Observe that $1-\theta^2 \le f_\varepsilon \le 1$. The density of m_ε is

$$\sigma_\varepsilon = \exp\left(-(n-1)h\rho_\varepsilon\right) \cdot \frac{1}{\vartheta_\varepsilon^n}\left(\frac{1}{1-\overline{r}^2}\right)^{\frac{n+1}{2}} f_\varepsilon^{\frac{n+1}{2}}.$$

In the sequel, we compute the flag curvature, the reduced S-curvature, and the reversibility of $(\mathbb{B}^n, F_\varepsilon, \mathsf{m}_\varepsilon)$ along the geodesics starting from O. Recall that $\vartheta_\varepsilon = k_\varepsilon(1-\theta)$. Concerning the flag curvature, we obtain

$$\mathbf{K}_{\rho_\varepsilon}(\overline{r}) \stackrel{\text{def}}{=} \mathbf{K}_\varepsilon(x,x) = -\frac{k_\varepsilon^2(1-\theta)^2(\overline{r}+\varepsilon)^4\left(e_0 - e_1\theta + e_2\theta^2\right)}{\left((\overline{r}+\varepsilon)^2 - \overline{r}(\overline{r}+2\varepsilon)\theta\right)^4},$$

where $e_0 = (\overline{r}+\varepsilon)^4$, $e_1 = 2\overline{r}^4 + 8\overline{r}^3\varepsilon + (10\overline{r}^2-3)\varepsilon^2 + 4\overline{r}\varepsilon^3$, and $e_2 = \overline{r}^4 + 4\overline{r}^3\varepsilon + (4\overline{r}^2-3)\varepsilon^2$.

Our goal is to choose k_ε such that relations (6.32) & (6.33) hold. For every $\varepsilon > 0$, one has

$$\lim_{\overline{r}\to\infty} \mathbf{K}_{\rho_\varepsilon}(\overline{r}) = -k_\varepsilon^2,$$

and

$$\lim_{\overline{r}\to 0} \mathbf{K}_{\rho_\varepsilon}(\overline{r}) = -k_\varepsilon^2 \cdot K_{\varepsilon,0}, \quad \text{where} \quad K_{\varepsilon,0} = (1-\theta)^2\left(1 + \frac{3\theta(1-\theta)}{\varepsilon^2}\right).$$

By a simple derivative test, we obtain that if $\varepsilon \le \sqrt{\frac{3(1-\theta)}{8\theta}} \stackrel{\text{def}}{=} \varepsilon_0$, then $\overline{r} \mapsto \mathbf{K}_{\rho_\varepsilon}(\overline{r})$ admits a unique maximum on $(0,\infty)$, which is given by

$$-k_\varepsilon^2 \cdot K_\varepsilon,$$

where

$$K_\varepsilon = \frac{6(1-\theta)^2\left((1-\theta)\left(3(1-\theta)-2\varepsilon^2\theta\right) + \sqrt{3(1-\theta)^3\left(3(1-\theta)-8\varepsilon^2\theta\right)}\right)^3}{\left(3(1-\theta)^2 + \sqrt{3(1-\theta)^3\left(3(1-\theta)-8\varepsilon^2\theta\right)}\right)^4} \le 1.$$

For simplicity define $K_\varepsilon=\infty$, when $\varepsilon>\varepsilon_0$. To ensure $\mathbf{K}_{\rho_\varepsilon}(\overline{r})\leq-\kappa^2$, $\forall\varepsilon>0,\overline{r}\geq 0$, choose

$$k_\varepsilon=\max\left\{\kappa,\frac{\kappa}{\sqrt{K_{\varepsilon,0}}},\frac{\kappa}{\sqrt{K_\varepsilon}}\right\}. \tag{6.34}$$

If $\varepsilon\to 0$, then

$$K_{\varepsilon,0}\to\infty,\quad K_\varepsilon\to 1,\quad\text{and}\quad k_\varepsilon\to\kappa.$$

Thus, for every $\overline{r}>0$ we obtain

$$\mathbf{K}_{\rho_\varepsilon}(\overline{r})\to-\kappa^2\quad\text{as}\quad\varepsilon\to 0.$$

The reduced S-curvature along ρ_ε can be computed as

$$\overline{\mathbf{S}}_{\rho_\varepsilon}(\overline{r})\overset{\text{def}}{=}\overline{\mathbf{S}}(x,x)=(n-1)h-\frac{\vartheta_\varepsilon(n+1)(\overline{r}+\varepsilon)\theta\varepsilon^2}{(\overline{r}+\varepsilon)^4-\overline{r}^2(\overline{r}+2\varepsilon)^2\theta^2}\leq(n-1)h.$$

We easily obtain that if $\overline{r}>0$, then

$$\overline{\mathbf{S}}_{\rho_\varepsilon}(\overline{r})\to(n-1)h\quad\text{as}\quad\varepsilon\to 0.$$

If $\overline{r}=0$, then

$$\overline{\mathbf{S}}_{\rho_\varepsilon}(\overline{r})\to-\infty\quad\text{as}\quad\varepsilon\to 0.$$

In addition, for every fixed $\varepsilon>0$ one has

$$\overline{\mathbf{S}}_{\rho_\varepsilon}(\overline{r})\to(n-1)h\quad\text{as}\quad\overline{r}\to\infty.$$

Finally, for the reversibility we obtain

$$\mathbf{rev}_{\rho_\varepsilon}(\overline{r})\overset{\text{def}}{=}\mathbf{rev}(x,x)=\frac{1+\frac{\overline{r}(\overline{r}+2\varepsilon)\theta}{(\overline{r}+\varepsilon)^2}}{1-\frac{\overline{r}(\overline{r}+2\varepsilon)\theta}{(\overline{r}+\varepsilon)^2}}\leq\frac{1+\theta}{1-\theta}=\lambda.$$

It follows that, if $\overline{r}>0$, then

$$\mathbf{rev}_{\rho_\varepsilon}(\overline{r})\to\lambda\quad\text{as}\quad\varepsilon\to 0,$$

and for every fixed $\varepsilon>0$ one has

$$\mathbf{rev}_{\rho_\varepsilon}(\overline{r})\to\lambda\quad\text{as}\quad\overline{r}\to\infty,\quad\text{and}\quad\mathbf{rev}_{\rho_\varepsilon}(\overline{r})\to 1\quad\text{as}\quad\overline{r}\to 0.$$

The latter limit ensures the continuity of F_ε at the origin. In addition,

$$\lambda_{F_\varepsilon}(\mathbb{B}^n)=\lambda,\quad\forall\varepsilon>0.$$

Indeed, for every $x\in\mathbb{B}^n$, and $y\in\mathbb{R}^n$ one has

$$\mathbf{rev}_\varepsilon(x,y)=\frac{F_\varepsilon(x,-y)}{F_\varepsilon(x,y)}\leq\frac{F_\varepsilon(x,-x)}{F_\varepsilon(x,x)}=\mathbf{rev}_\varepsilon(x,x)\leq\lambda,\quad\text{if } x\neq O,$$

and $\mathbf{rev}_\varepsilon(O,y)=1$, which concludes the proof. $\square$

We make the following observations.

Remark 6.11. (a) Although there exist various similarities between Theorem 6.5 and Theorem 6.10, we highlight a main difference, which is the monotonicity of the map $r \mapsto \mathbf{K}_{\rho_\varepsilon}(r)$. In case of non-positive curvature this function is increasing, hence its limit as $r \to \infty$ is its upper bound. In case of strong negative curvature, for sufficiently small ε, it increases to a global maximum $-k_{\varepsilon,\vartheta}^2 \cdot K_\varepsilon$, then decreases (in the proof, that maximum was adjusted to ensure the required upper bound).

It is an interesting question, if there exists a perturbation of the metric (6.31) (or some other construction), in which $\mathbf{K} \leq -\kappa^2$ along $\nabla\rho_\varepsilon$ with a strictly increasing flag curvature along $\nabla\rho_\varepsilon$, or if this phenomena is directly implied by the constraints on the geometry. We note that on each alternative prototypes of families of spaces that were examined during the research, the same phenomena occurred.

(b) As we anticipated in Remark 6.6/(b), the construction of the measure m_ε allows for non-zero upper bound of the reduced S-curvature, which is particularly important for our applications.

(c) Since $K_{\varepsilon,0} \to \infty$ as $\varepsilon \to 0$, there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that $k_\varepsilon = \frac{\kappa}{\sqrt{K_\varepsilon}}$, for every $\varepsilon \in (0, \varepsilon_1)$. Moreover, since $k_\varepsilon \to \kappa$ from *above*, there exist $\overline{\varepsilon} \in (0, \varepsilon_1)$ and $\overline{\kappa} > \kappa$, such that

$$\kappa \leq k_\varepsilon \leq \overline{\kappa}, \quad \forall \varepsilon \in (0, \overline{\varepsilon}). \tag{6.35}$$

(d) Similarly to Remark 6.6/(d), we have $\max\{F_\varepsilon^*(\pm D\rho_\varepsilon)\} = F_\varepsilon^*(D\rho_\varepsilon) = 1$, since

$$F_\varepsilon^*(-D\rho_\varepsilon) = \frac{1 - \frac{\overline{r}(\overline{r}+2\varepsilon)}{(\overline{r}+\varepsilon)^2}\theta}{1 + \frac{\overline{r}(\overline{r}+2\varepsilon)}{(\overline{r}+\varepsilon)^2}\theta} \stackrel{\text{def}}{=} h_\varepsilon(\overline{r}), \quad \text{where} \quad \frac{1-\theta}{1+\theta} \leq h_\varepsilon(\overline{r}) \leq 1. \tag{6.36}$$

6.3.2 Sufficient conditions for sharpness

In this section, we present sufficient conditions for the sharpness of Hardy inequalities on the spaces of $\mathcal{F}_{\kappa,h,\lambda}$, introduced in Theorem 6.10.

Let $(\mathbb{B}^n, F_\varepsilon, \mathsf{m}_\varepsilon) \subset \mathcal{F}_{\kappa,h,\lambda}$ for some $\varepsilon > 0$, $\lambda \in (1, \infty)$, $\kappa > 0$ and $h \in \mathbb{R}$. Then for every $x \in \mathbb{B}^n$, one has

$$\mathbf{K}_\varepsilon(x, x) \leq -\kappa^2, \quad \overline{\mathbf{S}}_\varepsilon(x, x) \leq (n-1)h, \quad \text{and} \quad \mathbf{rev}_\varepsilon(x, x) \leq \lambda = \lambda_{F_\varepsilon}(\mathbb{B}^n),$$

Denote by ρ_ε the distance from the origin. Applying the Laplace comparison theorem (see Theorem 2.3) one has

$$\Delta_{F_\varepsilon}(\rho_\varepsilon) \geq (n-1)\kappa \coth(\kappa\rho_\varepsilon) - (n-1)h \stackrel{\text{def}}{=} L^{\kappa,h}(\rho_\varepsilon).$$

Let $R \in (0, +\infty]$, $L(t) \leq L^{\kappa,h}(t)$ for every $t \in (0, R)$, and define

$$\Omega_\varepsilon = \{x \in \mathbb{B}^n \mid \rho_\varepsilon(x) < R\}.$$

Suppose that (L, W) is a (p, ρ_ε, w)-*Riccati pair* in $(0, R)$, that is $W\colon (0, R) \to (0, \infty)$ is continuous, $w\colon (0, R) \to (0, \infty)$ is of class C^1, and there exists $G\colon (0, R) \to (0, \infty)$ of class C^1 such that

$$(G(t)w(t))' + G(t)w(t)L(t) - (p-1)G(t)^{p'}w(t) \geq W(t)w(t), \quad \forall t \in (0, R). \tag{6.37}$$

By Theorem 6.1, for every $u \in C_0^\infty(\Omega_\varepsilon)$ one has

$$\lambda^p \int_{\Omega_\varepsilon} w(\rho_\varepsilon) F_\varepsilon^*(Du)^p \,\mathsf{dm}_\varepsilon \geq \int_{\Omega_\varepsilon} w(\rho_\varepsilon) W(\rho_\varepsilon) |u|^p \,\mathsf{dm}_\varepsilon. \tag{6.38}$$

Suppose that $G > 0$ satisfies the Riccati ODI (6.37) with equality, and define the function $v\colon (0, R) \to \mathbb{R}$ by

$$-(\log v(t))' = -\frac{v'(t)}{v(t)} = G(t)^{\frac{1}{p-1}}. \tag{6.39}$$

Since G is smooth and positive, v is smooth, positive, and decreasing. Introduce the notation

$$\mathcal{I}^{k,h}(f; a, b) = \int_a^b f(t) \sinh(kt)^{n-1} e^{-(n-1)ht} \,\mathrm{d}t, \tag{6.40}$$

where $k, h \in \mathbb{R}$ and $f\colon [a, b] \to \mathbb{R}$ is a continuous function, with $0 < a < b < R$. We have the following result.

Theorem 6.12 (see [58]). *Fix $\delta_0 > 0$ and let $t_1 = t_1(\delta), t_2 = t_2(\delta), t_3 = t_3(\delta)$, and $t_4 = t_4(\delta)$ be continuous functions admitting a (possibly infinite) limit at infinity, such that for every $\delta > \delta_0$ one has $0 < t_1 < t_2 < t_3 < t_4 < R$. Let $\varepsilon = \varepsilon(\delta)$ such that $\varepsilon \to 0$, as $\delta \to \infty$, and suppose that the following limits hold as $\delta \to \infty$:*

$$l_0^{k_\varepsilon,h}(\delta) \overset{def}{=} \frac{v(t_2)^p \cdot \mathcal{I}^{k_\varepsilon,h}(w; t_1, t_2)}{(t_2 - t_1)^p \cdot \mathcal{I}^{k_\varepsilon,h}(wWv^p; t_2, t_3)} + \frac{v(t_3)^p \cdot \mathcal{I}^{k_\varepsilon,h}(w; t_3, t_4)}{(t_4 - t_3)^p \cdot \mathcal{I}^{k_\varepsilon,h}(wWv^p; t_2, t_3)} \to 0,$$

$$l_1^{k_\varepsilon,h}(\delta) \overset{def}{=} \max_{t \in [t_2, t_3]} \frac{G(t)^{p'}}{W(t)} \to 1,$$

$$l_2^{k_\varepsilon,h}(\delta) \overset{def}{=} h_\varepsilon((1-\theta)\kappa t_2) \to \frac{1}{\lambda}, \quad \text{(see relation (6.36)).}$$

Then inequality (6.38) is sharp on $\mathcal{F}_{\kappa,h,\lambda}$ in the sense that it no longer holds on each FMMM of $\mathcal{F}_{\kappa,h,\lambda}$ if we multiply its right hand side by a constant $c > 1$.

Proof. Let $\varepsilon > 0$ and $T = (t_1, t_2, t_3, t_4)$. Define the limit function $u_\varepsilon^\star\colon \Omega \to (0, \infty)$ of (6.38) and its truncation $u_{\varepsilon,T}^\star$ by

$$u_\varepsilon^\star(x) = v(\rho_\varepsilon(x)) \quad \text{and} \quad u_{\varepsilon,T}^\star(x) = v_{\varepsilon,T}(\rho_\varepsilon(x)),$$

where

$$v_{\varepsilon,T}(t) = \begin{cases} 0, & \text{if } 0 < t \leq t_1, \\ v(t_2)\frac{t-t_1}{t_2-t_1}, & \text{if } t_1 \leq t \leq t_2, \\ v(t), & \text{if } t_2 \leq t \leq t_3, \\ v(t_3)\frac{t_4-t}{t_4-t_3}, & \text{if } t_3 \leq t \leq t_4, \\ 0, & \text{if } t_4 \leq t < R, \end{cases} \quad \text{and} \quad v_{\varepsilon,T}'(t) = \begin{cases} 0, & \text{if } 0 < t < t_1, \\ \frac{v(t_2)}{t_2-t_1}, & \text{if } t_1 < t < t_2, \\ v'(t), & \text{if } t_2 < t < t_3, \\ -\frac{v(t_3)}{t_4-t_3}, & \text{if } t_3 < t < t_4, \\ 0, & \text{if } t_4 < t < R. \end{cases} \tag{6.41}$$

Proceeding similarly to the proof of Theorem 6.7, we reduce the following integrals to one dimension:

$$I \overset{\text{def}}{=} \frac{1}{n\omega}\int_{\Omega_\varepsilon} w(\rho_\varepsilon)F_\varepsilon^*(Du_{\varepsilon,T}^\star)^p \,\mathsf{dm}_\varepsilon \quad \text{and} \quad J \overset{\text{def}}{=} \frac{1}{n\omega}\int_{\Omega_\varepsilon} w(\rho_\varepsilon)W(\rho_\varepsilon)(u_{\varepsilon,T}^\star)^p \,\mathsf{dm}_\varepsilon.$$

For simplicity, denote $r = |x|$, $\overline{r} = \operatorname{arctanh}(r)$, and $\vartheta_\varepsilon = k_\varepsilon(1-\theta)$. Introduce the following notations:

$$t \overset{\text{def}}{=} \rho_\varepsilon(x) = \overline{r}\cdot\frac{\overline{r}(1-\theta)+\varepsilon}{\vartheta_\varepsilon(\overline{r}+\varepsilon)} \quad \text{and} \quad e(t) \overset{\text{def}}{=} \exp(-(n-1)ht).$$

By a simple computation, we obtain that

$$r = \tanh\varphi_\varepsilon(t), \quad \text{where} \quad \varphi_\varepsilon(t) = \frac{t\vartheta_\varepsilon - \varepsilon + \sqrt{(t\vartheta_\varepsilon-\varepsilon)^2 + 4t\vartheta_\varepsilon\varepsilon(1-\theta)}}{2(1-\theta)},$$

$$\mathrm{d}r = \frac{1}{(\cosh\varphi_\varepsilon(t))^2}\varphi_\varepsilon'(t)\,\mathrm{d}t, \quad \text{where} \quad \varphi_\varepsilon'(t) = \frac{k}{2}\left(1 + \frac{t\vartheta_\varepsilon - \varepsilon + 2\varepsilon(1-\theta)}{\sqrt{(t\vartheta_\varepsilon-\varepsilon)^2+4t\vartheta_\varepsilon\varepsilon(1-\theta)}}\right),$$

$$\sigma_\varepsilon = e(t)\cdot\frac{1}{\vartheta_\varepsilon^n}(\cosh\varphi_\varepsilon(t))^{n+1}(f_\varepsilon(t))^{\frac{n+1}{2}}, \quad \text{where} \quad f_\varepsilon(t) = 1 - \frac{\varphi_\varepsilon^2(t)(\varphi_\varepsilon(t)+2\varepsilon)^2\theta^2}{(\varphi_\varepsilon(t)+\varepsilon)^4}.$$

It follows that

$$\sigma_\varepsilon r^{n-1}\,\mathrm{d}r = e(t)\cdot\varphi_\varepsilon'(t)\frac{1}{\vartheta_\varepsilon^n}(f_\varepsilon(t))^{\frac{n+1}{2}}(\sinh\varphi_\varepsilon(t))^{n-1}\,\mathrm{d}t \overset{\text{def}}{=} e(t)\cdot\psi_\varepsilon(t)\,\mathrm{d}t. \tag{6.42}$$

In the sequel, we provide suitable bounds for $\psi_\varepsilon(t)$. First, we observe that

$$(1-\theta)k_\varepsilon t \le \varphi_\varepsilon(t) \le k_\varepsilon t, \quad (1-\theta)k_\varepsilon \le \varphi_\varepsilon'(t) \le k_\varepsilon, \quad \text{and} \quad 1-\theta^2 \le f_\varepsilon(t) \le 1, \quad \forall t \ge 0.$$

Next, we show that there exists $\widetilde{\varepsilon} > 0$ and $\widetilde{c} > 0$ such that

$$\widetilde{c}\cdot\sinh(k_\varepsilon t) \le \sinh(\varphi_\varepsilon(t)) \le \sinh(k_\varepsilon t), \quad \forall t \ge 0, \varepsilon \in (0,\widetilde{\varepsilon}). \tag{6.43}$$

Since the hyperbolic sine is increasing, we obtain

$$\sinh((1-\theta)k_\varepsilon t) \le \sinh(\varphi_\varepsilon(t)) \le \sinh(k_\varepsilon t), \quad \forall t \ge 0, \varepsilon > 0,$$

thus, the second inequality of (6.43) holds. To verify the first inequality, define

$$\psi(k,t) = \frac{\sinh((1-\theta)kt)}{\sinh(kt)}, \quad \forall k > 0, t \ge 0.$$

Recall from Remark 6.11/(c) that there exists $\overline{\varepsilon} > 0$ and $\overline{\kappa} > \kappa$ such that

$$\kappa \le k_\varepsilon \le \overline{\kappa}, \quad \forall \varepsilon \in (0,\overline{\varepsilon}).$$

On the one hand, observe that $t \mapsto \psi(\overline{\kappa},t)$ is continuous and $\psi(\overline{\kappa},t) \to 1-\theta > 0$ as $t \to 0$, which implies that there exists $\widetilde{c_1} > 0$ and $\widetilde{t} > 0$ such that

$$\sinh((1-\theta)\overline{\kappa}t) \ge \widetilde{c_1}\cdot\sinh(\overline{\kappa}t), \quad \forall t \in [0,\widetilde{t}].$$

On the other hand, since $k \mapsto \psi(k,t)$ is decreasing, the above estimate holds for every $k_\varepsilon \in [\kappa, \overline{\kappa}]$. Hence,

$$\sinh((1-\theta)k_\varepsilon t) \geq \widetilde{c_1} \cdot \sinh(k_\varepsilon t), \quad \forall t \in [0, \widetilde{t}], \varepsilon \in (0, \overline{\varepsilon}).$$

Now we focus on the case when $t > \widetilde{t}$. Studying the asymptotic behavior of φ_ε we obtain

$$\varphi_\varepsilon(t) \geq tk_\varepsilon - \frac{\varepsilon\theta}{1-\theta} \stackrel{\text{def}}{=} tk_\varepsilon - s_\varepsilon, \quad \forall t > 0, \varepsilon \in (0, \overline{\varepsilon}).$$

Note that $k_\varepsilon \to \kappa$ and $s_\varepsilon \to 0$ as $\varepsilon \to 0$. Thus, since $\widetilde{t} > 0$, there exists an $\widetilde{\varepsilon} \in (0, \overline{\varepsilon})$ such that

$$\varphi_\varepsilon(\widetilde{t}) \geq \widetilde{t}k_\varepsilon - s_\varepsilon > s_\varepsilon > 0, \quad \forall \varepsilon \in (0, \widetilde{\varepsilon}). \tag{6.44}$$

Since $t \mapsto \varphi_\varepsilon(t)$ is increasing, we also have

$$\varphi_\varepsilon(t) \geq tk_\varepsilon - s_\varepsilon > s_\varepsilon > 0, \quad \forall \varepsilon \in (0, \widetilde{\varepsilon}), t > \widetilde{t}. \tag{6.45}$$

It follows that

$$\begin{aligned}\sinh(\varphi_\varepsilon(t)) &\geq \sinh(tk_\varepsilon - s_\varepsilon) = \sinh(tk_\varepsilon)\cosh(s_\varepsilon) - \cosh(tk_\varepsilon)\sinh(s_\varepsilon) \\ &= \sinh(tk_\varepsilon) \cdot (\cosh(s_\varepsilon) - \coth(tk_\varepsilon)\sinh(s_\varepsilon)).\end{aligned}$$

Since the hyperbolic cotangent is decreasing, using relation (6.45), we obtain that

$$\sinh(\varphi_\varepsilon(t)) \geq \sinh(tk_\varepsilon) \cdot (\cosh(s_\varepsilon) - \coth(2s_\varepsilon)\sinh(s_\varepsilon)) = \sinh(tk_\varepsilon) \cdot \frac{1}{2\cosh(s_\varepsilon)}$$

Define $s_{\widetilde{\varepsilon}} = \frac{\widetilde{\varepsilon}\theta}{1-\theta}$, the monotonicity of the hyperbolic cosine implies

$$\sinh(\varphi_\varepsilon(t)) \geq \sinh(tk_\varepsilon) \cdot \frac{1}{2\cosh(s_{\widetilde{\varepsilon}})} \stackrel{\text{def}}{=} \widetilde{c_2} \cdot \sinh(tk_\varepsilon).$$

Thus, $\widetilde{c} = \min\{\widetilde{c_1}, \widetilde{c_2}\}$ and $\widetilde{\varepsilon}$ yield inequality (6.43), for every $t \geq 0$.

Combining the above estimates, we obtain for every $t \geq 0$ and $0 < \varepsilon < \widetilde{\varepsilon}$ that

$$\begin{aligned}c\sinh(k_\varepsilon t)^{n-1} &\stackrel{\text{def}}{=} \frac{(1-\theta^2)^{\frac{n+1}{2}}(\widetilde{c})^{n-1}}{\vartheta_\varepsilon^{n-1}}\sinh(k_\varepsilon t)^{n-1} \\ &\leq \psi_\varepsilon(t) \leq \frac{k_\varepsilon}{\vartheta^n}\sinh(k_\varepsilon t)^{n-1} \stackrel{\text{def}}{=} C\sinh(k_\varepsilon t)^{n-1}.\end{aligned} \tag{6.46}$$

Since v is positive and decreasing, the positive homogeneity of F_ε^* and the relations from (6.41) imply

$$F_\varepsilon^*(Du_{\varepsilon,T}^\star) = F_\varepsilon^*(v_{\varepsilon,T}'(\rho_\varepsilon)D\rho_\varepsilon) = \begin{cases} \frac{v(t_2)}{t_2-t_1}F_\varepsilon^*(D\rho_\varepsilon), & \text{if } \rho_\varepsilon \in (t_1,t_2), \\ |v'(\rho_\varepsilon)|F_\varepsilon^*(-D\rho_\varepsilon) & \text{if } \rho_\varepsilon \in (t_2,t_3), \\ \frac{v(t_3)}{t_4-t_3}F_\varepsilon^*(-D\rho_\varepsilon) & \text{if } \rho_\varepsilon \in (t_3,t_4), \\ 0, & \text{if } \rho_\varepsilon \in (0,t_1)\cup(t_4,R). \end{cases} \tag{6.47}$$

According to Remark 6.11/(d), one has

$$F_\varepsilon^*(D\rho_\varepsilon)=1,\quad \text{and}\quad F_\varepsilon^*(-D\rho_\varepsilon)=\frac{1-\frac{\overline{r}(\overline{r}+2\varepsilon)}{(\overline{r}+\varepsilon)^2}\theta}{1+\frac{\overline{r}(\overline{r}+2\varepsilon)}{(\overline{r}+\varepsilon)^2}\theta}\stackrel{\text{def}}{=}h_\varepsilon(\overline{r}),\quad \text{where}\quad \frac{1-\theta}{1+\theta}\le h_\varepsilon(\overline{r})\le 1.$$

Observe that $h_\varepsilon(\overline{r})$ is decreasing in $\overline{r}$, increasing in ε, and approaches to $\frac{1-\theta}{1+\theta}=\frac{1}{\lambda}$ as $\varepsilon\to 0$. Clearly one has

$$\overline{r}=\varphi_\varepsilon(t)\ge(1-\theta)k_\varepsilon t\ge(1-\theta)\kappa t,$$

thus, we also obtain

$$h_\varepsilon(\overline{r})=h_\varepsilon(\varphi_\varepsilon(t))\le h_\varepsilon((1-\theta)\kappa t)\le 1. \tag{6.48}$$

As before, denote by ω the volume of the Euclidean unit ball of dimension n.

On the one hand, relations (6.42) and (6.47) yield

$$\begin{aligned} I=&\frac{v(t_2)^p}{(t_2-t_1)^p}\int_{t_1}^{t_2}w(t)\psi_\varepsilon(t)e(t)\,\mathrm{d}t+\int_{t_2}^{t_3}w(t)|v'(t)|^p h_\varepsilon^p(\varphi_\varepsilon(t))\psi_\varepsilon(t)e(t)\,\mathrm{d}t\\ &+\frac{v(t_3)^p}{(t_4-t_3)^p}\int_{t_3}^{t_4}w(t)h_\varepsilon^p(\varphi_\varepsilon(t))\psi_\varepsilon(t)e(t)\,\mathrm{d}t. \end{aligned}$$

Applying relations (6.46), (6.48), and using the definition of $\mathcal{I}^{k,h}$ from (6.40), we obtain

$$\begin{aligned} I\le&\frac{v(t_2)^p}{(t_2-t_1)^p}\int_{t_1}^{t_2}w(t)\psi_\varepsilon(t)e(t)\,\mathrm{d}t+\int_{t_2}^{t_3}w(t)|v'(t)|^p h_\varepsilon^p((1-\theta)\kappa t)\psi_\varepsilon(t)e(t)\,\mathrm{d}t\\ &+\frac{v(t_3)^p}{(t_4-t_3)^p}\int_{t_3}^{t_4}w(t)h_\varepsilon^p((1-\theta)\kappa t)\psi_\varepsilon(t)e(t)\,\mathrm{d}t\\ \le&\frac{Cv(t_2)^p}{(t_2-t_1)^p}\int_{t_1}^{t_2}w(t)\sinh(k_\varepsilon t)^{n-1}e(t)\,\mathrm{d}t+h_\varepsilon^p((1-\theta)\kappa t_2)\int_{t_2}^{t_3}w(t)|v'(t)|^p\psi_\varepsilon(t)e(t)\,\mathrm{d}t\\ &+\frac{Ch_\varepsilon^p((1-\theta)\kappa t_3)v(t_3)^p}{(t_4-t_3)^p}\int_{t_3}^{t_4}w(t)\sinh(k_\varepsilon t)^{n-1}e(t)\,\mathrm{d}t\\ \le&\frac{Cv(t_2)^p\cdot\mathcal{I}^{k_\varepsilon,h}(w;t_1,t_2)}{(t_2-t_1)^p}+h_\varepsilon^p((1-\theta)\kappa t_2)\int_{t_2}^{t_3}w(t)|v'(t)|^p\psi_\varepsilon(t)e(t)\,\mathrm{d}t\\ &+\frac{Cv(t_3)^p\cdot\mathcal{I}^{k_\varepsilon,h}(w;t_3,t_4)}{(t_4-t_3)^p}. \end{aligned}$$

By relation (6.39), for every $t\in[t_2,t_3]$ one has

$$\frac{1}{W(t)}\left|\frac{v'(t)}{v(t)}\right|^p=\frac{1}{W(t)}\left(-\frac{v'(t)}{v(t)}\right)^p=\frac{G(t)^{p'}}{W(t)}.$$

The above function is continuous, thus, it attains it maximum on $[t_2,t_3]$, hence we have

$$\begin{aligned} \int_{t_2}^{t_3}w(t)|v'(t)|^p\psi_\varepsilon(t)\,\mathrm{d}t&=\int_{t_2}^{t_3}\frac{1}{W(t)}\left|\frac{v'(t)}{v(t)}\right|^p w(t)W(t)(v(t))^p\psi_\varepsilon(t)\,\mathrm{d}t\\ &\le\max_{t\in[t_2,t_3]}\frac{G(t)^{p'}}{W(t)}\cdot\int_{t_2}^{t_3}w(t)W(t)(v(t))^p\psi_\varepsilon(t)e(t)\,\mathrm{d}t. \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} J &= \int_0^R w(t)W(t)(v_{\varepsilon,T}(t))^p \psi_\varepsilon(t)e(t)\,\mathrm{d}t \geq \int_{t_2}^{t_3} w(t)W(t)(v(t))^p \psi_\varepsilon(t)e(t)\,\mathrm{d}t \\ &\geq c \cdot \mathcal{I}^{k_\varepsilon,h}(wWv^p; t_2, t_3). \end{aligned} \tag{6.49}$$

Combining the above results, and using both inequalities of (6.49), it follows that

$$\begin{aligned} \frac{\lambda^p I}{J} &\leq \lambda^p h_\varepsilon^p((1-\theta)\kappa t_2) \cdot \max_{t\in[t_2,t_3]} \frac{G(t)^{p'}}{W(t)} \\ &\quad + \frac{\lambda^p C}{c}\left(\frac{v(t_2)^p \cdot \mathcal{I}^{k_\varepsilon,h}(w;t_1,t_2)}{(t_2-t_1)^p \cdot \mathcal{I}^{k_\varepsilon,h}(wWv^p;t_2,t_3)} + \frac{v(t_3)^p \cdot \mathcal{I}^{k_\varepsilon,h}(w;t_3,t_4)}{(t_4-t_3)^p \cdot \mathcal{I}^{k_\varepsilon,h}(wWv^p;t_2,t_3)}\right) \\ &= \lambda^p \cdot (l_2^{k_\varepsilon,h}(\delta))^p \cdot l_1^{k_\varepsilon,h}(\delta) + \frac{\lambda^p C}{c} \cdot l_0^{k_\varepsilon,h}(\delta). \end{aligned}$$

By our assumptions, the last relation approaches 1 as $\delta \to \infty$, thus, the inequality (6.38) is sharp on $\mathcal{F}_{\kappa,h,\lambda}$. ☐

Observe that in contrast with Theorem 6.7, the limits appearing in the statement of the above result are strongly dependent on ε. The reason for this stems from the observation of Remark 6.11/(a): Since the flag curvature is not increasing along $\nabla\rho$, we need to work with k_ε instead of κ, which explains the dependence on ε. Fortunately, by suitable choices of $\varepsilon(\delta)$, one can provide sharpness results in this setting, as well. Such an example is presented in the sequel.

6.3.3 The sharpness of a McKean-type spectral gap estimate

In this section, we prove a sharpness result concerning a McKean-type spectral gap estimate on $\mathcal{F}_{\kappa,h,\lambda}$, which can be stated as follows.

Theorem 6.13 (see [58]). *Let $n \geq 2$, $\lambda, p \in (1,\infty)$, and $\kappa > h \geq 0$. For every $(\mathbb{B}^n, F_\varepsilon, \mathsf{m}_\varepsilon) \in \mathcal{F}_{\kappa,h,\lambda}$ and $u \in C_0^\infty(\mathbb{B}^n)$ one has*

$$\lambda^p \int_{\mathbb{B}^n} F_\varepsilon^*(Du)^p \,\mathrm{d}\mathsf{m}_\varepsilon \geq c_{n,p,\kappa,h} \int_{\mathbb{B}^n} |u|^p \,\mathrm{d}\mathsf{m}_\varepsilon, \tag{6.50}$$

where

$$c_{n,p,\kappa,h} = \left(\frac{(n-1)(\kappa-h)}{p}\right)^p.$$

Moreover, $c_{n,p,\kappa,h}$ is sharp *in $\mathcal{F}_{\kappa,h,\lambda}$, in the sense that it is the greatest constant with this property.*

Proof. The validity of inequality (6.50) follows from Theorem 6.4. To prove the sharpness, we use Theorem 6.12. Recall the definition of $\widetilde{\varepsilon}$ from (6.44). Let $\delta > 1$, such that $\varepsilon = \frac{1}{\delta} \in (0, \widetilde{\varepsilon})$, and define

$$t_1(\delta) = \delta, \quad t_2(\delta) = 2\delta, \quad t_3(\delta) = 3\delta, \quad \text{and} \quad t_4(\delta) = 4\delta.$$

By a simple computation, we obtain

$$l_2^{k_\varepsilon,h} = h_\varepsilon((1-\theta)\kappa t_2) = \frac{1+4\delta^2(1-\theta)^2\kappa+4\delta^4(1-\theta)^3\kappa^2}{1+4\delta^2(1-\theta^2)\kappa+4\delta^4(1-\theta)^2(1+\theta)\kappa^2} \to \frac{1-\theta}{1+\theta} = \lambda,$$

as $\delta\to\infty$. Consider the following functions (see also the proof of Theorem 6.4):

$$w(t)=1, \quad W(t)=c_{n,p,\kappa,h}, \quad G(t)\equiv (c_{n,p,\kappa,h})^{\frac{p-1}{p}}, \quad \text{and} \quad v(t)^p = e^{-(n-1)t(\kappa-h)},$$

for every $t>0$. Since $k_\varepsilon>\kappa>h$, the following expressions are strictly increasing in t:

$$w(t)\sinh(k_\varepsilon t)^{n-1}e^{-(n-1)ht} = \sinh(k_\varepsilon t)^{n-1}e^{-(n-1)ht},$$
$$w(t)W(t)v(t)^p\sinh(k_\varepsilon t)^{n-1}e^{-(n-1)ht} = c_{n,p,\kappa,h}\sinh(k_\varepsilon t)^{n-1}e^{-(n-1)\kappa t}.$$

On the one hand, we have

$$\begin{aligned}\frac{v(t_2)^p\cdot\mathcal{I}^{k_\varepsilon,h}(w;t_1,t_2)}{(t_2-t_1)^p\cdot\mathcal{I}^{k_\varepsilon,h}(wWv^p;t_2,t_3)} &\leq \frac{v(t_2)^p}{(t_2-t_1)^p}\cdot\frac{(t_2-t_1)\sinh(k_\varepsilon t_1)^{n-1}e^{-(n-1)ht_1}}{(t_3-t_2)c_{n,p,\kappa,h}\sinh(k_\varepsilon t_3)^{n-1}e^{-(n-1)ht_3}}\\
&= \frac{e^{(n-1)\delta(\kappa+h)}\sinh(\delta k_\varepsilon)^{n-1}}{\delta^p c_{n,p,\kappa,h}\sinh(3\delta k_\varepsilon)^{n-1}}\\
&= \frac{e^{(n-1)\delta(\kappa+h-2k_\varepsilon)}}{\delta^p c_{n,p,\kappa,h}}\cdot\frac{e^{(n-1)\delta(2k_\varepsilon)}\sinh(\delta k_\varepsilon)^{n-1}}{\sinh(3\delta k_\varepsilon)^{n-1}} \to 0\cdot 1 = 0,\end{aligned}$$

as $\delta\to\infty$. On the other hand, observe that

$$\begin{aligned}\frac{v(t_3)^p\cdot\mathcal{I}^{k_\varepsilon,h}(w;t_3,t_4)}{(t_4-t_3)^p\cdot\mathcal{I}^{k_\varepsilon,h}(wWv^p;t_2,t_3)} &\leq \frac{v(t_3)^p}{(t_4-t_3)^p}\cdot\frac{(t_4-t_3)\sinh(\kappa t_3)^{n-1}e^{-(n-1)ht_3}}{(t_3-t_2)c_{n,p,\kappa,h}\sinh(\kappa t_3)^{n-1}e^{-(n-1)\kappa t_3}}\\
&= \frac{1}{\delta^p c_{n,p,\kappa,h}} \to 0, \quad \text{as } \delta\to\infty.\end{aligned}$$

Combining the above two limits, we obtain $l_1^{k_\varepsilon,h}(\delta)\to 0$ as $\delta\to\infty$.

Finally, since

$$\frac{G(t)^{p'}}{W(t)} = 1, \quad \forall t>0,$$

we also obtain $l_1^{k_\varepsilon,h}(\delta)=1$. Thus, Theorem 6.12 yields the sharpness of the inequality (6.50), which concludes the proof. □

Finally, for $p=2$ we obtain the following result.

Corollary 6.14 ([61])**.** *Let $n\geq 2$, $\lambda\in(1,\infty)$, and $\kappa>h\geq 0$. For every $(\mathbb{B}^n, F_\varepsilon, \mathsf{m}_\varepsilon)\in\mathcal{F}_{\kappa,h,\lambda}$ and $u\in C_0^\infty(\mathbb{B}^n)$ one has*

$$\lambda^2\int_{\mathbb{B}^n} F_\varepsilon^*(Du)^2\,\mathsf{dm}_\varepsilon \geq c_{n,\kappa,h}\int_{\mathbb{B}^n} u^2\,\mathsf{dm}_\varepsilon, \quad \textit{where} \quad c_{n,\kappa,h} = \left(\frac{(n-1)(\kappa-h)}{2}\right)^2. \tag{6.51}$$

Moreover, $c_{n,\kappa,h}$ is sharp *in $\mathcal{F}_{\kappa,h,\lambda}$, in the sense that it is the greatest constant with this property.*

Bibliography

[1] Adimurthi, N. Chaudhuri, and M. Ramaswamy. An improved Hardy-Sobolev inequality and its applications. *Proc. Amer. Math. Soc.*, 130(2):489–505, 2002.

[2] Adimurthi, M. Grossi, and S. Santra. Optimal Hardy–Rellich inequalities, maximum principle and related eigenvalue problem. *J. Funct. Anal.*, 240(1):36–83, 2006.

[3] K. Akutagawa and H. Kumura. Geometric relative Hardy inequalities and the discrete spectrum of Schrödinger operators on manifolds. *Calc. Var. Partial Differ. Equ.*, 48:67–88, 2013.

[4] W. Allegretto. On the equivalence of two types of oscillation for elliptic operators. *Pac. J. Math.*, 55(2):319–328, 1974.

[5] P.R.S. Antunes, D. Buoso, and P. Freitas. On the behavior of clamped plates under large compression. *SIAM J. Appl. Math.*, 79(5):1872–1891, 2019.

[6] M. Ashbaugh and R. Benguria. On Rayleigh's conjecture for the clamped plate and its generalization to three dimensions. *Duke Math. J.*, 78(1):1–17, 1995.

[7] A. A. Balinsky, W. D. Evans, and R. T. Lewis. *The analysis and geometry of Hardy's inequality.* Springer Verlag Universitext, 2015.

[8] D. Bao, S.-S. Chern, and Z. Shen. *An introduction to Riemann-Finsler geometry*, volume 200. Springer Science & Business Media, 2000.

[9] G. Barbatis, S. Filippas, and A. Tertikas. A unified approach to improved L^p Hardy inequalities with best constants. *Trans. Amer. Math. Soc.*, 356(6):2169–2196, 2004.

[10] E. Berchio, L. D'Ambrosio, D. Ganguly, and G. Grillo. Improved L^p-Poincaré inequalities on the hyperbolic space. *Nonlinear Anal.*, 157:146–166, 2017.

[11] E. Berchio, D. Ganguly, and G. Grillo. Sharp Poincaré–Hardy and Poincaré–Rellich inequalities on the hyperbolic space. *J. Funct. Anal.*, 272(4):1661–1703, 2017.

[12] E. Berchio, D. Ganguly, G. Grillo, and Y. Pinchover. An optimal improvement for the Hardy inequality on the hyperbolic space and related manifolds. *Proc. Roy. Soc. Edinburgh Sect. A*, 150(4):1699–1736, 2020.

[13] E. Berchio, D. Ganguly, and P. Roychowdhury. On some strong Poincaré inequalities on Riemannian models and their improvements. *J. Math. Anal. Appl.*, 490(1):124213, 2020.

[14] E. Berchio, D. Ganguly, and P. Roychowdhury. Hardy–Rellich and second order Poincaré identities on the hyperbolic space via Bessel pairs. *Calc. Var. Partial Differ. Equ.*, 61(4):24, Paper No. 130, 2022.

[15] G. Bol. Isoperimetrische Ungleichungen für Bereiche auf Flächen. *Jahresber. Dtsch. Math.-Ver.*, 51:219–257, 1941.

[16] D. Borisov and P. Freitas. The spectrum of geodesic balls on spherically symmetric manifolds. *Comm. Anal. Geom.*, 2017(3):507–544, 2017.

[17] H. Brezis and M. Marcus. Hardy's inequalities revisited. *Ann. Scuola Norm. Sup. Pisa Cl. Sci*, 25:217–237, 1997.

[18] H. Brezis and J. L. Vázquez. Blow-up solutions of some nonlinear elliptic problems. *Rev. Mat. Univ. Compl. Madrid*, 10:443–469, 1997.

[19] L. Caffarelli, R. Kohn, and L. Nirenberg. First order interpolation inequalities with weights. *Compos. Math.*, 53(3):259–275, 1984.

[20] G. Carron. Inégalités de Hardy sur les variétés riemanniennes non-compactes. *J. Math. Pures Appl.*, 76(10):883–891, 1997.

[21] F. G. de S. Carvalho and M. P de A. Cavalcante. On the fundamental tone of the p-Laplacian on Riemannian manifolds and applications. *J. Math. Anal. Appl.*, 506(2):8, Paper No. 125703, 2022.

[22] F. Catrina and Z.-Q. Wang. On the Caffarelli–Kohn–Nirenberg inequalities: Sharp constants, existence (and nonexistence), and symmetry of extremal functions. *Comm. Pure Appl. Math.*, 54(2):229–258, 2001.

[23] C. Cazacu, J. Flynn, and N. Lam. Sharp second order uncertainty principles. *J. Funct. Anal.*, 283(10):37, Paper No. 109659, 2022.

[24] I. Chavel. *Eigenvalues in Riemannian geometry*, volume 115 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1984.

[25] Y. Chen, N. C. Leung, and W. Zhao. Sharp Hardy inequalities via Riemannian submanifolds. *J. Geom. Anal.*, 32(7):204, 2022.

[26] Q.-M. Cheng and H. Yang. Estimates for eigenvalues on Riemannian manifolds. *J. Differ. Equ.*, 247(8):2270–2281, 2009.

[27] Q.-M. Cheng and H. Yang. Universal inequalities for eigenvalues of a clamped plate problem on a hyperbolic space. *Proc. Amer. Math. Soc.*, 139(2):461–471, 2011.

[28] S.-Y. Cheng. Eigenvalue comparison theorems and its geometric applications. *Math. Z.*, 143:289–297, 1975.

[29] C. B. Croke. A sharp four dimensional isoperimetric inequality. *Comment. Math. Helvetici*, 59(1):187–192, 1984.

[30] L. D'Ambrosio and S. Dipierro. Hardy inequalities on Riemannian manifolds and applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 31(3):449–475, 2014.

[31] E. B. Davies. *Heat kernels and spectral theory.* 92. Cambridge university press, 1989.

[32] E. B. Davies and A. M. Hinz. Explicit constant for Rellich inequalities in $L^p(\Omega)$. *Math. Z.*, 227:511–524, 1998.

[33] F. Della Pietra, G. di Blasio, and N. Gavitone. Anisotropic Hardy inequalities. *Proc. Roy. Soc. Edinburgh Sect. A*, 148(3):483–498, 2018.

[34] B. Devyver, M. Fraas, and Y. Pinchover. Optimal Hardy weight for second-order elliptic operator: An answer to a problem of Agmon. *J. Funct. Anal.*, 266(7):4422–4489, 2014.

[35] A. Dinghas. Einfacher Beweis der isoperimetrischen Eigenschaft der Kugel in Riemannschen Räumen konstanter Krümmung. *Math. Nachr.*, 2(3-4):148–162, 1949.

[36] N. T. Duy, N. Lam, and G. Lu. p-Bessel Pairs, Hardy's Identities and Inequalities and Hardy–Sobolev Inequalities with Monomial Weights. *J. Geom. Anal.*, 32(109):36, 2022.

[37] D. E. Edmunds and H. Triebel. Sharp Sobolev embeddings and related Hardy inequalities: the critical case. *Math. Nachr.*, 207:79–92, 1999.

[38] G. Faber. Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt. *Sitzungsber Bayer. Akad. Wiss. München, Math.-Phys. Kl.*, pages 169–172, 1923.

[39] C. Farkas, S. Kajántó, and A. Kristály. Sharp spectral gap estimates for higher-order operators on Cartan-Hadamard manifolds. *Commun. Contemp. Math.*, 27(03):21, 2450013, 2024.

[40] C. Farkas, S. Kajántó, and C. Varga. Lower semicontinuity of Kirchhoff-type energy functionals and spectral gaps on (sub)Riemannian manifolds. *Topol. Methods Nonlinear Anal.*, 61(2):743–760, 2023.

[41] C. L. Fefferman. The uncertainty principle. *Bull. Amer. Math. Soc. (N.S.)*, 9(2):129–206, 1983.

[42] S. Filippas, V. Maz'ya, and A. Tertikas. On a question of Brezis and Marcus. *Calc. Var.*, 25(4):491–501, 2006.

[43] S. Filippas, V. Maz'ya, and A. Tertikas. Critical Hardy–Sobolev inequalities. *J. Math. Pures Appl.*, 87(1):37–56, 2007.

[44] S. Filippas and A. Tertikas. Optimizing improved Hardy inequalities. *J. Funct. Anal.*, 192(1):186–233, 2002.

[45] J. Flynn, N. Lam, and G. Lu. Hardy-Poincaré-Sobolev type inequalities on hyperbolic spaces and related Riemannian manifolds. *J. Funct. Anal.*, 283(12):37, Paper No. 109714, 2022.

[46] J. Flynn, N. Lam, G. Lu, and S. Mazumdar. Hardy's identities and inequalities on Cartan-Hadamard manifolds. *J. Geom. Anal.*, 33(1):34, Paper No. 27, 2023.

[47] R. Frank. *Sobolev inequalities and uncertainty principles in mathematical physics, Part 1, Lectur Notes.* LMU Munich, 2011.

[48] S. Gallot, D. Hulin, and J. Lafontaine. *Riemmanian Geometry.* Universitext Springer-Verlag, Berlin, 2004.

[49] N. Ghoussoub and A. Moradifam. On the best possible remaining term in the Hardy inequality. *Proc. Natl. Acad. Sci. USA*, 105(37):13746–13751, 2008.

[50] N. Ghoussoub and A. Moradifam. Bessel pairs and optimal Hardy and Hardy–Rellich inequalities. *Math. Ann.*, 349(1):1–57, 2011.

[51] N. Ghoussoub and A. Moradifam. *Functional Inequalities: New Perspectives and New Applications*, volume 187 of *Mathematical Surveys and Monographs.* AMS, 2013.

[52] R. E. Greene and H.-H. Wu. *Function theory on manifolds which possess a pole*, volume 699. Springer, 2006.

[53] G. H. Hardy. Note on a theorem of Hilbert. *Math. Z.*, 6(3-4):314–317, 1920.

[54] E. Hebey. *Nonlinear analysis on manifolds: Sobolev spaces and inequalities: Sobolev spaces and inequalities*, volume 5 of *Courant Lect. Notes Math.* AMS, 2000.

[55] J. Heinonen, T. Kilpeläinen, and Martio O. *Nonlinear Potential Theory of Degenerate Elliptic Equations.* Clarendon Press, Oxford, 1993.

[56] L. Huang, A. Kristály, and W. Zhao. Sharp uncertainty principles on general Finsler manifolds. *Trans. Amer. Math. Soc.*, 373(11):8127–8161, 2020.

[57] S. Kajántó. Rellich inequalities via Riccati pairs on model space forms. *J. Math. Anal. Appl.*, 531(2, Part 1):13, Paper No. 127870, 2024.

[58] S Kajántó. Sharp Hardy and spectral gap inequalities on special irreversible Finsler manifolds. *Proc. R. Soc. Edinb. Sect. A.*, 2025. accepted.

[59] S. Kajántó, A. Kristály, I. R. Peter, and W. Zhao. A generic functional inequality and Riccati pairs: an alternative approach to Hardy-type inequalities. *Math. Ann.*, 390(3):3621–3663, 2024.

[60] S. Kajántó and A. Kristály. Unexpected Behaviour of Flag and S-Curvatures on the Interpolated Poincaré Metric. *J. Geom. Anal.*, 31:10246–10262, 2021.

[61] S. Kajántó and A. Kristály. Saturation phenomena of a nonlocal eigenvalue problem: the Riemannian case. *Optimization*, 73(11):3299–3322, 2024.

[62] B. Kleiner. An isoperimetric comparison theorem. *Invent. Math.*, 108(1):37–47, 1992.

[63] B. R. Kloeckner and G. Kuperberg. The Cartan–Hadamard conjecture and the Little Prince. *Rev. Mat. Iberoam.*, 35(4):1195–1258, 2019.

[64] I. Kombe and M. Özaydin. Improved Hardy and Rellich inequalities on Riemannian manifolds. *Trans. Am. Math. Soc.*, 361(12):6191–6203, 2009.

[65] I. Kombe and M. Özaydin. Hardy-Poincaré, Rellich and uncertainty principle inequalities on Riemannian manifolds. *Trans. Am. Math. Soc.*, 365(10):5035–5050, 2013.

[66] E. Krahn. Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises. *Math. Ann.*, 94(1):97–100, 1925.

[67] A. Kristály. Sharp uncertainty principles on Riemannian manifolds: the influence of curvature. *J. Math. Pures Appl.*, 119(9):326–346, 2018.

[68] A. Kristály. Fundamental tones of clamped plates in nonpositively curved spaces. *Adv. Math.*, 367:39, Paper No. 107113, 2020.

[69] A. Kristály. Lord Rayleigh's conjecture for vibrating clamped plates in positively curved spaces. *Geom. Funct. Anal.*, 32(4):881–937, 2022.

[70] A. Kristály. New features of the first eigenvalue on negatively curved spaces. *Adv. Calc. Var.*, 15(3):475–495, 2022.

[71] A. Kristály, Á. Mester, and I. I. Mezei. Sharp Morrey–Sobolev inequalities and eigenvalue problems on Riemannian–Finsler manifolds with nonnegative Ricci curvature. *Commun. Contemp. Math.* Paper No. 2250063, 2022.

[72] A. Kristály and S. Ohta. Caffarelli–Kohn–Nirenberg inequality on metric measure spaces with applications. *Math. Ann.*, 357(2):711–726, 2013.

[73] A. Kristály and D. Repovš. Quantitative Rellich inequalities on Finsler–Hadamard manifolds. *Commun. Contemp. Math.*, 18(6):17, Paper No. 1650020, 2016.

[74] A. Kristály and A. Szakál. Interpolation between Brezis-Vázquez and Poincaré inequalities on nonnegatively curved spaces: sharpness and rigidities. *J. Differ. Equ.*, 266(10):6621–6646, 2019.

[75] A Kristály, B. Li, and W. Zhao. Failure of famous functional inequalities on Finsler manifolds: The influence of S-curvature. *arXiv preprint: 2409.05497*, 2024.

[76] R. Küstner. Mapping properties of hypergeometric functions and convolutions of starlike or convex functions of order α. *Comput. Methods Funct. Theory*, 2:597–610, 2002.

[77] R. T. Lewis, J. Li, and Y. Li. A geometric characterization of a sharp Hardy inequality. *J. Funct. Anal.*, 262(7):3159–3185, 2012.

[78] X. Li, J. Mao, and L. Zeng. Eigenvalues of the bi-drifting Laplacian on the complete noncompact Riemannian manifolds. *Z. Angew. Math. Phys.*, 73(6):19, Paper No. 240, 2022.

[79] H. McKean. An upper bound to the spectrum of Δ on a manifold of negative curvature. *J. Diff. Geom.*, 4(3):359–366, 1970.

[80] C. Meng, H. Wang, and W. Zhao. Hardy type inequalities on closed manifolds via Ricci curvature. *Proc. Roy. Soc. Edinburgh Sect. A*, 151(3):993–1020, 2021.

[81] Á. Mester, I. R. Peter, and C. Varga. Sufficient criteria for obtaining Hardy inequalities on Finsler manifolds. *Mediterr. J. Math.*, 18(2):1–22, Paper No. 76, 2021.

[82] E. Mitidieri. A Rellich type identity and applications: Identity and applications. *Commun. Partial. Differ. Equ.*, 18(1-2):125–151, 1993.

[83] E. Mitidieri. A simple approach to Hardy inequalities. *Mathematical notes*, 67:479–486, 2000.

[84] W. Moss and J. Piepenbrink. Positive solutions of elliptic equations. *Pac. J. Math.*, 75(1):219–226, 1978.

[85] B. Muckenhoupt. Hardy's inequality with weights. *Studia Math.*, 44:31–38, 1972.

[86] N. S. Nadirashvili. Rayleigh's conjecture on the principal frequency of the clamped plate. *Arch. Rational Mech. Anal.*, 129(1):1–10, 1995.

[87] Q. A. Ngô and V. H. Nguyen. Sharp constant for Poincaré-type inequalities in the hyperbolic space. *Acta Math. Vietnam.*, 44:781–795, 2019.

[88] V. H. Nguyen. New sharp Hardy and Rellich type inequalities on Cartan–Hadamard manifolds and their improvements. *Proc. Roy. Soc. Edinburgh Sect. A*, 150(6):2952–2981, 2020.

[89] V. H. Nguyen. Sharp Caffarelli–Kohn–Nirenberg inequalities on Riemannian manifolds: the influence of curvature. *Proc. Roy. Soc. Edinburgh Sect. A*, 152(1):102–127, 2022.

[90] S. Ohta and K.-T. Sturm. Heat flow on Finsler manifolds. *Commun. Pure Appl. Math.*, 62(10):1386–1433, 2009.

[91] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds). *NIST Handbook of Mathematical Functions*. Cambridge Univ. Press, Cambridge, 2010.

[92] Y. Pinchover. Criticality and ground states for second-order elliptic equations. *J. Differ. Equ.*, 80(2):237–250, 1989.

[93] Y. Pinchover. On criticality and ground states of second order elliptic equations, II. *J. Differ. Equ.*, 87(2):353–364, 1990.

[94] H. Rademacher. A sphere theorem for non-reversible Finsler metrics. *Math. Ann.*, 328(3):373–387, 2004.

[95] A Rapcsák. Uber die bahntreuen Abbildungen metrisher Räume. *Publ. Math. Debr.*, 8:285–290, 1961.

[96] J. W. S. Rayleigh. The Theory of Sound. 2nd edition, revised and enlarged (in two volumes), 1945.

[97] F. Rellich. Halbbeschränkte Differentialoperatoren höherer Ordnung. In *Proc. Int. Congr. Math.*, volume 3, pages 243–250, 1954.

[98] M. Ruzhansky and D. Suragan. Hardy and Rellich inequalities, identities, and sharp remainders on homogeneous groups. *Adv. Math.*, 317:799–822, 2017.

[99] M. Ruzhansky and D. Suragan. *Hardy inequalities on homogeneous groups: 100 years of Hardy inequalities*. Birkhäuser Basel, 2019.

[100] A. Savo. On the lowest eigenvalue of the Hodge Laplacian on compact, negatively curved domains. *Ann. Global Anal. Geom.*, 35(1):39–62, 2009.

[101] Z. Shen. *Lectures on Finsler geometry.* World Scientific, 2001.

[102] A. Tertikas and N. B. Zographopoulos. Best constants in the Hardy–Rellich inequalities and related improvements. *Adv. Math.*, 209(2):407–459, 2007.

[103] Z.-Q. Wang and M. Willem. Caffarelli–Kohn–Nirenberg inequalities with remainder terms. *J. Funct. Anal.*, 203(2):550–568, 2003.

[104] G. N. Watson. A Treatise on the Theory of Bessel Functions, reprint of the second (1944) edition. *Cambridge Univ. Press*, 1995.

[105] B. Y. Wu and Y. L. Xin. Comparison theorems in Finsler geometry and their applications. *Math. Ann.*, 337(1):177–196, 2007.

[106] C. Xia. The Caffarelli–Kohn–Nirenberg inequalities on complete manifolds. *Math. Res. Lett.*, 14(5):875–885, 2007.

[107] Q. Yang, D. Su, and Y. Kong. Hardy inequalities on Riemannian manifolds with negative curvature. *Commun. Contemp. Math.*, 16:24, 2014.

[108] S.-T. Yin and Q. He. The first eigenvalue of Finsler p-Laplacian. *Differ. Geom. Appl.*, 35:30–49, 2014.

[109] W. Zhao. Hardy inequalities with best constants on Finsler metric measure manifolds. *J. Geom. Anal.*, 31(2):1992–2032, 2021.

UNIVERSITAS CLAUDIOPOLITANA
ROMANIA
PRESA UNIVERSITARĂ

ISBN 978-606-37-2744-3