Ildikó Ilona Mezei

NONSMOOTH ELLIPTIC PROBLEMS ON EUCLIDEAN AND RIEMANNIAN STRUCTURES

Hemivariational Inequalities and Differential Inclusions

Presa Universitară Clujeană

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2025

Referenți științifici:

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ISBN 978-606-37-2784-9

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https://libraria.ubbcluj.ro/

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Introduction

Over the past few decades, Nonlinear Analysis has undergone remarkable development motivated mostly by the large number of problems from Physics, Engineering, Optimization or Economy, that can be effectively solved using nonlinear methods. Within this domain, hemivariational inequalities, differ-ential inclusions and elliptic-type partial differential equations stand out for their fundamental importance.

The present work is devoted to the study of these class of problems through advanced methods in Nonlinear Analysis. It builds on the author's PhD thesis [68] and includes further contributions (either as coauthor or sole author) during the last two decades.

The first chapter presents the fundamental mathematical results, definitions, properties, and theorems that are used throughout this work.

The Chapter 2 focuses on problems defined on *bounded domains*. We study different hemivariational inequalities and differential inclusions on bounded domains. More precisely, in Section 2.1 we establish two existence results for the solutions of variational–hemivariational inequalities, in both the single-variable and two-variable cases, on bounded domains. These inequalities extend the classical framework of variational inequalities, but are considerably more general, in the sense that they are not equivalent to minimization problems and instead give rise to substationary problems. The theory of hemivariational inequalities was introduced by Panagiotopoulos in the 1980s ([80]) as a natural tool for modeling and solving a wide range of problems arising in mechanics, engineering, and economics. The analysis of hemivariational inequalities differs fundamentally from that of variational inequalities, due to the nonconvexity of the associated energy functional. In most cases these nonconvex energy functions are nonsmooth and therefore the methods of Nonsmooth Analysis are employed for the mathematical study and numerical treatment of them. Hemivariational inequalities use the notion of Clarke subdifferential of locally Lipschitz functions (see Clarke [28]), which is a generalization of classical derivative.

In the second section of Chapter 2 we prove a multiplicity result for a differential inclusion problem defined on the unit ball. By variational methods, we demonstrate that the solutions of this problem are invariant by spherical cap symmetrization. The main used tools are the symmetric version of Ekeland's variational principle proved by Squassina [92], the Mountain Pass Theorem and the nonsmooth version of the symmetric minimax principle due to Van Schaftingen [91].

The last section of Chapter 2 examines an elliptic differential inclusion problem on a bounded open domain with two nonsmooth nonlinear terms, considered in two different settings. We obtain infinitely many number of distinct solutions when the generalized gradient of one term oscillates near the origin or infinity, and the generalized gradient of the other term has order $p > 0$ near the origin or infinity.

The Chapter 3 treats hemivariational inequalities and elliptic problems on *unbounded domains*. The main problem arising here is the lack of compact embeddings of Sobolev spaces. In general, if Ω is unbounded, the Sobolev space $W^{1,p}(\Omega)$ (the space of all functions from $L^p(\Omega)$, for which $|\nabla u| \in L^p(\Omega)$) is not compactly embedded in any Lebesgue space $L^r(\Omega)$. There are two main approaches to overcome this difficulty. The first is to search for solutions in suitable subspaces of $W^{1,p}(\Omega)$, such as the weighted or double weighted Sobolev spaces. The second is to work within subspaces of $W^{1,p}(\Omega)$ whose elements possess specific symmetries. In both cases, there are results ensuring that these subspaces embed compactly into Lebesgue spaces. The compact embeddings are needed in order to obtain critical points for the energy functional attached to the initial problem. These critical points are the (weak) solutions of the problem.

In Section 3.1, multiplicity results are first proved for a double-eigenvalue quasilinear problem on an unbounded domain with nonlinear boundary conditions. The solutions are obtained in a *weighted* Sobolev space. We then consider semilinear homogeneous and inhomogeneous elliptic problems involving both superlinear and sublinear nonlinearities. In both cases, we obtain multiplicity results in *double weighted* Sobolev spaces. We use variational methods and some recent results from the theory of best approximation established by Ricceri in [88] and Tsar'kov in [96].

The Section 3.2 is dedicated to *symmetric solutions on striplike domains*. In the first subsection we guarantee two or three non-zero weak solutions to a quasilinear elliptic problem, which are invariant with respect to certain groups, whenever the nonlinear term verifies unusual growth assumptions. In the second subsection a perturbed gradient-type system is considered, having three distinct, axially symmetric weak solutions.

In the last section of this chapter we prove some multiplicity results for hemivariational inequalities defined on the whole space. By variational methods, we demonstrate that the solutions of these inequalities are invariant by spherical cap symmetrization.

Chapter 4 is devoted to a broad class of elliptic differential inclusions involving the Laplace–Beltrami operator together with a Hardy-type singular term. The problem is formulated in a non-Euclidean setting, more precisely, on *non-compact complete Riemannian manifolds*. Depending on the behavior of the nonlinear term and on the curvature of the underlying manifold, we establish both nonexistence results and existence/multiplicity results for the studied differential inclusion. The main analytical tools employed are drawn from nonsmooth variational analysis, complemented by techniques involving isometric group actions and eigenvalue properties on Riemannian manifolds.

Acknowledgements

Having outlined the content of this monograph, I wish to conclude this introduction by acknowledging the generous help and encouragement from my coauthors, colleagues, and those who have supported my teaching and research leading to this book.

I would like to express my gratitude to the late Professor Csaba Varga, my former PhD advisor and colleague for his confidence in me and for the invaluable guidance he constantly offered.

I am also grateful to Professor Alexandru Kristály for his careful reading of the manuscript and for the insightful suggestions he made. I greatly appreciate his guidance, support and encouragement throughout the years. I thank Professor Csaba Farkas for reading the manuscript, for his valuable suggestions and and advice, and for all the stimulating discussions we have had.

Cluj-Napoca
September, 2025

Ildikó Ilona Mezei

Chapter 1

Preliminaries

1.1 Locally Lipschitz functionals

Since the classical variational framework is based on smooth functionals and their Fréchet or Gateaux derivatives, difficulties arise when dealing with problems where the underlying energy functionals are only locally Lipschitz but not differentiable in the usual sense. To overcome this obstacle, the theory of hemivariational inequalities relies on generalized differential tools that extend the concept of the gradient to nonsmooth settings.

In particular, a central role is played by the Clarke subdifferential, which provides a natural substitute for the classical derivative when handling locally Lipschitz functions. This notion captures the generalized slope of a nonsmooth function and allows one to formulate variational principles even in the absence of differentiability.

Therefore, before turning to the analysis of hemivariational inequalities, we begin by recalling the basic notions of locally Lipschitz functions, Clarke's generalized directional derivative, and the Clarke subdifferential. These concepts constitute the foundation of nonsmooth variational analysis and will be systematically employed throughout the forthcoming sections.

In developing these notions, we will primarily rely on the monographs of Chang [26], Clarke [28] and Kristály and Varga [59].

Let X be a Banach space, X^* its dual. The euclidean norm in $\mathbb{R}^k$ and the duality pairing between the Banach space and its dual will be denoted by $|\cdot|$, resp. $\langle\cdot,\cdot\rangle$.

Let U be an open subset of X.

Definition 1.1.1. *A function $f:U\to\mathbb{R}$ is called* locally Lipschitz *if every point $u\in U$ possesses a neighborhood $N_u\subset U$ such that*

$$|f(u_1)-f(u_2)|\leq K||u_1-u_2||,\quad \forall u_1,u_2\in N_u,$$

for a constant $K>0$ depending on N_u.

Definition 1.1.2. *The* generalized directional derivative *of the locally Lipschitz function $f:U\to\mathbb{R}$ at the point $u\in U$ in the direction $v\in X$ is defined by*

$$f^0(u;v)=\limsup_{\substack{w\to u\\ t\searrow 0}}\frac{f(w+tv)-f(w)}{t}.$$

If $f:U\to\mathbb{R}$ is a function of class C^1 on U, then $h^0(u;v)=\langle h'(u),v\rangle$ for all $u,v\in X$.

Hereafter, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_*$ stand for the duality mapping on (X^*,X) and the norm on X^*, respectively.

Definition 1.1.3. *Let $f:U\to\mathbb{R}$ be a locally Lipschitz function. The* generalized gradient *or* Clarke subdifferential $\partial f(u)$ *of f at a point $u\in U$ is the subset of the dual space X^* defined as follows*

$$\partial f(u)=\{z\in X^*:\langle z,v\rangle\leq f^0(u;v),\forall v\in X\}.$$

For all $x\in X$, the functional $f^\circ(x,\cdot)$ is finite and positively homogeneous. Moreover, we have the following properties.

Proposition 1.1.1. ([28]) *Let X be a real Banach space, $U\subset X$ an open subset and $f,g:U\to\mathbb{R}$ be locally Lipschitz functions. The following properties hold:*

(a) *For every $x\in U$, $\partial f(x)$ is a nonempty, convex and weakly*-compact subset of $X^\star$ which is bounded by the Lipschitz constant $L>0$ of f near x;*

(b) $f^\circ(x;v) = \max\{\langle \xi, v\rangle : \ \xi \in \partial f(x)\}$ *for all* $v \in X$*;*

(c) $(f+g)^\circ(x;v) \leq f^\circ(x;v) + g^\circ(x;v)$ *for all* $x \in U$, $v \in X$;

(d) $\partial(f+g)(u) \subset \partial f(u) + \partial g(u)$ *for all* $u \in U$*;*

(e) $(-f)^\circ(x;v) = f^\circ(x;-v)$ *for all* $x \in U$*;*

(f) *The function* $(x,v) \mapsto f^\circ(x;v)$ *is upper semicontinuous;*

(g) *If* $j : X \to \mathbb{R}$ *is of class* C^1 *on* X*, then* $\partial(j+h)(u) = j'(u) + \partial h(u)$ *and* $(j+h)^0(u;v) = \langle j'(u), v\rangle + h^0(u;v)$ *for every* $u, v \in X$*.*

(h) *The set-valued map* $\partial f : U \to 2^{X^\star}$ *is weakly***-closed, that is, if* $\{x_i\} \subset U$ *and* $\{w_i\} \subset X^*$ *are sequences such that* $x_i \to x$ *strongly in* X *and* $w_i \in \partial f(x_i)$ *with* $w_i \rightharpoonup z$ *weakly** *in* X^**, then* $z \in \partial f(x)$*. In particular, if* X *is finite dimensional, then* ∂f *is upper semicontinuous, i.e., for every* $\epsilon > 0$ *there exists* $\gamma > 0$ *such that* $\partial f(x') \subseteq \partial f(x) + B_{X^*}(0,\epsilon)$*,* $\forall x' \in B_X(x,\gamma)$*;*

(i) $\partial(sf)(u) = s\partial f(u)$ *for every* $s \in \mathbb{R}$ *and* $u \in X$*.*

Proposition 1.1.2. ([26]) *The number* $\lambda_f(u) = \inf\limits_{w \in \partial f(u)} ||w||_{X^\star}$ *is well defined and*

$$\liminf_{u \to u_0} \lambda_f(u) \geq \lambda_f(u_0).$$

Definition 1.1.4. ([26]) *Let* $f : X \to \mathbb{R}$ *be a locally Lipschitz function. We say that* $u \in X$ *is a* critical point *(in the sense of Chang) of* f*, if* $\lambda_f(u) = 0$*, i.e.,* $0 \in \partial f(u)$*.*

Remark 1.1.1. ([28]) (a) $u \in X$ is a critical point of f if $f^\circ(u;v) \geq 0$ for all $v \in X$.

(b) If $x \in U$ is a local minimum or maximum of the locally Lipschitz function $f : X \to \mathbb{R}$ on an open set of a Banach space, then x is a critical point of f.

Proposition 1.1.3. (**Lebourg's mean value theorem** [28]) *Let X be a Banach space, $x, y \in X$ and $f : X \to \mathbb{R}$ be Lipschitz on an open set containing the line segment $[x, y]$. Then there is a point $a \in (x, y)$ such that*

$$f(y) - f(x) \in \langle \partial f(a), y - x \rangle.$$

Proposition 1.1.4. (**Chain Rule** [28]) *Let X be Banach space, let us consider the composite function $f = g \circ h$ where $h : X \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ are given functions. Let denote h_i, $i \in \{1, ..., n\}$ be the component functions of h. We assume h_i is locally Lipschitz near x and g is too near $h(x)$. Then f is locally Lipschitz near x as well. Let us denote by α_i the elements of ∂g, and let $\alpha = (\alpha_1, ..., \alpha_n)$; then*

$$\partial f(x) \subset \overline{\text{co}}\{\sum \alpha_i \xi_i : \xi_i \in \partial h_i(x), \alpha \in \partial g(h(x))\},$$

where $\overline{\text{co}}$ denotes the weak-closed convex hull.

1.2 Different continuities of real-valued and set-valued functions

In this section we are using mainly the monographs of Aubin and Frankowska [6] and of Kristály and Varga [59].

Definition 1.2.1. *We say that $f : U \to \mathbb{R}$ is* sequentially weakly continuous, *if for every sequence $\{u_n\} \subset U$ converging weakly to a $u_0 \in U$, we have that $f(u_n)$ converges to $f(u_0)$, as $n \to \infty$.*

Definition 1.2.2. *We say that $f : U \to \mathbb{R}$ is* upper semicontinuous at $u_0 \in U$, *if for every $\varepsilon > 0$ there exists a neighborhood N_u of u_0 such that $f(u) < f(u_0) + \varepsilon$ for all u in U. Equivalently, this can be expressed as*

$$\limsup_{u \to u_0} f(u) \le f(u_0).$$

Definition 1.2.3. *The set-valued function $F : X \to \mathbb{R}$ is* upper semicontinuous *in x if and only if for any neighborhood U of $F(x)$, there exists $\eta > 0$ such that for every $x' \in B_X(x, \eta)$, $F(x') \subset U$.*

Definition 1.2.4. *Let K be a convex subset of X.*

(i) *A set-valued mapping $\mathcal{F} : K \rightsquigarrow X^*$ is said to be* upper demicontinuous at $x_0 \in K$ *if for any $h \in X$, the real-valued function $x \mapsto \sigma(\mathcal{F}(x), h) = \sup\limits_{x^* \in \mathcal{F}(x)} \langle x^*, h\rangle$ is upper semicontinuous at x_0. $\mathcal{F}$ is upper demicontinuous on K if it is upper demicontinuous in every $x \in K$.*

(ii) *$\mathcal{F} : K \rightsquigarrow X^*$ is said to be* upper demicontinuous from the line segments in K *if the application $t \mapsto \sigma(\mathcal{F}(tx + (1-t)y), h)$ is upper semicontinuous on the interval $[0, 1]$, $\forall x, y \in K, h \in X$.*

(iii) *$F : K \to X^*$ is said to be* w^*-demicontinuous in u_0 *if for any sequence $\{u_n\} \subset K$ converging to u_0 (in the strong topology), the image sequence $\{F(u_n)\}$ converges to $F(u_0)$ in the weak*-topology in X^*.*

Remark 1.2.1. If $\mathcal{F}(x) = \{F(x)\}$, that is, if $\mathcal{F}$ is a single valued map, then $\mathcal{F}$ is upper demicontinuous at $u_0 \in K$ if and only if the operator $F : K \to X^*$ is w^*-demicontinuous at $u_0 \in K$.

Definition 1.2.5. *We say that $f : U \to \mathbb{R}$ is* lower semicontinuous at $u_0 \in U$, *if for every $\varepsilon > 0$ there exists a neighborhood N_u of u_0 such that $f(u) > f(u_0) + \varepsilon$ for all u in U. Equivalently, this can be expressed as*

$$\liminf_{u \to u_0} f(u) \geq f(u_0).$$

Remark 1.2.2. The $h \mapsto \sigma(\mathcal{F}(x), h)$ is a lower semicontinuous sublinear function.

Let Y, Z be two metric spaces and a set-valued map (with nonempty values) $F : Y \rightsquigarrow Z$.

Definition 1.2.6. *F is* lower semicontinuous at $y \in Y$ *if and only if for any $z \in F(y)$ and for any sequence $\{y_n\}$, converging to y, there exists a sequence $\{z_n\}$, $z_n \in F(y_n)$ converging to z. F is said to be lower semicontinuous if it is lower semicontinuous at every point $y \in Y$.*

Let us consider a function $f : Graph(F) \to \mathbb{R}$. We define the *marginal function* $g : Y \to \mathbf{R} \cup \{+\infty\}$ by $g(y) = \sup_{z \in F(y)} f(y, z)$. We have the foolowig result.

Lemma 1.2.1. (**Maximum Theorem** [6, Theorem 1.4.16]) *If f and F are lower semicontinuous on Y, then the marginal function is also lower semicontinuous.*

Definition 1.2.7. *We say that for a mapping $G : X \to 2^Y$ with nonempty values is* upper hemicontinuous *if for any $p \in Y^*$, the functional*

$$x \mapsto \sup_{y \in G(x)} \langle p, y \rangle$$

is upper semicontinuous.

Remark 1.2.3. If G is upper semicontinuous, then it is upper hemicontinuous.

In the sequel, we present two auxiliary results concerning the upper semicontinuity of certain mappings. These results are used in the proofs of our main results in Section 2.1. The first was proved by Buzogány, Mezei, and Cs. Varga in [16], and the second by Buzogány, Mezei, and V. Varga in [17]. For completeness, we include both results here along with their proofs.

Lemma 1.2.2. ([16, Lemma 1.]) *Let K be a subset of the Banach space X and $\mathcal{A} : K \rightsquigarrow X^*$ be an upper demicontinuous set-valued map with bounded values. Then the function $K \times K \ni (x, y) \mapsto \sigma(\mathcal{A}(x), y - x)$ is upper semicontinuous.*

Proof. First, we prove that the function $x \mapsto \|\mathcal{A}(x)\| = \sup_{x^* \in \mathcal{F}(x)} \|x^*\|$ is locally bounded on K. Indeed, let $x_0 \in K$ be an arbitrary fixed element and assume that the above function is not locally bounded in x_0. Then there exist sequences $\{x_n\} \subset K$ respectively $x_n^* \in \mathcal{A}(x_n)$ such that $x_n \to x_0$ and $\|x_n^*\| \to \infty$. By Banach-Steinhaus theorem, there exists $v \in X$ such that $\langle x_n^*, v \rangle$ is not bounded. Therefore, taking a subsequence $\{x_m^*\}$ of $\{x_n^*\}$, we may suppose that $\langle x_m^*, v\rangle \to \infty$. Then $\sigma(\mathcal{A}(x_m), v) \to \infty$. By the upper demicontinuity of $\mathcal{A}$ and the boundedness of $\mathcal{A}(x_0)$, we get

$$\limsup_{m\to\infty} \sigma(\mathcal{A}(x_m), v) \leq \sigma(\mathcal{A}(x_0), v) = \sup\{\langle x^*, v\rangle | x^* \in \mathcal{A}(x_0)\} \leq$$

$$\leq \sup\{||x^*|| \cdot ||v|| \; |x^* \in \mathcal{A}(x_0)\} =$$
$$= ||v|| \sup\{||x^*|| \; |x^* \in \mathcal{A}(x_0)\} = \|\mathcal{A}(x_0)\| \cdot \|v\| < \infty,$$

contradiction. Therefore, $x \mapsto \|\mathcal{A}(x)\|$ is locally bounded on K.

Now, let $u, v \in K$ be fixed, and $\{u_n, v_n\}$ a sequence in $K \times K$, converging to (u, v). Let $L_u > 0$ be the local bound for $\mathcal{A}$ in a neighborhood N_u of u. Since $u_n \to u$, there exists a number $n_u \in \mathbb{N}$ such that for $n \geq n_u$, we have $u_n \in N_u$. By the subadditivity we have

$$\sigma(\mathcal{A}(u_n), v_n - u_n) \leq$$
$$\leq \sigma(\mathcal{A}(u_n), v_n - v) + \sigma(\mathcal{A}(u_n), v - u) + \sigma(\mathcal{A}(u_n), u - u_n) \leq$$
$$\leq \sigma(\mathcal{A}(u_n), v - u) + \|\mathcal{A}(u_n)\| \cdot [\|u - u_n\| + \|v - v_n\|] \leq$$
$$\leq \sigma(\mathcal{A}(u_n), v - u) + L_u[\|u - u_n\| + \|v - v_n\|], \; \forall\, n \geq n_u.$$

Using again the upper demicontinuity of $\mathcal{A}$, we obtain

$$\limsup_{n\to\infty} \sigma(\mathcal{A}(u_n), v_n - u_n) \leq$$
$$\leq \limsup_{n\to\infty} \sigma(\mathcal{A}(u_n), v - u) + L_u \limsup_{n\to\infty} [\|u - u_n\| + \|v - v_n\|] \leq$$
$$\leq \sigma(\mathcal{A}(u), v - u),$$

i.e. the function $(x, y) \mapsto \sigma(\mathcal{A}(x), y - x)$ is upper semicontinuous in (u, v). Since u and v were arbitrary fixed, the proof is complete. $\square$

Lemma 1.2.3. *Let $\mathcal{F} : K \rightsquigarrow X^*$ be an upper demicontinuous set-valued map with bounded values, i.e.* $\sup\limits_{x^*\in\mathcal{F}(x)} \|x^*\| < \infty$, $\forall x \in K$. *Then the function* $u \mapsto \sigma(\mathcal{F}(u), v - u)$ *is upper semicontinuous,* $\forall\, v \in K$.

Proof. In Lemma 1.2.2 we proved that the function $x \mapsto \|\mathcal{F}(x)\| = \sup\limits_{x^*\in\mathcal{F}(x)} \|x^*\|$ is locally bounded on K.

Now, let $u \in K$ be fixed, and $\{u_n\}$ be a sequence in K converging to $u \in K$. Let $L_u > 0$ be the local bound for $\mathcal{F}$ in a neighborhood N_u of u. Since $u_n \to u$, there exists a number $n_u \in \mathbb{N}$ such that for $n \geq n_u$, we have $u_n \in N_u$. By the subadditivity (remark 1.2.2), we have that

$$\begin{aligned}\sigma(\mathcal{F}(u_n), v - u_n) &\leq \sigma(\mathcal{F}(u_n), v - u) + \sigma(\mathcal{F}(u_n), u - u_n) \leq \\ &\leq \sigma(\mathcal{F}(u_n), v - u) + \|\mathcal{F}(u_n)\| \cdot \|u - u_n\| \leq \\ &\leq \sigma(\mathcal{F}(u_n), v - u) + L_u\|u - u_n\|,\end{aligned}$$

for all $n \geq n_u$.

Using the upper demicontinuity of $\mathcal{F}$, we obtain

$$\begin{aligned}\limsup_{n\to\infty} \sigma(\mathcal{F}(u_n), v - u_n) &\leq \limsup_{n\to\infty} \sigma(\mathcal{F}(u_n), v - u) + L_u \limsup_{n\to\infty} \|u - u_n\| \\ &\leq \sigma(\mathcal{F}(u), v - u).\end{aligned}$$

Therefore the function $x \mapsto \sigma(\mathcal{F}(x), y-x)$ is upper semicontinuous in $u \in K$, for every $y \in K$. Since u was arbitrarily fixed, the proof is complete. $\square$

Definition 1.2.8. *Let K be a convex subset of X and let Z be a topological vector space. The set-valued map $F : K \rightsquigarrow Z$ (with nonempty values) is* convex *if and only if we have*

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2), \forall\ x_1, x_2 \in K,\ \ \forall\ \lambda \in [0, 1].$$

Remark 1.2.4. $F : K \rightsquigarrow Z$ is convex if and only if for every $x_1, \dots, x_n \in K$, for all $\lambda_1, \dots, \lambda_n \geq 0$, $n \in \mathbb{N}^*$, such that $\sum_{i=1}^{n} \lambda_i = 1$, we have

$$\sum_{i=1}^{n} \lambda_i F(x_i) \subseteq F\left(\sum_{i=1}^{n} \lambda_i x_i\right).$$

Definition 1.2.9. *Let X be a nonempty convex subset of E. A functional $\phi : X \times X \to \overline{\mathbb{R}}$ is said to be* 0-diagonally concave in the second argument *(in short 0-DCV), if for any finite subset $\{y_1, y_2, \dots, y_m\} \subset X$ and any $x = \sum_{i=1}^{m} \lambda_i y_i$ ($\lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1$), we have $\sum_{i=1}^{m} \lambda_i \Phi(x, y_i) \leq 0$.*

Remark 1.2.5. If Φ is concave in the second argument and $\Phi(x, y) \leq 0$ for each $x \in X$ then Φ is 0-DCV.

Definition 1.2.10. *The mapping $F : K \subseteq X \rightsquigarrow X^*$ is* monotone *if*

$$\langle f_1 - f_2, u - v\rangle \geq 0,\ \forall u, v \in K,\ \forall f_1 \in F(u), \forall f_2 \in F(v).$$

1.3 Palais-Smale condition, Mountain Pass Theorem

The Palais-Smale condition (henceforth denoted by (PS)) is a fundamental compactness criterion in the calculus of variations and critical point theory, particularly for studying the existence of critical points of functionals defined on unbounded domains. It ensures that certain sequences, called Palais-Smale sequences, along which the functional's value is bounded and its derivative tends to zero, possess convergent subsequences. This compactness property is essential because, in unbounded settings, bounded sequences do not automatically have convergent subsequences.

The Palais-Smale condition plays a crucial role in minimax theorems, which are powerful tools for locating critical points of functionals. Minimax methods work by examining the extremal behavior of the functional, such as taking the maximum of a set of minima or the minimum of a set of maxima over certain subsets of the domain. The condition ensures that the sequences constructed via these variational procedures converge to actual critical points, rather than "escaping" to infinity or failing to converge. Consequently, the Palais-Smale condition provides the necessary compactness to rigorously justify the application of variational methods in proving the existence of solutions to nonlinear problems.

Definition 1.3.1. *Suppose E is a Banach space. We denote by $C^1(E, \mathbb{R})$ the set of functionals that are Fréchet differentiable and whose Fréchet derivatives are continuous on E. For $I \in C^1(E, \mathbb{R})$, we say that I satisfies the* **Palais-Smale condition** *if any sequence $u_m \subset E$ for which $I(u_m)$ is bounded and $I'(u_m) \to 0$ as $m \to \infty$ possesses a convergent subsequence.*

As we mentioned, the (PS) condition is a convenient way to build some "compactness" into the functional I. Indeed observe that (PS) implies that $K_c \equiv \{u \in E : I(u) = c \text{ and } I'(u) = 0\}$, i.e. the set of critical points having critical value c, is compact for any $c \in \mathbb{R}$ (see also [4, pp. 59-103], [25]).

The Mountain Pass Theorem, which is due to Ambrosetti and Rabinowitz [3] is one of the most powerful tools in Nonlinear Analysis for proving the existence of critical points of energy functionals. A version of the Mountain Pass Theorem (see [85]) asserts that if a continuously differentiable function has two local minima, then (under some natural assumptions) such a function has a third critical point.

Theorem 1.3.1. (**Mountain Pass Theorem** [86]) *Let E be a Banach space and $I \in C^1(E, \mathbb{R})$ a functional, satisfying the Palais-Smale condition. Suppose $I(0) = 0$ and*

(a) there are constants $\alpha > 0$ and $\rho > 0$ such that $I(u) \geq \alpha$, for every $||u|| = \rho$;

(b) there is an $e \in E$ with $||e|| > \rho$ and $I(e) \leq 0$.

Then the number

$$c = \inf_{g \in \Gamma} \max_{v \in g([0,1])} I(v),$$

where

$$\Gamma = \{g \in C([0,1], E) : g(0) = 0, g(1) = e\},$$

is a critical value of I, with $c \geq \alpha$.

1.4 Principle of Symmetric Criticality

Many functionals in calculus of variations and mathematical physics are symmetric, meaning they remain unchanged under certain transformations (ex. rotations, reflections). The Principle of Symmetric Criticality (PSC) allows us to search for critical points within the smaller space of symmetric functions, knowing that any critical point found there is also a critical point in the full space.

PSC is widely used in nonlinear PDEs and variational problems, where symmetry reduces the complexity of the problem.

Definition 1.4.1. *Let G be a group with identity element e, and X a topological space. A group action α of G on X is a function $\alpha : G \times X \to X$ that satisfies the following two axioms:*

1) Identity: $\alpha(e, u) = u$;

2) Compatibility: $\alpha(g, \alpha(h, u)) = \alpha(gh, u)$

for all $g, h \in G$ and all $u \in X$.

Remark 1.4.1. For simplicity we shall often write gu instead of $\alpha(g, u)$.

Definition 1.4.2. *A function $h : X \to \mathbb{R}$ is called G-invariant if $h(gu) = h(u)$ for every $u \in X$ and $g \in G$.*

Definition 1.4.3. *A subset M of X is called G-invariant if*

$$gM = \{gu : u \in M\} \subseteq M, \forall g \in G.$$

Theorem 1.4.1. (**Principle of symmetric criticality** [79]) *Assume that the action of the topological group G on the reflexive and strictly convex Banach space X is isometric. If $\varphi \in C^1(X, \mathbb{R})$ is invariant and u is a critical point of φ restricted to the space of invariant points, then u is critical point of φ.*

This smooth version of the principle of symmetric criticality provided by Palais [79] was later extended to various nonsmooth settings.

Let G be a compact Lie group acting *linear isometrically* on the real Banach space $(X, \|\cdot\|)$, i.e., the action $G \times X \to X$, $(\sigma, u) \mapsto \sigma u$ is continuous and for every $\sigma \in G$ the map $u \mapsto \sigma u$ is linear such that $\|\sigma u\| = \|u\|$ for every $u \in X$. Let

$$\mathsf{Fix}_X(G) = \{u \in X : \sigma u = u, \ \forall \sigma \in G\} \tag{1.1}$$

be the *set of fixed points* of G over X.

According to Krawcewicz and Marzantowicz [56] (see also Costea, Kristály and Varga [29, Section 3.4]), the principle of symmetric criticality for locally Lipschitz functions can be stated as follows.

Theorem 1.4.2. **(Principle of symmetric criticality for locally Lipschitz functionals [56])** *Let G be a compact Lie group acting linear isometrically on the real Banach space $(X, \|\cdot\|)$ and $h : X \to R$ be a G-invariant, locally Lipschitz functional. If $h|_G$ denotes the restriction of h to $\mathsf{Fix}_X(G)$ and $u \in \mathsf{Fix}_X(G)$ is a critical point of $h|_G$ then u is also a critical point of h.*

1.5 Abstract framework of symmetrization

In this section we present the definitions of different symmetrizations. More properties and examples can be found in Van Shaftingen [91] and Squassina [92].

Definition 1.5.1. *For $f : A \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and $c \in \overline{\mathbb{R}}$ let*

$$\{f > c\} = \{x \in A | f(x) > c\}.$$

Definition 1.5.2. *Let $P \in \partial B(0,1) \cap \mathbb{R}^N$. The* spherical cap symmetrization of the set *A with respect to P is the unique set A^* such that $A^* \cap \{0\} = A \cap \{0\}$ and for any $r \geq 0$,*

$$A^* \cap \partial B(0,r) = B_g(rP, \rho) \cap \partial B(0,r) \text{ for some } \rho \geq 0,$$

$$\mathcal{H}^{N-1}(A^* \cap \partial B(0,r)) = \mathcal{H}^{N-1}(A \cap \partial B(0,r)),$$

where $\mathcal{H}^{N-1}$ is the outer Hausdorff $(N-1)$-dimensional measure and by $B_g(rP, \rho)$ we denote the geodesic ball on the sphere $\partial B(0,r)$ of center rP and radius ρ. By definition $B_g(rP, 0) = \emptyset$.

Definition 1.5.3. *The* spherical cap symmetrization of a function *$u : \Omega \to \overline{\mathbb{R}}$ is the unique function $u^* : \Omega^* \to \overline{\mathbb{R}}$ such that, for all $c \in \mathbb{R}$,*

$$\{u^* > c\} = \{u > c\}^*.$$

Definition 1.5.4. *A subset H of $\mathbb{R}^N$ is called a* polarizer *if it is a closed affine half-space of $\mathbb{R}^N$, namely the set of points x which satisfy $\alpha \cdot x \leq \beta$ for some $\alpha \in \mathbb{R}^N$ and $\beta \in \mathbb{R}$ with $|\alpha| = 1$. Given x in $\mathbb{R}^N$ and a polarizer*

H the reflection of x with respect to the boundary of H is denoted by x_H. The polarization of a function $u : \mathbb{R}^N \to \mathbb{R}^+$ *by a polarizer H is the function $u^H : \mathbb{R}^N \to \mathbb{R}^+$ defined by*

$$u^H(x) = \begin{cases} \max\{u(x), u(x_H)\}, & \text{if } x \in H \\ \min\{u(x), u(x_H)\}, & \text{if } x \in \mathbb{R}^N \setminus H. \end{cases} \tag{1.2}$$

The polarization $C^H \subset \mathbb{R}^N$ of a set $C \subset \mathbb{R}^N$ is defined as the unique set which satisfies $\chi_{C^H} = (\chi_C)^H$, where χ denotes the characteristic function. The polarization u^H of a positive function u defined on $C \subset \mathbb{R}^N$ is the restriction to C^H of the polarization of the extension $\tilde{u} : \mathbb{R}^N \to \mathbb{R}^+$ of u by zero outside C. The polarization of a function which may change sign is defined by $u^H := |u|^H$, for any given polarizer H.

Following Van Schaftingen [91], consider the abstract framework below:

Let X, V and W be three real Banach spaces, with $X \subset V \subset W$ and let $S \subset X$. For the clarity, we present some crucial abstract symmetrization and polarization results of this work and of Squassina [92]. Let us first introduce the following main assumption.

Definition 1.5.5. *Let $\mathcal{H}_\star$ be a pathconnected topological space and denote by $h : S \times \mathcal{H}_\star \to S$, $(u, H) \mapsto u^H$, the polarization map. Let $\star : S \to V, u \mapsto u^\star$, be any symmetrization map. Assume that the following properties hold.*

1) The embeddings $X \hookrightarrow V$ and $V \hookrightarrow W$ are continuous;

2) h is continuous;

3) $(u^\star)^H = (u^H)^\star = u^\star$ and $(u^H)^H = u^H$ for all $u \in S$ and $H \in \mathcal{H}_\star$;

4) for all $u \in S$ there exists a sequence $(H)_m \subset \mathcal{H}_\star$ such that $u^{H_1 \dots H_m} \to u^\star$ in V;

5) $\|u^H - v^H\|_V \le \|u - v\|_V$ for all u, $v \in S$ and $H \in \mathcal{H}_\star$.

Since there exists a map $\Theta : (X, \|\cdot\|_V) \to (S, \|\cdot\|_V)$ which is Lipschitz continuous, with Lipschitz constant $C_\Theta > 0$, and such that $\Theta|_S = Id|_S$, both maps $h : S \times \mathcal{H}_\star \to S$ and $\star : S \to V$ can be extended to $h : X \times \mathcal{H}_\star \to S$ and $\star : X \to V$ by setting $u = (\Theta(u))^H$ and $u^\star = (\Theta(u))^\star$ for every $u \in X$ and $H \in \mathcal{H}_\star$.

The previous properties, in particular 4) and 5), and the definition of Θ easily yield that

$$\|u^H - v^H\|_V \leq C_\Theta \|u - v\|_V, \qquad \|u^\star - v^\star\|_V \leq C_\Theta \|u - v\|_V \tag{1.3}$$

for all u, $v \in X$ and for all $H \in \mathcal{H}_\star$.

Some known examples of spherical cap symmetrization with Dirichlet boundary and of Schwarz symmetrization are given by J. Van Schaftingen in [91].

We recall three results which will play an essential role in the Section 2.

Proposition 1.5.1. ([37, Proposition 3.3]) *Let $G : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such as $G(x, u) = G(y, u)$ for a.e. x, $y \in \mathbb{R}^N$, with $|x| = |y|$, and all $u \in \mathbb{R}$. Then, for all $H \in \mathcal{H}_\star$*

$$\int_{\mathbb{R}^N} G(x, u(x))dx = \int_{\mathbb{R}^N} G(x, u^H(x))dx$$

along any $u : \mathbb{R}^N \to \mathbb{R}_0^+$, with $G(\cdot, u(\cdot)) \in L^1(\mathbb{R}^N)$.

Remark 1.5.1. The statement of the above proposition remains valid if we choose $\Omega = \Omega^H \subset \mathbb{R}$ instead of the whole space $\mathbb{R}^N$ (Van Schaftingen [91, Proposition 2.19]).

In the paper of Cs. Varga and V. Varga [98] a quantitative deformation lemma is proved for locally Lipschitz functions. Van Schaftingen in [91] Theorem 3.5 , proves a symmetric version of this variational principle for C^1 functionals. Using the mentioned results with slight modifications, one can prove the following symmetric variational principle for locally Lipschitz functionals.

Theorem 1.5.1. (**Symmetric variational principle**)

Let $(X, V, \star, \mathcal{H}_\star, S)$ satisfy the assumptions of Definition 1.5.5. Denote by $\kappa > 0$ any constant with the property $\|u\|_V \leq \kappa \|u\|_X$ for all $u \in X$. Let $e \in X \setminus \{0\}$ be fixed and

$$\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e\}.$$

Consider also the locally Lipschitz functional $\Phi : X \to \mathbb{R}$*, which satisfies:*

1) $\infty > c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \Phi(\gamma(t)) > a := \max\{\Phi(0), \Phi(e)\}$,

2) $\Phi(u^H) \leq \Phi(u)$*, for all* $u \in S$ *and* $H \in \mathcal{H}_\star$.

Then for every $0 < \varepsilon < \frac{c-a}{2}$, $\delta > 0$ *and* $\gamma \in \Gamma$*, with the properties*

i) $\sup_{t \in [0,1]} \Phi(\gamma(t)) \leq c + \varepsilon$*;*

ii) $\gamma([0,1]) \subset S$*;*

iii) $\{\gamma(0), \gamma(1)\}^{H_0} = \{\gamma(0), \gamma(1)\}$ *for some* $H_0 \in \mathcal{H}_\star$*,*

there exists $u_\varepsilon \in X$ *such that*

a) $c - 2\varepsilon \leq \Phi(u_\varepsilon) \leq c + 2\varepsilon$*;*

b) $\|u_\varepsilon - u_\varepsilon^\star\|_V \leq 2(2\kappa + 1)\delta$*;*

c) $\lambda_\Phi(u) \leq 8\varepsilon/\delta$.

Chapter 2

Hemivariational inequalities, differential inclusions on bounded domains

2.1 Existence results

In this section we establish two existence results concerning some hemivariational inequalities. As consequences of these results we reobtain some classical results on Banach spaces, such as the Schauder fixed point theorem. These results are contained in the papers of Buzogány, Mezei and Cs. Varga [16] and of Buzogány, Mezei and V. Varga [17].

Through this section we always assume that X is a Banach space, X^* its dual, Ω is a bounded open set in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary. Furthermore we consider the following hypotheses:

$(\mathbf{H_T})$ $T : X \to L^p(\Omega, \mathbb{R}^k)$ is a linear, continuous operator, where $p \geq 1$, $k \geq 1$;

$(\mathbf{H_j})$ $j : \Omega \times \mathbb{R}^k \to \mathbb{R}$ is a Carathéodory function which is locally Lipschitz with respect to the second variable and satisfying the following growth condition: there exist $h_1 \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R})$ and $h_2 \in L^\infty(\Omega, \mathbb{R})$ such that

$$|w| \leq h_1(x) + h_2(x)|y|^{p-1}$$

for a.e. $x \in \Omega$, for every $y \in \mathbb{R}^k$ and for all $w \in \partial j(x,y)$, where $\partial j(x,y)$ is the Clarke generalized gradient of j.

The euclidean norm in $\mathbb{R}^k$ and the duality pairing between the Banach space and its dual will be denoted by $|\cdot|$, resp. $\langle \cdot, \cdot \rangle$.

Using the hypotheses (H_T) and (H_j), Panagiotopoulos, Fundo and Rădulescu proved the following result in [82]. We will make use of this lemma in the present chapter, and due to its importance, we include it along with its complete proof.

Lemma 2.1.1. ([82, Lemma 1.]) *If T and j satisfy the (H_T) and (H_j) respectively and V_1, V_2 are non-empty subsets of X, then the mapping $V_1 \times V_2 \to \mathbb{R}$ defined by*

$$(u,v) \mapsto \int_\Omega j_y^0(x, Tu(x), Tv(x))dx, \;\; (u,v) \in V_1 \times V_2 \tag{2.1}$$

is upper semicontinuous.

Proof. Let $\{(u_m, u_n)\}_{m\in\mathbb{N}} \subset V_1 \times V_2$ be a sequence converging to $(u,v) \in V_1 \times V_2$, as $m \to \infty$. Since $T : V \to L^p(\Omega, \mathbb{R}^k)$ is continuous, it follows that

$$Tu_m \to Tu, \;\; Tv_m \to Tv \;\; \text{in } L^p(\Omega, \mathbb{R}^k), \quad \text{as } m \to \infty.$$

There exist a subsequence $\{(Tu_n, Tv_n)\}$ of the sequence $\{(Tu_m, Tv_m)\}$ such that

$$\limsup_{m\to\infty} \int_\Omega j^0(x, Tu_m(x); Tv_m(x))dx = \lim_{m\to\infty} \int_\Omega j^0(x, Tu_n(x); Tv_n(x))dx.$$

By Proposition 4.11 in [44], one may suppose the existence of two functions $Tu_0, Tv_0 \in L^p(\Omega\mathbb{R}^k)$, and of two subsequences denoted again by the same symbols, such that:

$$|Tu_n(x)| \le Tu_0(x), \;\; |Tv_n(x)| \le Tv_0(x),$$

$$Tu_n(x) \to Tu(x), \;\; Tv_n(x) \to Tv(x), \quad \text{as } n \to \infty$$

for a.e. $x \in \Omega$. On the other hand, for each x where the condition (H_j) holds true and for each $y, h \in \mathbb{R}^k$, there exists $z \in \partial j(x,y)$ such that

$$j^0(x, y; h) = \langle z, h \rangle = \max\{\langle w, h \rangle \; w \in \partial j(x,y)\},$$

(see [28, Prop.2.1.2]). Now, by (H_j)

$$|j^0(x,y;h)| \leq |z||h| \leq (h_1(x) + h_2(x)|y|^{p-1})|h|.$$

Consequently, denoting $F(x) = (h_1(x) + h_2(x)|Tu_0(x)|^{p-1})|Tv_0(x)|$, we find that

$$|j^0(x, Tu_n(x); Tv_n(x))| \leq F(x),$$

for all $n \in \mathbb{N}$ and for a.e. $x \in \Omega$.

From Hölder's inequalities and from the condition (H_j) for the functions h_1 and h_2 it follows that $F \in L^1(\Omega, \mathbb{R})$. Fatou's Lemma yields

$$\lim_{n\to\infty} \int_\Omega j^0(x, Tu_n(x); Tv_n(x))dx \leq \limsup_{n\to\infty} \int_\Omega j^0(x, Tu_n(x); Tv_n(x))dx.$$

Next, by upper-semicontinuity of the mapping $j^0(x, .; ,)$ (see Clarke [28, Prop.2.1.1]) we get that

$$\limsup_{n\to\infty} j^0(x, Tu_n(x); Tv_n(x)) \leq j^0(x, Tu_0(x); Tv_0(x)),$$

for a.e. $x \in \Omega$. Hence

$$\limsup_{m\to\infty} j^0(x, Tu_m(x); Tv_m(x)) \leq j^0(x, Tu_0(x); Tv_0(x)),$$

which proves the upper-semicontinuity of the mapping defined by (2.1). $\square$

2.1.1 Variational-hemivariational inequality with one variable

Let K be a subset of X, $\mathcal{A} : K \rightsquigarrow X^*$ and $F : K \rightsquigarrow K$ two multivalued mappings (with nonempty values). Under hypotheses (H_T) and (H_j) the main problem of this subsection is the following:

(HVI) Find $u \in F(u)$, such that

$$\sigma(\mathcal{A}(u), v - u) + \int_\Omega j_y^0(x, Tu(x); Tv(x) - Tu(x))dx \geq 0,$$

for all $v \in F(u)$, where $j_y^0(x, Tu(x); Tv(x) - Tu(x))$ is the generalized Clarke derivative of $j(x, \cdot)$ at the point $Tu(x)$ with the direction $Tv(x) - Tu(x)$ and $\sigma(\mathcal{A}(u), v - u)$ denotes the support function of the set $\mathcal{A}(u)$, which is

$$\sigma(\mathcal{A}(u), v - u) = \sup_{x^* \in \mathcal{A}(u)} \langle x^*, v - u \rangle.$$

Before presenting the main result of this subsection, Theorem 2.1.1, we first introduce the following lemma, proved by Abdelkhaled El-Arni in [5]. This lemma serves as a key tool in the proof of Theorem 2.1.1, and we therefore find it useful to include it here.

Lemma 2.1.2. ([5]) *Let K be a convex, closed subset of a Banach space X. If*

(i) $F : K \rightsquigarrow K$ is upper semicontinuous with closed convex nonempty values;

(ii) $\Phi : K \times K \to \overline{\mathbb{R}}$ is lower semicontinuous in first argument on every compact convex subset of X and 0-diagonally concave in the second argument and $\Phi(x, x) \leq 0, \ \forall x \in K$;

(iii) the set $\{x \in K : \sup_{y \in F(x)} \Phi(x, y) \leq 0\}$ is closed in K;

(iv) there exists a nonempty compact convex subset $Z \subset K$ such that for all $x \in K \setminus Z$ exists $y \in Z \cap F(x)$ with $\Phi(x, y) > 0$.

Then there exists $x_0 \in F(x_0)$ such that $\sup_{y \in F(x_0)} \Phi(x_0, y) \leq 0$.

Now, we can state the main result of this subsection.

Theorem 2.1.1. ([16]) *Let K be a convex closed subset of a Banach space X, $\mathcal{A} : K \rightsquigarrow X^*$ is upper demicontinuous on K with bounded values, $T : X \to L^p(\Omega, \mathbb{R}^k)$ is satisfying (H_T) and $j : \Omega \times \mathbb{R}^k \to \mathbb{R}$ satisfies (H_j). Let $F : K \rightsquigarrow K$ be a multivalued upper-hemicontinuous and lower semicontinuous mapping with convex, closed and nonempty values. Moreover, we suppose* *that there exists a compact subset K_0 of K such that for all $x \in K \setminus K_0$ there exists $y_x \in K_0 \cap F(x)$ such that*

$$\sigma(\mathcal{A}(x), y_x - x) + \int_\Omega j_y^0(\omega, Tx(\omega), Ty_x(\omega) - Tx(\omega))d\omega < 0.$$

Then (HVI) has at least one solution.

Proof. Let $\Phi : K \times K \to \mathbb{R}$, defined by

$$\Phi(u, v) = -\sigma(\mathcal{A}(u), v - u) - \int_\Omega j_y^0(x, Tu(x); Tv(x) - Tu(x))dx.$$

We will verify the hypotheses of Lemma 2.1.2.

(ii) From Lemma 2.1.1 and Lemma 1.2.2 we obtain that Φ is lower semicontinuous in the first argument. Using the facts that $h \mapsto \sigma(\mathcal{A}(x), h)$ and $h \mapsto \int_\Omega j_y^0(x, Tu(x); h)dx$ are sublinear functions (therefore convex), from Remark 1.2.5 we obtain that Φ is 0-DCV in the second argument, taking into account that $\Phi(u, u) = 0$, $\forall u \in K$.

(iii) Since $(u, v) \to \Phi(u, v)$ and F are lower semicontinuous, using the Maximum Theorem (Lemma 1.2.1), we get that $x \mapsto \sup_{y \in F(x)} \Phi(x, y)$ is lower semicontinuous function. Therefore, the level set

$$\{x \in K : \sup_{y \in F(x)} \Phi(x, y) \leq 0\}$$

is closed in K.

(iv) is exactly the our coercivity hypothesis.

Therefore, there exists $x_0 \in F(x_0)$ such that $\sup_{y \in F(x_0)} \Phi(x_0, y) \leq 0$, i.e.

$$\sigma(\mathcal{A}(x_0), v - x_0) + \int_\Omega j_y^0(\omega, Tx_0(\omega); Tv(\omega) - Tx_0(\omega))d\omega \geq 0, \ \forall v \in F(x_0),$$

which completes the proof. $\square$

As a consequence of this theorem, we obtain a result of Pangiotopoulos, Fundo, Rădulescu, proved in [82, Theorem 1].

Corollary 2.1.1. *Let K be a compact and convex subset of a Banach space X and let j satisfying the condition* (j). *If $A : K \to X^*$ is w^*-demicontinuous, then there exists an $u \in K$ such that for all $v \in K$*

$$\langle Au, v - u \rangle + \int_\Omega j_y^0(x, Tu(x); Tv(x) - Tu(x))dx \geq 0.$$

Proof. We choose $F(u) = K$, which is continuous with convex and closed values. Then using Remark 1.2.1 and applying the above theorem, we conclude this corollary. $\square$

2.1.2 Variational-hemivariational inequality with two variables

In this subsection we consider the set-valued mappings with nonempty values $\mathcal{A} : K \times K \rightsquigarrow X^*$ and $G : K \times X \rightsquigarrow \mathbb{R}$, where K is a subset of X. Under hypotheses (H_T) and (H_j) the main problem of this subsection is the following differential inclusion problem

(P) Find $u \in K$ such that, for every $v \in K$

$$\sigma(\mathcal{A}(u,u), v-u) + G(u, v-u) + \int_\Omega j_y^0(x, Tu(x); Tv(x) - Tu(x))dx \subseteq \mathbb{R}_+.$$

Here $\sigma(\mathcal{A}(w,u), v-u)$ denotes the support function of the set $\mathcal{A}(w,u)$, i.e.

$$\sigma(\mathcal{A}(w,u), v-u) = \sup_{x^* \in \mathcal{A}(w,u)} \langle x^*, v-u \rangle.$$

The problem (P) can be formulated in the form of a hemivariational inequality

(P′) Find $u \in K$ such that, for every $v \in K$

$$\sigma(\mathcal{A}(u,u), v-u) + \inf G(u, v-u) + \int_\Omega j_y^0(x, Tu(x); Tv(x) - Tu(x))dx \geq 0.$$

Furthermore, we assume that the following hypotheses are satisfied on the mappings G and $\mathcal{A}$.

(H_G)(1) $G(u, 0) \subseteq \mathbb{R}_+, \ \forall\, u \in K$;

(2) $G(u, \cdot)$ is convex, $\forall u \in K$;

(3) $G(\cdot, \cdot)$ is lower semicontinuous on $K \times X$;

(4) $G(u, \cdot)$ is subhomogenous, i.e.

$tG(u, y) \subseteq G(u, ty), \ \forall t \in [0, 1], \ \forall u \in K, \ \forall y \in X.$

$(H_{\mathcal{A}})(1)$ $\mathcal{A}$ has bounded values, i.e. $\sup\limits_{x^*\in\mathcal{A}(u,v)} \|x^*\| < \infty$, $\forall u, v \in K$;

(2) $\mathcal{A}(v, \cdot) : K \rightsquigarrow X^*$ is upper demicontinuous on K, $\forall v \in K$;

(3) $\mathcal{A}(\cdot, u) : K \rightsquigarrow X^*$ is upper demicontinuous from the line segments in K, $\forall u \in K$.

(4) $\mathcal{A}(\cdot, u)$ has the monotonicity property

$$\sigma(\mathcal{A}(v, u), v - u) \geq \sigma(\mathcal{A}(u, u), v - u), \forall u, v \in K.$$

The main result of this subsection is the following

Theorem 2.1.2. ([17]) *Let K be a convex, closed subset of a Banach space X and $\mathcal{A} : K \times K \rightsquigarrow X^*$, $G : K \times X \rightsquigarrow \mathbb{R}$, $T : X \to L^p(\Omega, \mathbb{R}^k)$ and $j : \Omega \times \mathbb{R}^k \to \mathbb{R}$ satisfying $(H_{\mathcal{A}})$, (H_G), (H_T) and (H_j) respectively. In addition, if*

(H_{coer}) there exists a compact subset K_0 of K and $u_0 \in K$ such that for all $u \in K \setminus K_0$ we have

$$\left\{\sigma(\mathcal{A}(u, u), u_0 - u) + G(u, u_0 - u) + \int_\Omega j_y^0(x, Tu(x), Tu_0(x) - Tu(x))dx\right\} \cap \mathbb{R}_-^* \neq \emptyset,$$

then (P) has at least one solution.

Proof. For $w \in K$, let

$$T_1(w) = \{u \in K : \sigma(\mathcal{A}(u, u), w - u) + \inf\, G(u, w - u) + \\ + \int\limits_\Omega j_y^0(x, Tu(x), Tw(x) - Tu(x))dx \geq 0\};$$

$$T_2(w) = \{u \in K_0 : \sigma(\mathcal{A}(w, u), w - u) + \inf\, G(u, w - u) + \\ + \int\limits_\Omega j_y^0(x, Tu(x), Tw(x) - Tu(x))dx \geq 0\}.$$

Step 1. We prove that $T_1(u_0) \subseteq K_0$, where u_0 is from the condition (H_{coer}).

Suppose that there exists $u \in T_1(u_0) \subset K$ such that $u \notin K_0$. From the definition of $T_1(u_0)$, we have that

$$\sigma(\mathcal{A}(u,u), u_0 - u) + \inf G(u, u_0 - u) + \int_\Omega j^0(x, Tu(x), Tu_0(x) - Tu(x))dx \geq 0.$$

But this contradicts the (H_{coer}). Therefore $T_1(u_0) \subseteq K_0$.

Step 2. We prove that $T_1 : K \rightsquigarrow K$ is KKM-mapping, i.e.

$$\forall w_1, ..., w_n \in K : co\{w_1, ..., w_n\} \subseteq \bigcup_{i=1}^{n} T_1(w_i).$$

We argue by contradiction, we suppose that there exist

$$\lambda_1, \ldots, \lambda_n \geq 0, \ \sum_{i=1}^{n} \lambda_i = 1$$

such that $\overline{w} = \sum\limits_{i=1}^{n} \lambda_i w_i \notin T_1(w_i)$, for $i = \overline{1,n}$. Therefore

$$\sigma(\mathcal{A}(\overline{w}, \overline{w}), w_i - \overline{w}) + \inf\, G(\overline{w}, w_i - \overline{w}) +$$
$$+ \int_\Omega j^0(x, T\overline{w}(x), Tw_i(x) - T\overline{w}(x))dx < 0, \ i = \overline{1,n}.$$

Let $\mathcal{I} = \{i = \overline{1,n} : \lambda_i \neq 0\}$. Multiplying the above inequalities by λ_i for $i \in \mathcal{I}$ and using the homogenity of T and the positive homogenity of j^0, we have

$$\sigma(\mathcal{A}(\overline{w}, \overline{w}), \lambda_i w_i - \lambda_i \overline{w}) + \lambda_i \inf\, G(\overline{w}, w_i - \overline{w}) +$$
$$+ \int_\Omega j^0(x, T\overline{w}(x), T(\lambda_i w_i)(x) - T(\lambda_i \overline{w})(x))dx < 0, \forall i \in \mathcal{I}.$$

Adding the above relations for $i \in \mathcal{I}$ and using that $h \mapsto \sigma(\mathcal{A}(\overline{w}, \overline{w}), h)$ and $h \mapsto j^0(x, T\overline{w}(x), h)$ are subadditive, we get

$$\sigma\left(\mathcal{A}(\overline{w}, \overline{w}), \sum_{i \in \mathcal{I}} \lambda_i w_i - \sum_{i \in \mathcal{I}} \lambda_i \overline{w}\right) + \sum_{i \in \mathcal{I}} \lambda_i \inf\, G(\overline{w}, w_i - \overline{w}) +$$
$$+ \int_\Omega j^0(x, T\overline{w}(x), \sum_{i \in \mathcal{I}} T(\lambda_i w_i)(x) - \sum_{i \in \mathcal{I}} T(\lambda_i \overline{w})(x))dx < 0.$$

On the other hand since $z \mapsto \inf G(\overline{w}, z)$ is convex for all $\overline{w} \in K$, T is additive and the condition $(H_G)(1)$ holds, we get

$$\sigma\left(\mathcal{A}(\overline{w}, \overline{w}), \sum_{i\in\mathcal{I}} \lambda_i w_i - \sum_{i\in\mathcal{I}} \lambda_i \overline{w}\right) + \sum_{i\in\mathcal{I}} \lambda_i \inf G(\overline{w}, w_i - \overline{w}) +$$

$$+ \int_\Omega j^0(x, T\overline{w}(x), \sum_{i\in\mathcal{I}} T(\lambda_i w_i)(x) - \sum_{i\in\mathcal{I}} T(\lambda_i \overline{w})(x))dx \geq 0,$$

which is absurd. Therefore, T_1 is KKM-mapping.

<u>*Step 3.*</u> We prove that $\bigcap_{w\in K} \overline{T_1(w)} \neq \emptyset$. Here, $\overline{T_1(w)}$ is the closure of $T_1(w)$. Indeed, from Step 1, we have that $T_1(u) \subseteq K_0$. Since K_0 is compact, $\overline{T_1(u_0)}$ is also compact. Using the Step 2 and applying Lemma 2 of Ky Fan, we obtain that $\bigcap_{w\in K} \overline{T_1(w)} \neq \emptyset$.

<u>*Step 4.*</u> $\bigcap_{w\in K} T_1(w) = \bigcap_{w\in K} T_2(w)$.

(a) Let $u \in \bigcap_{w\in K} T_1(w)$ i.e. $\forall w \in K$. we have

$$\sigma(\mathcal{A}(u,u), w-u) + \inf G(u, w-u) + \int_\Omega j^0(x, Tu(x), Tw(x) - Tu(x))dx \geq 0.$$

From the condition (H_{coer}), we have that $u \in K_0$, and from the monotonicity property $(H_{\mathcal{A}})(4)$ of $\mathcal{A}$, we have

$$\sigma(\mathcal{A}(w,u), w-u) + \inf G(u, w-u) + \int_\Omega j^0(x, Tu(x), Tw(x) - Tu(x))dx \geq 0,$$

for every $w \in K$, therefore $u \in \bigcap_{w\in K} T_2(w)$.

(b) Let $u \in \bigcap_{w\in K} T_2(w)$. Then $\sigma(\mathcal{A}(v,u), v-u) + \inf G(u, v-u) + \int_\Omega j^0(x, Tu(x), Tv(x) - Tu(x))dx \geq 0$, $\forall v \in K$. Let $v \in K$ be an arbitrary element, and $v_t = tv + (1-t)u$, for $t \in [0,1]$. Clearly, $v_t \in K$. We have for every $t \in [0,1]$

$$\sigma(\mathcal{A}(v_t, u), v_t - u) + \inf G(u, v_t - u) + \int_\Omega j^0(x, Tu(x), Tv_t(x) - Tu(x))dx \geq 0.$$

From the linearity of T, we have

$$\sigma(\mathcal{A}(v_t,u),t(v-u))+\inf\,G(u,t(v-u))+$$
$$+\int_\Omega j^0(x,Tu(x),t(Tv(x)-Tu(x)))dx\geq 0,\ \forall t\in[0,1].$$

From the (H_G) (4) and from the fact that $j_y^0(x,Tu(x),\cdot)$ is positive homogeneous, we obtain

$$\sigma(\mathcal{A}(v_t,u),v-u)+\inf\,G(u,v-u)+\int_\Omega j^0(x,Tu(x),Tv(x)-Tu(x))dx\geq 0,$$

for every $t\in(0,1]$. Using $(H_{\mathcal{A}})$ (3), we have that $\limsup\limits_{t\to 0^+}\sigma(\mathcal{A}(v_t,u),v-u)\leq\sigma(\mathcal{A}(u,u),v-u)$. Therefore,

$$\sigma(\mathcal{A}(u,u),v-u)+\inf\,G(u,v-u)+\int_\Omega j^0(x,Tu(x),Tv(x)-Tu(x))dx\geq 0.$$

Since $v\in K$ was arbitrarily choosen, it follows that $u\in\bigcap\limits_{w\in K}T_1(w)$.

<u>*Step 5.*</u> $\bigcap\limits_{w\in K}\overline{T_1(w)}=\bigcap\limits_{w\in K}T_2(w)$.

Clearly, $\bigcap\limits_{w\in K}T_2(w)\subseteq\bigcap\limits_{w\in K}\overline{T_1(w)}$ from Step 4. Conversely, let

$$v\in\bigcap_{w\in K}\overline{T_1(w)}.$$

We prove that $v\in\bigcap\limits_{w\in K}T_2(w)$.

Since K_0 is compact, from Step 1. it follows that $\overline{T_1(u_0)}\subset K_0$, we have that $\bigcap\limits_{w\in K}\overline{T_1(w)}\subseteq K_0$. Therefore, $v\in K_0\cap\overline{T_1(w)}$, $\forall w\in K$.

Now, let $u\in K$ be a fixed element. Since $v\in\overline{T_1(u)}$, there exists a sequence $\{v_n\}$ from $T_1(u)$ such that $v_n\to v$. Since $v_n\in T_1(u)$, we have

$$\sigma(\mathcal{A}(v_n,v_n),u-v_n)+\inf\,G(v_n,u-v_n)+\int_\Omega j^0(x,Tv_n(x),Tu(x)-Tv_n(x))dx\geq 0.$$

From $(H_{\mathcal{A}})$(4), we have

$$\sigma(\mathcal{A}(u,v_n),u-v_n)+\inf\,G(v_n,u-v_n)+\int_\Omega j^0(x,Tv_n(x),Tu(x)-Tv_n(x))dx\geq 0.$$

From $(H_{\mathcal{A}})(1)$ and (2), applying Lemma 1.2.3 we obtain that $v \mapsto \sigma(\mathcal{A}(u,v), u-v)$ is upper semicontinuous, therefore

$$\limsup_{n\to\infty} \sigma(\mathcal{A}(u,v_n), u-v_n) \leq \sigma(\mathcal{A}(u,v), u-v).$$

From Lemma 1.2.1 (with $F = G$, $Y := K \times X$, $Z := \mathbb{R}$, $f((y_1,y_2),z) = -z$, where $z \in G(y_1,y_2)$) and $(H_G)(3)$ we have that $v \mapsto \inf G(v, u-v)$ is upper semicontinuous, therefore

$$\limsup_{n\to\infty} \inf G(v_n, u-v_n) \leq \inf G(v, u-v).$$

Using the Lemma 2.1.1 we get the following inequality

$$\limsup_{n\to\infty} \int_\Omega j^0(x, Tv_n(x), Tu(x) - Tv_n(x))dx \leq$$

$$\leq \int_\Omega j^0(x, Tv(x), Tu(x) - Tv(x))dx.$$

Summarizing the above relations, we get

$$\sigma(\mathcal{A}(u,v), u-v) + \inf G(v, u-v) +$$

$$+ \int_\Omega j^0(x, Tv(x), Tu(x) - Tv(x))dx \geq 0,$$

i.e. $v \in T_2(u)$. Since u was arbitrary, we have that $v \in \bigcap_{u\in K} T_2(u)$.

Step 6. From Steps 3, 4 and 5, we have that $\bigcap_{w\in K} T_1(w) \neq \emptyset$, which means that there exists at least one $u \in \bigcap_{w\in K} T_1(w)$, which is a solution for (P). $\square$

Remark 2.1.1. If K is compact in the above theorem, the hypothesis (H_{coer}) can be omitted.

In the following we present some applications of the Theorem 2.1.2.

As a first application, we can deduce easily the Schauder resp. Brouwer fixed point theorem on Banach spaces. For the completeness, we give the proof.

Corollary 2.1.2. *Let K be a compact, convex subset of a Banach space X and $f : K \to K$ be a continuous function. Then f has at least one fixed point.*

Proof. Let $\mathcal{A} \equiv 0$, $j \equiv 0$, $T \equiv 0$ and $G : K \times X \rightsquigarrow \mathbb{R}$ defined by $G(u,v) = [\|u+v-f(u)\| - \|u-f(u)\|, \infty)$.

We verify (H_G). Clearly, $G(u,0) = [0,\infty) = \mathbb{R}_+$ and $v \rightsquigarrow G(u,v)$ is convex, $\forall\, v \in K$. Since f is continuous, the function $(u,v) \mapsto \|u+v-f(u)\| - \|u-f(u)\|$ is continuous also. Therefore, it's easy to prove that $(u,v) \rightsquigarrow G(u,v)$ is lower semicontinuous on $K \times X$. The subhomogenity of $G(u,\cdot)$ for $t=0$ and $t=1$ is trivial. Otherwise, this follows from the triangle inequality.

Therefore, the hypotheses of (H_G) are fulfilled, so from Theorem 2.1.2 it follows that there exists $u_0 \in K$ such that

$$[\|v-f(u_0)\| - \|u_0 - f(u_0)\|, \infty) = G(u_0, v-u_0) \subseteq \mathbb{R}_+,\ \forall\, v \in K.$$

In particular, we have $\|v-f(u_0)\| - \|u_0-f(u_0)\| \geq 0$, $\forall\, v \in K$. Let $v := f(u_0)$. We have $-\|u_0 - f(u_0)\| \geq 0$, i.e. $u_0 = f(u_0)$. $\square$

Corollary 2.1.3. (Brouwer fixed point theorem) *Let $f : K \to K$ be a continuous function, K being a compact, convex subset of $\mathbb{R}^n$. Then f has at least one fixed point.*

As an other application we can prove a result obtained by Panagiotopoulos, Fundo and Rădulescu, namely

Corollary 2.1.4. ([82, Theorem 1.])*Let K be a compact and convex subset of a Banach space X and j and T satisfying (H_j) and (H_T) respectively. If the operator $A : K \to X^*$ is w^*-demicontinuous, then there exists $u \in K$ such that*

$$(PFR) \qquad \langle Au, v-u\rangle + \int_\Omega j_y^0(x, Tu(x), Tv(x) - Tu(x))dx \geq 0,\ \forall v \in K.$$

Proof. Let $\mathcal{A} : K \times K \rightsquigarrow X^*$ defined by $\mathcal{A}(v,u) = \{A(u)\}$, $\forall u,v \in K$ and $G \equiv 0$. Let $v \in K$ be fixed. From Remark 1.2.1, $\mathcal{A}(v,\cdot)$ is upper demicontinuous on K (with bounded values). Therefore, $(H_\mathcal{A})$ holds. Since $\sigma(\mathcal{A}(u,u), v-u) = \langle Au, v-u\rangle$, the assertion follows easily from Theorem 2.1.2. $\square$

The following result is of Browder's type, see [15].

Corollary 2.1.5. *Let K be a convex, closed subset of a Banach space, $\mathcal{A} : K \times K \rightsquigarrow X^*$ be an operator satisfying $(H_\mathcal{A})$. Suppose that there exists a compact subset $K_0 \subset K$ and $u_0 \in K$ such that $\sigma(\mathcal{A}(u,u), u_0 - u) < 0$, $\forall u \in K \setminus K_0$. Then there exists $u \in K$ such that*

$$\sigma(\mathcal{A}(u,u), v-u) \geq 0,\ \forall v \in K.$$

Proof. We apply Theorem 2.1.2 for $G \equiv 0,\ j \equiv 0$ and $T \equiv 0$. □

Remark 2.1.2. Similar results were obtained by Chen in [27].

Finally let X_1 and X_2 two Banach spaces, $K_1 \subseteq X_1$, $K_2 \subseteq X_2$ two nonempty closed, convex sets. Let $F_i : K_1 \times K_2 \to X_i^*,\ i = 1,2$ two operators. Our aim is to give existence result for the following problem:

Find $(u_1, u_2) \in K_1 \times K_2$ such that

$$(NP) \qquad \begin{cases} \langle F_1(u_1,u_2), x - u_1\rangle \geq 0,\ \forall x \in K_1, \\ \\ \langle F_2(u_1,u_2), y - u_2\rangle \geq 0,\ \forall y \in K_2 \end{cases}$$

The above problem is originated from the Nash equilibrium points (see [42] and [43]).

Theorem 2.1.3. ([17]) *Suppose that*

(i) for every $x_i \in K_i,\ i = 1,2$ the mappings $F_1(\cdot, x_2) : K_1 \to X_1^*$ and $F_2(x_1, \cdot) : K_2 \to X_2^*$ are monotone, w^*-demicontinuous and upper demicontinuous on the line segments in K_1 respective K_2;

(ii) there exist $K_i^0 \subseteq K_i,\ i = 1,2$ compact sets and $x_i^0 \in K_i^0$ such that for every $(x_1, x_2) \in (K_1 \times K_2) \setminus (K_1^0 \times K_2^0)$

$$\langle F_1(x_1,x_2), x_1^0 - x_1\rangle + \langle F_2(x_1,x_2), x_2^0 - x_2\rangle < 0.$$

Then (NP) has at least a solution.

Proof. First let $j \equiv 0$, $G \equiv 0$ and $T \equiv 0$ in Theorem 2.1.2. Moreover, let $X := X_1 \times X_2$, $K := K_1 \times K_2$ and $\mathcal{A} : K \times K \rightsquigarrow X^*$ be a single-valued map, defined by

$$\mathcal{A}((x,y),(z,t)) = (F_1(x,t), F_2(z,y)),\ \forall (x,y),\ (z,t) \in K.$$

Clearly, $\mathcal{A}$ satisfies $(H_{\mathcal{A}})$. Let $K_0 := K_1^0 \times K_2^0$ and $u_0 := (x_1^0, x_2^0) \in K_0$. The K_0 and u_0 satisfy the (H_{coer}) condition from Theorem 2.1.2. Therefore, there exists $u = (u_1,u_2) \in K$ such that $\langle \mathcal{A}(u,u), w - u\rangle \geq 0,\ \forall w \in K$. This is equivalent with

$$\langle F_1(u_1,u_2), w_1 - u_1\rangle + \langle F_2(u_1,u_2), w_2 - u_2\rangle \geq 0,\ \forall w_i \in K_i,\ i = \overline{1,2}.$$

Substituting $w_2 := u_2$ and $w_1 := u_1$ respectively, we obtain that $u = (u_1,u_2) \in K$ is a solution for (NP). □

Remark 2.1.3. *If K_1 and K_2 are compact sets, the hypothesis (ii) from the Theorem 2.1.3, can be omitted.*

Remark 2.1.4. *From the Theorem 2.1.3 we obtain also the Brouwer fixed point theorem (see Corollary 2.1.3), choosing $K = K_1 = K_2$, $F_1(u_1, u_2) = -f(u_1) + u_2$ and $F_2(u_1, u_2) = u_2 - u_1$.*

2.2 Multiple symmetric solutions for differential inclusions

In this section we prove a multiplicity result for a differential inclusion problem defined on the unit ball. By variational methods, we demonstrate that the solutions of this problem are invariant by spherical cap symmetrization. We shall follow the paper of Mezei, Molnár and Vas [70].

Consider the following semi-linear elliptic differential inclusion problem, coupled with the homogeneous Dirichlet boundary condition:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u \in \lambda\partial F(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}^1_\lambda$}$$

where λ is a positive parameter, $1 < p < N$, $\Omega \subset \mathbb{R}^N$ is the unit ball, $\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, and $\partial F(x, s)$ stands for the generalized gradient of the locally Lipschitz function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ at the point $s \in \mathbb{R}$ with respect to the second variable (see for details Section 2). Here and in the sequel $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$.

Such problems arise mostly in mathematical physics, where solutions of elliptic problems correspond to certain equilibrium state of the physical system. This is the reason why problems of this type were intensively studied by several authors in the last years.

The purpose of this section is ensure the existence of multiple spherical cap symmetric solutions for the problem ($\mathcal{P}^1_\lambda$), where the natural functional space is the Sobolev space $W^{1,p}_0(\Omega)$, endowed with its standard norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p + \int_\Omega |u(x)|^p \right)^{1/p}.$$

In order to obtain our result, we need the following assumptions on the function F:

$(\mathbf{F_1})$ $\displaystyle\lim_{|s|\to 0} \frac{\max\{|\xi| : \xi \in \partial F(x,s)\}}{|s|^{p-1}} = 0;$

$(\mathbf{F_2})$ $\displaystyle\lim_{|s|\to +\infty} \frac{\max\{|\xi| : \xi \in \partial F(x,s)\}}{|s|^{p-1}} = 0;$

$(\mathbf{F_3})$ There exists an $u_0 \in W_0^{1,p}(\Omega)$, $u_0 \neq 0$ such that

$$\int_\Omega F(x, u_0(x))dx > 0.$$

$(\mathbf{F_4})$ $F(x,s) = F(y,s)$ for a.e. $x, y \in \Omega$, with $|x| = |y|$ and all $s \in \mathbb{R}$;

$(\mathbf{F_5})$ $F(x,s) \leq F(x,-s)$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^-$.

The main result of the subsection is the following:

Theorem 2.2.1. ([70]) *Assume that $1 < p < N$. Let $\Omega \subset \mathbb{R}^N$ be the unit ball and $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function with $F(x,0) = 0$, satisfying $(\mathbf{F_1})$-$(\mathbf{F_5})$. Then,*

(a) there exists a λ_F such that, for every $0 < \lambda \leq \lambda_F$ the problem $(\mathcal{P}_\lambda^1)$ has only the trivial solution;

(b) there exists a λ_1 such that, for every $\lambda > \lambda_1$ the problem $(\mathcal{P}_\lambda^1)$ has at least two weak solutions in $W_0^{1,p}(\Omega)$, invariants by spherical cap symmetrization.

Remark 2.2.1. Choosing $p = 3$, the function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$F(x,s) = \begin{cases} |x|(s^4 - s^2), & \text{if } |s| \leq 1 \\ |x| \ln s^2, & \text{if } |s| > 1. \end{cases}$$

fulfills the hypotheses $(\mathbf{F_1})$-$(\mathbf{F_5})$.

Definition 2.2.1. *We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution to problem $(\mathcal{P}_\lambda^1)$ if there exists $\xi_F \in \partial F(x, u(x))$ for a.e. $x \in \Omega$ such that for all $v \in W_0^{1,p}(\Omega)$ we have*

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv)dx = \lambda \int_\Omega \xi_F v(x)dx. \tag{2.2}$$

We consider the functionals $I, \mathcal{F} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p + |u|^p)dx, \qquad \mathcal{F}(u) = \int_\Omega F(x, u(x))dx.$$

Now, we can define the energy functional associated to the problem $(\mathcal{P}_\lambda^1)$ by

$$\mathcal{E}_\lambda(u) = I(u) - \lambda\mathcal{F}(u).$$

Remark 2.2.2. If Ω is bounded, using [77, Theorem 1.3], we have

$$\partial\mathcal{F}(u) \subset \int_\Omega \partial F(x, u(x))dx.$$

Hence, the critical points of the energy functional $\mathcal{E}_\lambda$ are exactly the (weak) solutions of the problem $(\mathcal{P}_\lambda^1)$. So, instead of seeking for the solutions of the problem $(\mathcal{P}_\lambda^1)$, it is enough to look for the critical points of the energy functional $\mathcal{E}_\lambda$.

Before proving our main result, we prove that the functional E_λ is coercive and it satisfies the non-smooth Palais-Smale condition on $W_0^{1,p}(\Omega)$.

Lemma 2.2.1. *The functional $\mathcal{E}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ is coercive for every $\lambda \geq 0$, that is, $\mathcal{E}_\lambda(u) \to \infty$ as $\|u\| \to \infty$, for all $u \in W_0^{1,p}(\Omega)$.*

Proof. Let us fix a $\lambda \geq 0$. In particular, from $(\mathbf{F_1})$, there exists a $\delta_1 > 0$ such that

$$|\xi| \leq \frac{1}{2} \cdot \frac{1}{p} c_p^{-p} \frac{1}{1+\lambda} |s|^{p-1}, |s| < \delta_1,$$

where c_p is the best Sobolev constant in the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ $(q \in [1, p^*)])$.

Due to $(\mathbf{F_2})$, it follows that for every $\varepsilon > 0$ there exists $\delta_2 = \delta_2(\varepsilon) > 0$, such that

$$\max\{|\xi| : \xi \in \partial F(x, s)\} \leq \varepsilon |s|^{p-1}, |s| > \delta_2.$$

Moreover, if $\varepsilon = \frac{1}{2} \cdot \frac{1}{p} c_p^{-p} \cdot \frac{1}{1+\lambda}$, then for every $\xi \in \partial F(x, s)$ one, has

$$|\xi| \leq \frac{1}{2} \cdot \frac{1}{p} c_p^{-p} \frac{1}{1+\lambda} |s|^{p-1}, |s| > \delta_2.$$

Since the set-valued mapping ∂F is upper-semicontinuous, then there exists $C_F = \sup\{\partial F(x, [\delta_2, \delta_1])\}$, thus

$$|\xi| \le \frac{1}{p} c_p^{-p} \frac{1}{1+\lambda} |s|^{p-1} + C_F, \quad \text{for all } s \in \mathbb{R}. \tag{2.3}$$

Now we can use Lebourg's mean value theorem (see Theorem 1.1.3), obtaining that:

$$|F(x,s)| = |F(x,s) - F(x,0)| \le |\xi_\theta s| \text{ for some } \xi_\theta \in \partial F(x, \theta s), \theta \in (0,1).$$

Combining this inequality with the relation (2.3), we get

$$|F(x,s)| \le \frac{1}{p} c_p^{-p} \frac{1}{1+\lambda} |s|^p + C_F|s|.$$

Moreover,

$$\mathcal{E}_\lambda(u) \ge \frac{1}{p}\|u\|^p - \frac{1}{p}\frac{\lambda}{1+\lambda}\left(\frac{\|u\|_p^p}{c_p^p}\right) - \lambda C_F \|u\|_1.$$

Therefore,

$$\begin{aligned} \mathcal{E}_\lambda(u) &\ge \frac{1}{p}\|u\|^p - \frac{1}{p}\frac{\lambda}{1+\lambda}\|u\|^p - \lambda \cdot C_1\|u\| \\ &= \frac{1}{p}\left(1 - \frac{\lambda}{1+\lambda}\right)\|u\|^p - \lambda C_1 \|u\| \to \infty \end{aligned}$$

as $\|u\| \to \infty$,where C_1 is a constant, which concludes our proof.

Lemma 2.2.2. *For every $\lambda > 0$, $\mathcal{E}_\lambda$ satisfies the non-smooth Palais-Smale condition.*

Proof. Let $\lambda > 0$ be fixed. We consider a Palais-Smale sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ for $\mathcal{E}_\lambda$, i.e., for some $\varepsilon_n \to 0^+$, we have

$$\mathcal{E}_\lambda^o(u_n; u - u_n) \ge -\varepsilon_n \|u - u_n\| \tag{2.4}$$

and $\{\mathcal{E}_\lambda(u_n)\}$ is bounded in $W_0^{1,p}(\Omega)$. Since $\mathcal{E}_\lambda$ is coercive, the sequence $\{u_n\}$ is bounded. Therefore taking a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and $u_n \to u$ strongly in L^p (note that $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, see Brezis [19]). One clearly has,

$$\langle I'(u_n), u - u_n\rangle = \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n\right)(\nabla u - \nabla u_n) + \int_\Omega |u_n|^{p-2}u_n(u - u_n),$$

and

$$\langle I'(u), u_n - u\rangle = \int_\Omega \left(|\nabla u|^{p-2}\nabla u\right)(\nabla u_n - \nabla u) + \int_\Omega |u|^{p-2}u(u_n - u).$$

Adding these two relations and from the fact that $|v-w|^p \leq (|v|^{p-2}v - |w|^{p-2}w)(v-w)$, one can conclude that

$$\langle I'(u_n), u - u_n\rangle + \langle I'(u), u_n - u\rangle =$$

$$\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u - \nabla u_n) + \int_\Omega (|u_n|^{p-2}u_n - |u|^{p-2}u)(u - u_n)$$

$$\leq \int_\Omega (-|\nabla u_n - \nabla u|^p - |u_n - u|^p) = -\|u_n - u\|^p. \tag{2.5}$$

On the other hand, by the relations

$$\mathcal{E}^o_\lambda(u_n; u - u_n) = \langle I'(u_n); u - u_n\rangle + \lambda\mathcal{F}^o(u_n; u_n - u),$$

$$\mathcal{E}^o_\lambda(u; u_n - u) = \langle I'(u); u_n - u\rangle + \lambda\mathcal{F}^o(u; u - u_n),$$

and the inequalities (2.4) and (2.5), we have

$$\|u_n - u\|^p \leq \varepsilon_n\|u - u_n\| - \mathcal{E}^o_\lambda(u; u_n - u)+$$

$$+\lambda(\mathcal{F}^o(u_n; u_n - u) + \mathcal{F}^o(u; u - u_n)) \tag{2.6}$$

Since the sequence $\{u_n\}$ is bounded in $W^{1,p}_0(\Omega)$, we clearly have

$$\lim_{n\to\infty} \varepsilon_n\|u - u_n\| = 0. \tag{2.7}$$

Now fix $w^* \in \partial\mathcal{E}_\lambda(u)$. In particular, by the definition, we have $\langle w^*; u_n - u\rangle \leq \mathcal{E}^o_\lambda(u; u_n - u)$. Since $u_n \rightharpoonup u$ weakly in $W^{1,p}_0(\Omega)$, we obtain

$$\liminf_{n\to\infty} \mathcal{E}^o_\lambda(u; u_n - u) \geq 0. \tag{2.8}$$

Now, for the remaining two terms in the estimation (2.6), we use the fact that

$$\mathcal{F}^o(u; v) \leq \int_\Omega F^o(x, u(x); v(x))dx, \forall u, v \in W^{1,p}_0(\Omega).$$

Therefore,

$$\begin{aligned}\mathcal{F}^o(u_n; u_n - u) &\leq \int_\Omega F^o(x, u_n(x); u_n(x) - u(x))dx \\ &= \int_\Omega \max\{\xi(u_n(x) - u(x)) : \xi \in \partial F(x, u_n(x))\}dx \\ &\leq \int_\Omega |u_n(x) - u(x)| \cdot \max\{|\xi| : \xi \in \partial F(x, u_n(x))\}dx.\end{aligned}$$

From the upper semi-continuity property of ∂F, one has

$$\sup_{\substack{n\in\mathbb{N}\\ x\in\Omega}} \{|\xi| : \xi \in \partial F(x, u_n(x)\} < \infty.$$

Proceeding in the same way for $\mathcal{F}^o(u; u - u_n)$ and adding the outcomes, we obtain

$$\mathcal{F}^o(u_n; u_n - u) + \mathcal{F}^o(u; u - u_n) \leq K \cdot \int_\Omega |u_n(x) - u(x)| = K||u_n - u||_{L^1},$$

where K is a constant. Since $u_n \to u$ strongly in $L^1(\Omega)$, we have that

$$\limsup_{n\to\infty}(\mathcal{F}^o(u_n; u_n - u) + \mathcal{F}^o(u; u - u_n)) \leq 0. \tag{2.9}$$

Now, combining the inequalities (2.7), (2.8) and (2.9), we obtain

$$\limsup_{n\to\infty} \|u - u_n\|^p \leq 0,$$

which means that $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$.

From the symmetric Ekeland's variational principle, given by Squassina in [92, Theorem 2.8], we can state the following corollary for locally Lipschitz functions.

Lemma 2.2.3. *Let $(X, V, \star, \mathcal{H}_\star, S)$ satisfy the assumptions given in Definition 1.5.5, with $V = L^p(\Omega)$, $X = W_0^{1,p}(\Omega)$ and with the further property that if $(u_n)_n \subset W_0^{1,p}(\Omega)$ such that $u_n \to u$ in $L^p(\Omega)$, then $u_n^\star \to u^\star$ in $L^p(\Omega)$. Assume that $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ is a locally Lipschitz functional bounded from below such that*

$$\Phi(u^H) \leq \Phi(u) \quad \textit{for all } u \in S \textit{ and } H \in \mathcal{H}_\star.$$

and for all $u \in W_0^{1,p}(\Omega)$ there exists $\xi \in S$, with $\Phi(\xi) \leq \Phi(u)$.

If Φ satisfies the $(PS)_{\inf \Phi}$ condition, then there exists $v \in W_0^{1,p}(\Omega)$, such that $\Phi(v) = \inf \Phi$ and $v = v^\star$ in $L^p(\Omega)$.

Proof. Put $\inf \Phi = d$. For the minimizing sequence $(u_n)_n$ we consider the following sequence:

$$\varepsilon_n = \begin{cases} \Phi(u_n) - d, & \text{if } \Phi(u_n) - d > 0 \\ \dfrac{1}{n}, & \text{if } \Phi(u_n) - d = 0. \end{cases}$$

Then $\Phi(u_n) \leq d + \varepsilon_n$ and $\varepsilon_n \to 0$ as $n \to \infty$. By [92, Theorem 2.8], there exists a sequence $(v_n)_n \subset W_0^{1,p}(\Omega)$ such that:

a) $\Phi(v_n) \leq \Phi(u_n)$;

b) $\lambda_\Phi(u_n) \to 0$;

c) $\|v_n - v_n^\star\|_p \to 0$;

Since Φ satisfies the $(PS)_d$ condition, there exists $v \in W_0^{1,p}(\Omega)$ such that $v_n \to v$ in $W_0^{1,p}(\Omega)$. Hence $v_n \to v$ in $L^p(\Omega)$ (because $W_0^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$) and so $v_n^\star \to v^\star$ in $L^p(\Omega)$ by assumption. In particular,

$$\|v - v^\star\|_p \leq \|v - v_n\|_p + \|v_n - v_n^\star\|_p + \|v_n^\star - v^\star\|_p \to 0.$$

Therefore $v = v^\star$ in $L^p(\Omega)$, as stated.

Lemma 2.2.4. *One has,*

$$\mathcal{E}_\lambda(u^H) \leq \mathcal{E}_\lambda(u).$$

Proof. One has that $\|\nabla u^H\|_p = \|\nabla u\|_p$, and $\|u^H\|_p \leq \|u\|_p$ (see Van Schaftingen [91]). On the other hand, due to Proposition 1.5.1, one has

$$\int_\Omega F(x, u(x))dx = \int_\Omega F(x, u^H(x))dx,$$

therefore

$$\mathcal{E}_\lambda(u^H) \leq \mathcal{E}_\lambda(u).$$

□

Now we can prove the Theorem 2.2.1.

Proof of Theorem 2.2.1: (a) Suppose that $u \in W_0^{1,p}(\Omega)$ is a weak solution of $(\mathcal{P}_\lambda^1)$. Now, if we put $v = u$ as the test function in the relation (2.2), we obtain

$$||u||^p = \int_\Omega (|\nabla u|^p + |u|^p)dx = \lambda \int_\Omega \xi_F u dx \leq \lambda c_F \int_\Omega |u|^p dx \leq \lambda c_F c_p^p ||u||^p,$$

where $c_F = \max_{s>0} \dfrac{\max\{|\xi| : \xi \in \partial F(x,s)\}}{s^{p-1}} > 0$. Therefore, if $\lambda < \frac{1}{c_F c_p^p}$, then $u = 0$.
(b) By Lemma 2.2.3 there exists the global minimum $v_\lambda = v_\lambda^*$ of the energy functional $\mathcal{E}_\lambda$.

We now turn to establish the existence of the second nontrivial solution of $(\mathcal{P}_\lambda^1)$.

From the assumption $(\mathbf{F_3})$, one has

$$\mathcal{E}_\lambda(u_0) = \frac{1}{p}\|u_0\|^p - \lambda \int_\Omega F(x, u_0(x))dx = A - \lambda B,$$

where $A = \frac{1}{p}\|u_0\|^p > 0$, and $B = \int_\Omega F(x, u_0(x))dx > 0$. Consequently, there exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$, we have that $h(\lambda) = A - \lambda B < 0$, therefore

$$\mathcal{E}_\lambda(u_0) = \frac{1}{p}\|u_0\|^p - \lambda \int_\Omega F(x, u_0(x))dx < 0.$$

In fact, we may choose,

$$\lambda_0 = \frac{1}{p} \inf \left\{ \frac{\|u\|^p}{\mathcal{F}(u)} : u \in W_0^{1,p}(\Omega), \mathcal{F}(u) > 0 \right\}.$$

Now, fix $\lambda > \lambda_0$. From $(\mathbf{F_1})$ it follows that for fixed $\frac{1}{p\lambda c_p^p} > \varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that

$$\max\{|\xi| : \xi \in \partial F(x,s)\} \leq \varepsilon |s|^{p-1}, |s| < \delta,$$

therefore for every $\xi \in \partial F(x,s)$, $|s| \leq \delta$ one has,

$$|\xi| \leq \varepsilon \cdot |s|^{p-1}. \tag{2.10}$$

Using the Lebourg's mean value theorem (see Theorem 1.1.3), we obtain:

$$|F(x,s)| = |F(x,s) - F(x,0)| \leq |\xi_\theta s| \text{ for some } \xi_\theta \in \partial F(x, \theta s), \theta \in (0,1),$$

which means that using the (2.10) iequality, we have

$$|F(x,s)| \leq \varepsilon \cdot |s|^p,$$

whenever $|s| \leq \delta$.

Thus, if $u \in W_0^{1,p}(\Omega)$ with $\|u\| = \rho < \min\left\{\frac{\delta}{c_p}, \|u_0\|\right\}$, then

$$\begin{aligned}
\mathcal{E}_\lambda(u) &= \frac{1}{p}||u||^p - \lambda \mathcal{F}(u) \\
&\geq \frac{1}{p}||u||^p - \varepsilon \lambda c_p^p \|u\|^p \\
&= ||u||^p \left(\frac{1}{p} - \varepsilon \lambda c_p^p\right) \\
&= \rho^p \left(\frac{1}{p} - \varepsilon \lambda c_p^p\right) > 0.
\end{aligned}$$

Since $\mathcal{E}_\lambda$ satisfies the Palais-Smale condition and

$$\inf_{\|u\|=\rho} \mathcal{E}_\lambda(u) > 0 = \mathcal{E}_\lambda(0) > \mathcal{E}_\lambda(u_0),$$

we are in the position to apply the Mountain Pass theorem, which means that $c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \mathcal{E}_\lambda(\gamma(t))$ is a critical value of $\mathcal{E}_\lambda$, therefore there exists a critical point u such that $\mathcal{E}_\lambda(u) = c$.

From the definition of c, we have

$$\sup_{t\in[0,1]} \mathcal{E}_\lambda(\gamma(t)) \leq c + \frac{1}{n^2}.$$

From the above inequality and from the fact that we can choose $\gamma(0) = 0$ and $\gamma(1) = u = u^H$, we can apply Theorem 1.5.1 for $\varepsilon = \frac{1}{n^2}$, and $\delta = \frac{1}{n}$. Thus, there exists $u_n \in W_0^{1,p}(\Omega)$ such that

(a) $|\mathcal{E}_\lambda(u_n) - c| \leq \frac{2}{n^2}$;

(b) $\|u_n - u_n^*\|_p \leq 2(2\kappa + 1)\frac{1}{n}$;

(c) $\lambda_{\mathcal{E}_\lambda}(u_n) \leq \frac{8}{n}$.

Since $\mathcal{E}_\lambda$ satisfies the Palais-Smale condition, up to a subsequence u_n converges to u in $W_0^{1,p}(\Omega)$, with $\mathcal{E}_\lambda(u) = c$, $\lambda_{\mathcal{E}_\lambda}(u) = 0$ and $u = u^*$. This means that u is a critical point for the energy functional $\mathcal{E}_\lambda$, different from the critical point obtained in (a) and it is invariant by spherical cap symmetrization as well. □

2.3 Infinitely many solutions for differential inclusions

Motivated by mechanical problems where external forces are non-smooth, we consider the differential inclusion problem

$$\begin{cases} -\Delta u(x) \in \partial F(u(x)) + \lambda \partial G(u(x)) & \text{in} \quad \Omega; \\ u \geq 0, & \text{in} \quad \Omega; \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{$\mathcal{D}_\lambda$}$$

where $\lambda \geq 0$, $p > 0$, $\Omega \subset \mathbb{R}^n$ is a bounded open domain, ∂F and ∂G stand for the generalized gradients of the locally Lipschitz functions $F, G : \mathbb{R}_+ \to \mathbb{R}$. Hereafter we will use the notation $\mathbb{R}_+ = [0, \infty)$.

In this section we provide a quite complete picture on the number of solutions of ($\mathcal{D}_\lambda$) whenever ∂F oscillates near the origin/infinity and ∂G is a generic perturbation of order $p > 0$ at the origin/infinity, respectively.

We follow Kristály, Mezei and Szilák [53] which extends in several aspects the results in Kristály and Moroşanu [55].

2.3.1 Localization: a generic result

First of all, We consider the following differential inclusion problem

$$\begin{cases} -\triangle u(x) + ku(x) \in \partial A(u(x)), \quad u(x) \geq 0 & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{D_A^k}$$

where $k > 0$ and

(H_A^1): $A : [0, \infty) \to \mathbb{R}$ is a locally Lipschitz function with $A(0) = 0$, and there is $M_A > 0$ such that

$$\max\{|\partial A(s)|\} := \max\{|\xi| : \xi \in \partial A(s)\} \leq M_A$$

for every $s \geq 0$;

(H_A^2): there are $0 < \delta < \eta$ such that $\max\{\xi : \xi \in \partial A(s)\} \leq 0$ for every $s \in [\delta, \eta]$.

For simplicity, we extend the function A by $A(s) = 0$ for $s \leq 0$; the extended function is locally Lipschitz on the whole $\mathbb{R}$. The natural energy functional $\mathcal{T}$: $H_0^1(\Omega) \to \mathbb{R}$ associated with the differential inclusion problem (D_A^k) is defined by

$$\mathcal{T}(u) = \frac{1}{2}\|u\|_{H_0^1}^2 + \frac{k}{2}\int_\Omega u^2 dx - \int_\Omega A(u(x))dx.$$

The energy functional $\mathcal{T}$ is well defined and locally Lipschitz on $H_0^1(\Omega)$, while its critical points in the sense of Chang (see Definition 1.1.4) are precisely the weak solutions of the differential inclusion problem

$$\begin{cases} -\triangle u(x) + ku(x) \in \partial A(u(x)), & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega. \end{cases} \tag{$\mathrm{D}_A^{k,0}$}$$

Note that at this stage we have no information on the sign of u.

Indeed, if $0 \in \partial\mathcal{T}(u)$, then for every $v \in H_0^1(\Omega)$ we have

$$\int_\Omega \nabla u(x)\nabla v(x)dx - k\int_\Omega u(x)v(x)dx - \int_\Omega \xi_x(x)v(x)dx = 0,$$

where $\xi_x \in \partial A(u(x))$ a.e. $x \in \Omega$, see e.g. Motreanu and Panagiotopoulos [77]. By using the divergence theorem for the first term at the left hand side (and exploring the Dirichlet boundary condition), we obtain that

$$\int_\Omega \nabla u(x)\nabla v(x)dx = -\int_\Omega \mathrm{div}(\nabla u(x))v(x)dx = -\int_\Omega \Delta u(x)v(x)dx.$$

Accordingly, we have that

$$-\int_\Omega \Delta u(x)v(x)dx + k\int_\Omega u(x)v(x) = \int_\Omega \xi_x v(x)dx$$

for every test function $v \in H_0^1(\Omega)$ which means that $-\Delta u(x) + ku(x) \in \partial A(u(x))$ in the weak sense in Ω, as claimed before.

Let us consider the number $\eta \in \mathbb{R}$ from (H_A^2) and the set

$$W^\eta = \{u \in H_0^1(\Omega) : \|u\|_{L^\infty} \leq \eta\}.$$

Our localization result reads as follows (see [56, Theorem 2.1] for its smooth form):

Theorem 2.3.1. ([53]) *Let $k > 0$ and assume that hypotheses* (H_A^1) *and* (H_A^2) *hold. Then*

(i) *the energy functional $\mathcal{T}$ is bounded from below on W^η and its infimum is attained at some $\tilde{u} \in W^\eta$;*

(ii) *$\tilde{u}(x) \in [0, \delta]$ for a.e. $x \in \Omega$;*

(iii) *$\tilde{u}$ is a weak solution of the differential inclusion* (D_A^k).

Proof. The proof is similar to that of Kristály and Moroşanu [55]; for completeness, we provide its main steps.

(i) Due to (H_A^1), it is clear that the energy functional $\mathcal{T}$ is bounded from below on $H_0^1(\Omega)$. Moreover, due to the compactness of the embedding $H_0^1(\Omega) \subset L^q(\Omega)$, $q \in [2, 2^*)$, it turns out that $\mathcal{T}$ is sequentially weak lower semi-continuous on $H_0^1(\Omega)$. In addition, the set W^η is weakly closed, being convex and closed in $H_0^1(\Omega)$. Thus, there is $\tilde{u} \in W^\eta$ which is a minimum point of $\mathcal{T}$ on the set W^η, cf. Zeidler [100].

(ii) We introduce the set $L = \{x \in \Omega : \tilde{u}(x) \notin [0, \delta]\}$ and suppose indirectly that $m(L) > 0$. Define the function $\gamma : \mathbb{R} \to \mathbb{R}$ by $\gamma(s) = \min(s_+, \delta)$, where $s_+ = \max(s, 0)$. Now, set $w = \gamma \circ \tilde{u}$. It is clear that γ is a Lipschitz function and $\gamma(0) = 0$. Accordingly, based on the superposition theorem of Marcus and Mizel [73], one has that $w \in H_0^1(\Omega)$. Moreover, $0 \le w(x) \le \delta$ for a.e. Ω. Consequently, $w \in W^\eta$.

Let us introduce the sets

$$L_1 = \{x \in L : \tilde{u}(x) < 0\} \quad \text{and} \quad L_2 = \{x \in L : \tilde{u}(x) > \delta\}.$$

In particular, $L = L_1 \cup L_2$, and by definition, it follows that $w(x) = \tilde{u}(x)$ for all $x \in \Omega \setminus L$, $w(x) = 0$ for all $x \in L_1$, and $w(x) = \delta$ for all $x \in L_2$. In addition, one has

$$\begin{aligned}
\mathcal{T}(w) - \mathcal{T}(\tilde{u}) &= \frac{1}{2}\left[\|w\|_{H_0^1}^2 - \|\tilde{u}\|_{H_0^1}^2\right] + \frac{k}{2}\int_\Omega \left[w^2 - \tilde{u}^2\right] - \\
&\quad - \int_\Omega [A(w(x)) - A(\tilde{u}(x))] \\
&= -\frac{1}{2}\int_L |\nabla \tilde{u}|^2 + \frac{k}{2}\int_L [w^2 - \tilde{u}^2] - \int_L [A(w(x)) - A(\tilde{u}(x))].
\end{aligned}$$

On account of $k > 0$, we have

$$k \int_L [w^2 - \tilde{u}^2] = -k \int_{L_1} \tilde{u}^2 + k \int_{L_2} [\delta^2 - \tilde{u}^2] \leq 0.$$

Since $A(s) = 0$ for all $s \leq 0$, we have

$$\int_{L_1} [A(w(x)) - A(\tilde{u}(x))] = 0.$$

By means of the Lebourg's mean value theorem 1.1.3, for a.e. $x \in L_2$, there exists $\theta(x) \in [\delta, \tilde{u}(x)] \subseteq [\delta, \eta]$ such that

$$A(w(x)) - A(\tilde{u}(x)) = A(\delta) - A(\tilde{u}(x)) = a(\theta(x))(\delta - \tilde{u}(x)),$$

where $a(\theta(x)) \in \partial A(\theta(x))$. Due to (H_A^2), it turns out that

$$\int_{L_2} [A(w(x)) - A(\tilde{u}(x))] \geq 0.$$

Therefore, we obtain that $\mathcal{T}(w) - \mathcal{T}(\tilde{u}) \leq 0$. On the other hand, since $w \in W^\eta$, then $\mathcal{T}(w) \geq \mathcal{T}(\tilde{u}) = \inf_{W^\eta} \mathcal{T}$, thus every term in the difference $\mathcal{T}(w) - \mathcal{T}(\tilde{u})$ should be zero; in particular,

$$\int_{L_1} \tilde{u}^2 = \int_{L_2} [\tilde{u}^2 - \delta^2] = 0.$$

The latter relation implies in particular that $m(L) = 0$, which is a contradiction, completing the proof of (ii).

(iii) Since $\tilde{u}(x) \in [0, \delta]$ for a.e. $x \in \Omega$, an arbitrarily small perturbation $\tilde{u} + \epsilon v$ of $\tilde{u}$ with $0 < \epsilon \ll 1$ and $v \in C_0^\infty(\Omega)$ still implies that $\mathcal{T}(\tilde{u} + \varepsilon v) \geq \mathcal{T}(\tilde{u})$; accordingly, $\tilde{u}$ is a minimum point for $\mathcal{T}$ in the strong topology of $H_0^1(\Omega)$, thus $0 \in \partial \mathcal{T}(\tilde{u})$, cf. Remark 1.1.1. Consequently, it follows that $\tilde{u}$ is a weak solution of the differential inclusion (D_A^k). □

In the sequel, we need a truncation function of $H_0^1(\Omega)$, see also [55]. To construct this function, let $B(x_0, r) \subset \Omega$ be the n-dimensional ball with radius $r > 0$ and center $x_0 \in \Omega$. For $s > 0$, define

$$w_s(x) = \begin{cases} 0, & \text{if} \quad x \in \Omega \setminus B(x_0, r); \\ s, & \text{if} \quad x \in B(x_0, r/2); \\ \frac{2s}{r}(r - |x - x_0|), & \text{if} \quad x \in B(x_0, r) \setminus B(x_0, r/2). \end{cases} \tag{2.11}$$

Note that that $w_s \in H_0^1(\Omega)$, $\|w_s\|_{L^\infty} = s$ and

$$\|w_s\|_{H_0^1}^2 = \int_\Omega |\nabla w_s|^2 = 4r^{n-2}(1-2^{-n})\omega_n s^2 \equiv C(r,n)s^2 > 0; \tag{2.12}$$

hereafter ω_n stands for the volume of $B(0,1) \subset \mathbb{R}^n$.

2.3.2 Nonlinearities with oscillation near origin

We assume:

(F_0^0) $F(0) = 0$;

(F_1^0) $-\infty < \liminf_{s\to 0^+} \frac{F(s)}{s^2}$; $\limsup_{s\to 0^+} \frac{F(s)}{s^2} = +\infty$;

(F_2^0) $l_0 := \liminf_{s\to 0^+} \frac{\max\{\xi:\xi\in\partial F(s)\}}{s} < 0$.

(G_0^0) $G(0) = 0$;

(G_1^0) There exist $p > 0$ and $\underline{c}, \overline{c} \in \mathbb{R}$ such that

$$\underline{c} = \liminf_{s\to 0^+} \frac{\min\{\xi : \xi \in \partial G(s)\}}{s^p} \le \limsup_{s\to 0^+} \frac{\max\{\xi : \xi \in \partial G(s)\}}{s^p} = \overline{c}.$$

Remark 2.3.1. Hypotheses (F_1^0) and (F_2^0) imply a strong oscillatory behavior of ∂F near the origin. Moreover, it turns out that $0 \in \partial F(0)$; indeed, if we assume the contrary, by the upper semicontinuity of ∂F we also have that $0 \notin \partial F(s)$ for every small $s > 0$. Thus, by (F_2^0) we have that $\partial F(s) \subset (-\infty, 0]$ for these values of $s > 0$. By using (F_0^0) and Lebourg's mean value theorem (see Proposition 1.1.3), it follows that $F(s) = F(s) - F(0) = \xi s \le 0$ for some $\xi \in \partial F(\theta s) \subset (-\infty, 0]$ with $\theta \in (0,1)$. The latter inequality contradicts the second assumption from (F_1^0). Similarly, one obtains that $0 \in \partial G(0)$ by exploring (G_0^0) and (G_1^0), respectively.

In conclusion, since $0 \in \partial F(0)$ and $0 \in \partial G(0)$, it turns out that $0 \in H_0^1(\Omega)$ is a solution of the differential inclusion ($\mathcal{D}_\lambda$). Clearly, we are interested in nonzero solutions of ($\mathcal{D}_\lambda$).

Example 2.3.1. Let us consider $F_0(s) = \displaystyle\int_0^s f_0(t)$, $s \ge 0$, where $f_0(t) = \sqrt{t}(\frac{1}{2} + \sin t^{-1})$, $t > 0$ and $f_0(0) = 0$, or some of its jumping variants. One can prove that $\partial F_0 = f_0$ verifies the assumptions (F_0^0) – (F_2^0). For a fixed $p > 0$, let $G_0(s) = \ln(1 + s^{p+2})\max\{0, \cos s^{-1}\}$, $s > 0$ and $G_0(0) = 0$. It is clear that G_0 is not of class C^1 and verifies (G_1^0) with $\underline{c} = -1$ and $\overline{c} = 1$, respectively; see Figure 2.1 representing both f_0 and G_0 (for $p = 2$).

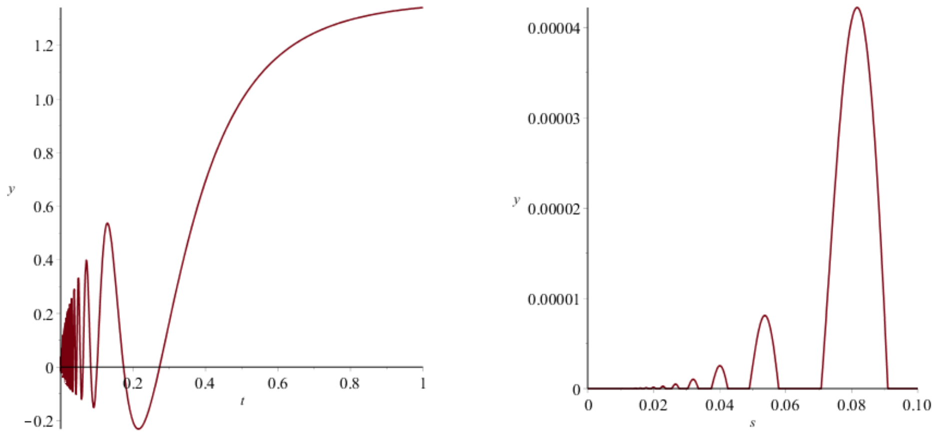

Figure 2.1: Graphs of f_0 and G_0 around the origin, respectively.

In the sequel, we provide a quite complete picture about the competition concerning the terms $s \mapsto \partial F(s)$ and $s \mapsto \partial G(s)$, respectively. First, we are going to show that when $p \geq 1$ then the 'leading' term is the oscillatory function ∂F; roughly speaking, one can say that the effect of $s \mapsto \partial G(s)$ is negligible in this competition. More precisely, we prove the following result.

Theorem 2.3.2. ([53]) (Case $p \geq 1$) *Assume that $p \geq 1$ and the locally Lipschitz functions $F, G : \mathbb{R}_+ \to \mathbb{R}$ satisfy $(F_0^0) - (F_2^0)$ and $(G_0^0) - (G_1^0)$. If*

(i) *either $p = 1$ and $\lambda \bar{c} < -l_0$ (with $\lambda \geq 0$),*

(ii) *or $p > 1$ and $\lambda \geq 0$ is arbitrary,*

then the differential inclusion problem $(\mathcal{D}_\lambda)$ admits a sequence $\{u_i\}_i \subset H_0^1(\Omega)$ of distinct weak solutions such that

$$\lim_{i\to\infty} \|u_i\|_{H_0^1} = \lim_{i\to\infty} \|u_i\|_{L^\infty} = 0.$$

In the case when $p < 1$, the perturbation term ∂G may compete with the oscillatory function ∂F; namely, we have:

Theorem 2.3.3. ([53]) (Case $0 < p < 1$) *Assume $0 < p < 1$ and that the locally Lipschitz functions $F, G : \mathbb{R}_+ \to \mathbb{R}$ satisfy $(F_0^0) - (F_2^0)$ and $(G_0^0) - (G_1^0)$. Then, for every $k \in \mathbb{N}$, there exists $\lambda_k > 0$ such that the differential inclusion $(\mathcal{D}_\lambda)$ has at least k distinct weak solutions $\{u_{1,\lambda}, ..., u_{k,\lambda}\} \subset H_0^1(\Omega)$ whenever $\lambda \in [0, \lambda_k]$. Moreover,*

$$\|u_{i,\lambda}\|_{H_0^1} < i^{-1} \quad \textit{and} \quad \|u_{i,\lambda}\|_{L^\infty} < i^{-1} \quad \textit{for any } i = \overline{1,k};\ \lambda \in [0, \lambda_k]. \tag{2.13}$$

Before proving the main theorems of this subsection, we study the differential inclusion problem

$$\begin{cases} -\triangle u(x) + ku(x) \in \partial A(u(x)), \quad u(x) \geq 0 & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{D_A^k}$$

where $k > 0$ and the locally Lipschitz function $A : \mathbb{R}_+ \to \mathbb{R}$ verifies

(H_0^0): $A(0) = 0$;

(H_1^0): $-\infty < \liminf_{s\to 0^+} \frac{A(s)}{s^2}$ and $\limsup_{s\to 0^+} \frac{A(s)}{s^2} = +\infty$;

(H_2^0): there are two sequences $\{\delta_i\}$, $\{\eta_i\}$ with $0 < \eta_{i+1} < \delta_i < \eta_i$, $\lim_{i\to\infty} \eta_i = 0$, and

$$\max\{\partial A(s)\} := \max\{\xi : \xi \in \partial A(s)\} \leq 0$$

for every $s \in [\delta_i, \eta_i]$, $i \in \mathbb{N}$.

Theorem 2.3.4. ([53]) *Let $k > 0$ and assume hypotheses* (H_0^0), (H_1^0) *and* (H_2^0) *hold. Then there exists a sequence $\{u_i^0\}_i \subset H_0^1(\Omega)$ of distinct weak solutions of the differential inclusion problem* (D_A^k) *such that*

$$\lim_{i\to\infty} \|u_i^0\|_{H_0^1} = \lim_{i\to\infty} \|u_i^0\|_{L^\infty} = 0. \tag{2.14}$$

Proof. We may assume that $\{\delta_i\}_i$, $\{\eta_i\}_i \subset (0, 1)$. For any fixed number $i \in \mathbb{N}$, we define the locally Lipschitz function $A_i : \mathbb{R} \to \mathbb{R}$ by

$$A_i(s) = A(\tau_{\eta_i}(s)), \tag{2.15}$$

where $A(s) = 0$ for $s \leq 0$ and $\tau_\eta : \mathbb{R} \to \mathbb{R}$ denotes the truncation function $\tau_\eta(s) = \min(\eta, s)$, $\eta > 0$. For further use, we introduce the energy functional $\mathcal{T}_i : H_0^1(\Omega) \to \mathbb{R}$ associated with problem ($\mathrm{D}_{A_i}^k$).

We notice that for $s \geq 0$, the chain rule (see Proposition 1.1.4) gives

$$\partial A_i(s) = \begin{cases} \partial A(s) & \text{if} \quad s < \eta_i, \\ \overline{\text{co}}\{0, \partial A(\eta_i)\} & \text{if} \quad s = \eta_i, \\ \{0\} & \text{if} \quad s > \eta_i. \end{cases}$$

It turns out that on the compact set $[0, \eta_i]$, the upper semicontinuous set-valued map $s \mapsto \partial A_i(s)$ attains its supremum (see Proposition 1.1.1); therefore, there exists $M_{A_i} > 0$ such that

$$\max |\partial A_i(s)| := \max\{|\xi| : \xi \in \partial A_i(s)\} \leq M_{A_i}$$

for every $s \geq 0$, i.e., $(\mathrm{H}^1_{A_i})$ holds. The same is true for $(\mathrm{H}^2_{A_i})$ by using (H^0_2) on $[\delta_i, \eta_i]$, $i \in \mathbb{N}$.

Accordingly, the assumptions of Theorem 2.3.1 are verified for every $i \in \mathbb{N}$ with $[\delta_i, \eta_i]$, thus there exists $u_i^0 \in W^{\eta_i}$ such that

$$u_i^0 \text{ is the minimum point of the functional } \mathcal{T}_i \text{ on } W^{\eta_i}, \tag{2.16}$$

$$u_i^0(x) \in [0, \delta_i] \text{ for a.e. } x \in \Omega, \tag{2.17}$$

$$u_i^0 \text{ is a solution of } (\mathrm{D}^k_{A_i}). \tag{2.18}$$

On account of relations (2.15), (2.17) and (2.18), u_i^0 is a weak solution also for the differential inclusion problem (D^k_A).

We are going to prove that there are infinitely many distinct elements in the sequence $\{u_i^0\}_i$. To conclude it, we first prove that

$$\mathcal{T}_i(u_i^0) < 0 \quad \text{for all } i \in \mathbb{N}; \text{ and} \tag{2.19}$$

$$\lim_{i \to \infty} \mathcal{T}_i(u_i^0) = 0. \tag{2.20}$$

The left part of (H^0_1) implies the existence of some $l_0 > 0$ and $\zeta \in (0, \eta_1)$ such that

$$A(s) \geq -l_0 s^2 \text{ for all } s \in (0, \zeta). \tag{2.21}$$

One can choose $L_0 > 0$ such that

$$\frac{1}{2}C(r,n) + \left(\frac{k}{2} + l_0\right) m(\Omega) < L_0 (r/2)^n \omega_n, \tag{2.22}$$

where $r > 0$ and $C(r,n) > 0$ come from (2.12). Based on the right part of (H^0_1), one can find a sequence $\{\tilde{s}_i\}_i \subset (0, \zeta)$ such that $\tilde{s}_i \leq \delta_i$ and

$$A(\tilde{s}_i) > L_0 \tilde{s}_i^2 \quad \text{for all } i \in \mathbb{N}. \tag{2.23}$$

Let $i \in \mathbb{N}$ be a fixed number and let $w_{\tilde{s}_i} \in H^1_0(\Omega)$ be the function from (2.11) corresponding to the value $\tilde{s}_i > 0$. Then $w_{\tilde{s}_i} \in W^{\eta_i}$, and due to (2.21), (2.23) and (2.12) one has

$$\begin{aligned}
\mathcal{T}_i(w_{\tilde{s}_i}) &= \frac{1}{2}\|w_{\tilde{s}_i}\|^2_{H^1_0} + \frac{k}{2}\int_\Omega w^2_{\tilde{s}_i} - \int_\Omega A_i(w_{\tilde{s}_i}(x))dx \\
&= \frac{1}{2}C(r,n)\tilde{s}_i^2 + \frac{k}{2}\int_\Omega w^2_{\tilde{s}_i} - \int_{B(x_0,r/2)} A(\tilde{s}_i)dx - \\
&\quad - \int_{B(x_0,r)\setminus B(x_0,r/2)} A(w_{\tilde{s}_i}(x))dx \\
&\leq \left[\frac{1}{2}C(r,n) + \frac{k}{2}m(\Omega) - L_0(r/2)^n\omega_n + l_0 m(\Omega)\right]\tilde{s}_i^2.
\end{aligned}$$

Accordingly, with (2.16) and (2.22), we conclude that

$$\mathcal{T}_i(u_i^0) = \min_{W^{\eta_i}} \mathcal{T}_i \leq \mathcal{T}_i(w_{\tilde{s}_i}) < 0 \tag{2.24}$$

which completes the proof of (2.19).

Now, we prove (2.20). For every $i \in \mathbb{N}$, by using the Lebourg's mean value theorem, relations (2.15) and (2.17) and (H_0^0) , we have

$$\mathcal{T}_i(u_i^0) \geq -\int_\Omega A_i(u_i^0(x))dx = -\int_\Omega A_1(u_i^0(x))dx \geq -M_{A_1} m(\Omega)\delta_i.$$

Since $\lim_{i\to\infty} \delta_i = 0$, the latter estimate and (2.24) provides relation (2.20).

Based on (2.15) and (2.17), we have that $\mathcal{T}_i(u_i^0) = \mathcal{T}_1(u_i^0)$ for all $i \in \mathbb{N}$. This relation with (2.19) and (2.20) means that the sequence $\{u_i^0\}_i$ contains infinitely many distinct elements.

We now prove (2.14). One can prove the former limit by (2.17), i.e. $\|u_i^0\|_{L^\infty} \leq \delta_i$ for all $i \in \mathbb{N}$, combined with $\lim_{i\to\infty} \delta_i = 0$. For the latter limit, we use $k > 0$, (2.24), (2.15) and (2.17) to get for all $i \in \mathbb{N}$ that

$$\begin{aligned}\frac{1}{2}\|u_i^0\|_{H_0^1}^2 &\leq \frac{1}{2}\|u_i^0\|_{H_0^1}^2 + \frac{k}{2}\int_\Omega (u_i^0)^2 < \int_\Omega A_i(u_i^0(x)) = \int_\Omega A_1(u_i^0(x)) \\ &\leq M_{A_1} m(\Omega)\delta_i,\end{aligned}$$

which completes the proof. □

Now, we are in position to prove the main theorems of this subsection.

Proof of Theorem 2.3.2. We split the proof into two parts.

(i) *Case* $p = 1$. Let $\lambda \geq 0$ with $\lambda\bar{c} < -l_0$ and fix $\tilde{\lambda}_0 \in \mathbb{R}$ such that $\lambda\bar{c} < \tilde{\lambda}_0 < -l_0$. With these choices we define

$$k := \tilde{\lambda}_0 - \lambda\bar{c} > 0 \text{ and } A(s) := F(s) + \frac{\tilde{\lambda}_0}{2}s^2 + \lambda\left(G(s) - \frac{\bar{c}}{2}s^2\right), \ \forall s \in [0,\infty). \tag{2.25}$$

It is clear that $A(0) = 0$, i.e., (H_0^0) is verified. Since $p = 1$, by (G_1^0) one has

$$\underline{c} = \liminf_{s\to 0^+} \frac{\min\{\partial G(s)\}}{s} \leq \limsup_{s\to 0^+} \frac{\max\{\partial G(s)\}}{s} = \bar{c}.$$

In particular, for sufficiently small $\epsilon > 0$ there exists $\gamma = \gamma(\epsilon) > 0$ such that

$$\max\{\partial G(s)\} - \bar{c}s < \epsilon s, \ \forall s \in [0, \gamma],$$

and
$$\min\{\partial G(s)\} - \underline{c}s > -\epsilon s, \ \forall s \in [0, \gamma].$$

For $s \in [0, \gamma]$, Lebourg's mean value theorem and $G(0) = 0$ implies that there exists $\xi_s \in \partial G(\theta_s s)$ for some $\theta_s \in [0, 1]$ such that $G(s) - G(0) = \xi_s s$. Accordingly, for every $s \in [0, \gamma]$ we have that

$$(\underline{c} - \epsilon)s^2 \leq G(s) \leq (\overline{c} + \epsilon)s^2. \tag{2.26}$$

By (2.26) and (F_1^0) we have that

$$\begin{aligned} \liminf_{s\to 0^+} \frac{A(s)}{s^2} &\geq \liminf_{s\to 0^+} \frac{F(s)}{s^2} + \frac{\tilde{\lambda}_0 - \lambda\overline{c}}{2} + \lambda \liminf_{s\to 0^+} \frac{G(s)}{s^2} \\ &\geq \liminf_{s\to 0^+} \frac{F(s)}{s^2} + \frac{\tilde{\lambda}_0 - \lambda\overline{c}}{2} + \lambda\underline{c} > -\infty \end{aligned}$$

and
$$\limsup_{s\to 0^+} \frac{A(s)}{s^2} \geq \limsup_{s\to 0^+} \frac{F(s)}{s^2} + \frac{\tilde{\lambda}_0 - \lambda\overline{c}}{2} + \lambda \liminf_{s\to 0^+} \frac{G(s)}{s^2} = +\infty,$$

i.e., (H_1^0) is verified.

Since
$$\partial A(s) \subseteq \partial F(s) + \tilde{\lambda}_0 s + \lambda(\partial G(s) - \overline{c}s), \tag{2.27}$$

and $\lambda \geq 0$, we have that

$$\max\{\partial A(s)\} \leq \max\{\partial F(s) + \tilde{\lambda}_0 s\} + \lambda \max\{\partial G(s) - \overline{c}s\}. \tag{2.28}$$

Since
$$\limsup_{s\to 0^+} \frac{\max\{\partial G(s)\}}{s} = \overline{c},$$

cf. (G_1^0), and
$$\liminf_{s\to 0^+} \frac{\max\{\partial F(s)\}}{s} = l_0 < 0,$$

cf. (F_2^0), it turns out by (2.28) that

$$\begin{aligned} \liminf_{s\to 0^+} \frac{\max\{\partial A(s)\}}{s} &\leq \liminf_{s\to 0^+} \frac{\max\{\partial F(s)\}}{s} + \tilde{\lambda}_0 - \lambda\overline{c} + \\ &\quad + \lambda \limsup_{s\to 0^+} \frac{\max\{\partial G(s)\}}{s} \leq l_0 + \tilde{\lambda}_0 < 0. \end{aligned}$$

Therefore, one has a sequence $\{s_i\}_i \subset (0,1)$ converging to 0 such that $\frac{\max\{\partial A(s_i)\}}{s_i} < 0$ i.e., $\max\{\partial A(s_i)\} < 0$ for all $i \in \mathbb{N}$. By using the upper semicontinuity of $s \mapsto \partial A(s)$, we may choose two numbers $\delta_i, \eta_i \in (0,1)$ with $\delta_i < s_i < \eta_i$ such that $\partial A(s) \subset \partial A(s_i) + [-\epsilon_i, \epsilon_i]$ for every $s \in [\delta_i, \eta_i]$, where $\epsilon_i := -\max\{\partial A(s_i)\}/2 > 0$. In particular, $\max\{\partial A(s)\} \leq 0$ for all $s \in [\delta_i, \eta_i]$. Thus, one may fix two sequences $\{\delta_i\}_i, \{\eta_i\}_i \subset (0,1)$ such that $0 < \eta_{i+1} < \delta_i < s_i < \eta_i$, $\lim_{i\to\infty} \eta_i = 0$, and $\max\{\partial A(s)\} \leq 0$ for all $s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$. Accordingly, (H_2^0) is verified as well. Let us apply Theorem 2.3.4 with the choice (2.25), i.e., there exists a sequence $\{u_i\}_i \subset H_0^1(\Omega)$ of different elements such that

$$\begin{cases} -\triangle u_i(x) + (\tilde{\lambda}_0 - \lambda\bar{c})u_i(x) \in \partial F(u_i(x)) + \tilde{\lambda}_0 u_i(x) + \\ \qquad\qquad +\lambda(\partial G(u_i(x)) - \bar{c}u_i(x)) & x \in \Omega, \\ u_i(x) \geq 0 & x \in \Omega, \\ u_i(x) = 0 & x \in \partial\Omega, \end{cases}$$

where we used the inclusion (2.27). In particular, u_i solves problem $(\mathcal{D}_\lambda)$, $i \in \mathbb{N}$, which completes the proof of (i).

(ii) *Case* $p > 1$. Let $\lambda \geq 0$ be arbitrary fixed and choose a number $\lambda_0 \in (0, -l_0)$. Let

$$k := \lambda_0 > 0 \quad \text{and} \quad A(s) := F(s) + \lambda G(s) + \lambda_0 \frac{s^2}{2} \ \text{ for every } s \in [0,\infty). \tag{2.29}$$

Since $F(0) = G(0) = 0$, hypothesis (H_0^0) clearly holds. By (G_1^0) one has

$$\underline{c} = \liminf_{s\to 0^+} \frac{\min\{\partial G(s)\}}{s^p} \leq \limsup_{s\to 0^+} \frac{\max\{\partial G(s)\}}{s^p} = \bar{c}.$$

In particular, since $p > 1$, then

$$\lim_{s\to 0^+} \frac{\min\{\partial G(s)\}}{s} = \lim_{s\to 0^+} \frac{\max\{\partial G(s)\}}{s} = 0 \tag{2.30}$$

and for sufficiently small $\epsilon > 0$ there exists $\gamma = \gamma(\epsilon) > 0$ such that

$$\max\{\partial G(s)\} - \bar{c}s^p < \epsilon s^p, \ \forall s \in [0,\gamma]$$

and

$$\min\{\partial G(s)\} - \underline{c}s^p > -\epsilon s^p, \ \forall s \in [0,\gamma].$$

For a fixed $s \in [0,\gamma]$, by Lebourg's mean value theorem and $G(0) = 0$ we conclude again that $G(s) - G(0) = \xi_s s$. Accordingly, for sufficiently small $\epsilon > 0$ there exists

$\gamma = \gamma(\epsilon) > 0$ such that $(\underline{c} - \epsilon)s^{p+1} \leq G(s) \leq (\overline{c} + \epsilon)s^{p+1}$ for every $s \in [0, \gamma]$. Thus, since $p > 1$,

$$\lim_{s\to 0^+} \frac{G(s)}{s^2} = \lim_{s\to 0^+} \frac{G(s)}{s^{p+1}} s^{p-1} = 0.$$

Therefore, by using (2.29) and (F_1^0), we conclude that

$$\liminf_{s\to 0^+} \frac{A(s)}{s^2} = \liminf_{s\to 0^+} \frac{F(s)}{s^2} + \lambda \lim_{s\to 0^+} \frac{G(s)}{s^2} + \frac{\lambda_0}{2} > -\infty,$$

and

$$\limsup_{s\to 0^+} \frac{A(s)}{s^2} = \infty,$$

i.e., (H_0^1) holds. Since

$$\partial A(s) \subseteq \partial F(s) + \lambda \partial G(s) + \lambda_0 s,$$

and $\lambda \geq 0$, we have that

$$\max\{\partial A(s)\} \leq \max\{\partial F(s)\} + \max\{\lambda \partial G(s) + \lambda_0 s\}.$$

Since

$$\limsup_{s\to 0^+} \frac{\max\{\partial G(s)\}}{s^p} = \overline{c},$$

cf. (G_1^0), and

$$\liminf_{s\to 0^+} \frac{\max\{\partial F(s)\}}{s} = l_0,$$

cf. (F_2^0), by relation (2.30) it turns out that

$$\begin{aligned}
\liminf_{s\to 0^+} \frac{\max\{\partial A(s)\}}{s} &= \liminf_{s\to 0^+} \frac{\max\{\partial F(s)\}}{s} + \lambda \lim_{s\to 0^+} \frac{\max\{\partial G(s)\}}{s} + \lambda_0 \\
&= l_0 + \lambda_0 < 0,
\end{aligned}$$

and the upper semicontinuity of ∂A implies the existence of two sequences $\{\delta_i\}_i$ and $\{\eta_i\}_i \subset (0,1)$ such that $0 < \eta_{i+1} < \delta_i < s_i < \eta_i$, $\lim_{i\to\infty} \eta_i = 0$, and $\max\{\partial A(s)\} \leq 0$ for all $s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$. Therefore, hypothesis (H_2^0) holds. Now, we can apply Theorem 2.3.4, i.e., there is a sequence $\{u_i\}_i \subset H_0^1(\Omega)$ of different elements such that

$$\begin{cases}
-\triangle u_i(x) + \lambda_0 u_i(x) \in \partial A(u_i(x)) \subseteq \partial F(u_i(x)) + & \\
\qquad\qquad + \lambda \partial G(u_i(x)) + \lambda_0 u_i(x) & x \in \Omega, \\
u_i(x) \geq 0 & x \in \Omega, \\
u_i(x) = 0 & x \in \partial\Omega,
\end{cases}$$

which means that u_i solves problem $(\mathcal{D}_\lambda)$, $i \in \mathbb{N}$. This completes the proof of Theorem 2.3.2. $\square$

Proof of Theorem 2.3.3. The proof is done in two steps:
(i) Let $\lambda_0 \in (0, -l_0)$, $\lambda \geq 0$ and define

$$k := \lambda_0 > 0 \;\text{ and }\; A^\lambda(s) := F(s) + \lambda G(s) + \lambda_0 \frac{s^2}{2} \text{ for every } s \in [0, \infty).$$

One can observe that $\partial A^\lambda(s) \subseteq \partial F(s) + \lambda_0 s + \lambda \partial G(s)$ for every $s \geq 0$. On account of (F_2^0), there is a sequence $\{s_i\}_i \subset (0,1)$ converging to 0 such that

$$\max\{\partial A^{\lambda=0}(s_i)\} \leq \max\{\partial F(s_i)\} + \lambda_0 s_i < 0.$$

Thus, due to the upper semicontinuity of $(s, \lambda) \mapsto \partial A^\lambda(s)$, we can choose three sequences $\{\delta_i\}_i, \{\eta_i\}_i, \{\lambda_i\}_i \subset (0,1)$ such that $0 < \eta_{i+1} < \delta_i < s_i < \eta_i, \lim_{i\to\infty} \eta_i = 0$, and

$$\max\{\partial A^\lambda(s)\} \leq 0 \text{ for all } \lambda \in [0, \lambda_i], s \in [\delta_i, \eta_i],\; i \in \mathbb{N}.$$

Without any loss of generality, we may choose

$$\delta_i \leq \min\{i^{-1}, 2^{-1} i^{-2}[1 + m(\Omega)(\max_{s\in[0,1]} |\partial F(s)| + \max_{s\in[0,1]} |\partial G(s)|)]^{-1}\}. \tag{2.31}$$

For every $i \in \mathbb{N}$ and $\lambda \in [0, \lambda_i]$, let $A_i^\lambda : [0, \infty) \to \mathbb{R}$ be defined as

$$A_i^\lambda(s) = A^\lambda(\tau_{\eta_i}(s)), \tag{2.32}$$

and the energy functional $\mathcal{T}_{i,\lambda} : H_0^1(\Omega) \to \mathbb{R}$ associated with the differential inclusion problem$(\mathrm{D}^k_{A_i^\lambda})$ is given by

$$\mathcal{T}_{i,\lambda}(u) = \frac{1}{2}\|u\|^2_{H_0^1} + \frac{k}{2}\int_\Omega u^2 dx - \int_\Omega A_i^\lambda(u(x))dx.$$

One can easily check that for every $i \in \mathbb{N}$ and $\lambda \in [0, \lambda_i]$, the function A_i^λ verifies the hypotheses of Theorem 2.3.1. Accordingly, for every $i \in \mathbb{N}$ and $\lambda \in [0, \lambda_i]$:

$$\mathcal{T}_{i,\lambda} \text{ attains its infinum on } W^{\eta_i} \text{ at some } u^0_{i,\lambda} \in W^{\eta_i} \tag{2.33}$$

$$u^0_{i,\lambda}(x) \in [0, \delta_i] \text{ for a.e. } x \in \Omega; \tag{2.34}$$

$$u^0_{i,\lambda} \text{ is a weak solution of } (\mathrm{D}^k_{A_i^\lambda}). \tag{2.35}$$

By the choice of the function A^λ and $k > 0$, $u^0_{i,\lambda}$ is also a solution to the differential inclusion problem $(\mathrm{D}^k_{A^\lambda})$, so $(\mathcal{D}_\lambda)$.

(ii) It is clear that for $\lambda = 0$, the set-valued map $\partial A^\lambda_i = \partial A^0_i$ verifies the hypotheses of Theorem 2.3.4. In particular, $\mathcal{T}_i := \mathcal{T}_{i,0}$ is the energy functional associated with problem $(\mathrm{D}^k_{A^0_i})$. Consequently, the elements $u^0_i := u^0_{i,0}$ verify not only (2.33)-(2.35) but also

$$\mathcal{T}_i(u^0_i) = \min_{W^{\eta_i}} \mathcal{T}_i \le \mathcal{T}_i(w_{\tilde{s}_i}) < 0 \text{ for all } i \in \mathbb{N}. \tag{2.36}$$

Similarly to Kristály and Moroşanu [55], let $\{\theta_i\}_i$ be a sequence with negative terms such that $\lim_{i\to\infty} \theta_i = 0$. Due to (2.36) we may assume that

$$\theta_i < \mathcal{T}_i(u^0_i) \le \mathcal{T}_i(w_{\tilde{s}_i}) < \theta_{i+1}. \tag{2.37}$$

Let us choose

$$\lambda_i^{'} = \frac{\theta_{i+1} - \mathcal{T}_i(w_{\tilde{s}_i})}{m(\Omega)\max_{s\in[0,1]}|G(s)| + 1} \text{ and } \lambda_i^{''} = \frac{\mathcal{T}_i(u^0_i) - \theta_i}{m(\Omega)\max_{s\in[0,1]}|G(s)| + 1}\ ,\ i \in \mathbb{N},$$

and for a fixed $k \in \mathbb{N}$, set

$$\lambda^0_k = \min(1, \lambda_1, ..., \lambda_k, \lambda_1^{'}, ..., \lambda_k^{'}, \lambda_1^{''}, ..., \lambda_k^{''}) > 0.$$

Having in our mind these choices, for every $i \in \{1, ..., k\}$ and $\lambda \in [0, \lambda^0_k]$ one has

$$\begin{aligned} \mathcal{T}_{i,\lambda}(u^0_{i,\lambda}) &\le \mathcal{T}_{i,\lambda}(w_{\tilde{s}_i}) = \frac{1}{2}\|w_{\tilde{s}_i}\|^2_{H^1_0} - \int_\Omega F(w_{\tilde{s}_i}(x))dx - \lambda \int_\Omega G(w_{\tilde{s}_i}(x))dx \\ &= \mathcal{T}_i(w_{\tilde{s}_i}) - \lambda \int_\Omega G(w_{\tilde{s}_i}(x))dx \\ &< \theta_{i+1}, \end{aligned} \tag{2.38}$$

and due to $u^0_{i,\lambda} \in W^{\eta_i}$ and to the fact that u^0_i is the minimum point of $\mathcal{T}_i$ on the set W^{η_i}, by (2.37) we also have

$$\mathcal{T}_{i,\lambda}(u^0_{i,\lambda}) = \mathcal{T}_i(u^0_{i,\lambda}) - \lambda \int_\Omega G(u^0_{i,\lambda}(x))dx \ge \mathcal{T}_i(u^0_i) - \lambda \int_\Omega G(u^0_{i,\lambda}(x))dx > \theta_i. \tag{2.39}$$

Therefore, by (2.38) and (2.39), for every $i \in \{1, ..., k\}$ and $\lambda \in [0, \lambda^0_k]$, one has

$$\theta_i < \mathcal{T}_{i,\lambda}(u^0_{i,\lambda}) < \theta_{i+1},$$

thus

$$\mathcal{T}_{1,\lambda}(u^0_{1,\lambda}) < ... < \mathcal{T}_{k,\lambda}(u^0_{k,\lambda}) < 0.$$

We notice that $u^0_i \in W^{\eta_1}$ for every $i \in \{1, ..., k\}$, so $\mathcal{T}_{i,\lambda}(u^0_{i,\lambda}) = \mathcal{T}_{1,\lambda}(u^0_{i,\lambda})$ because of (2.32). Therefore, we conclude that for every $\lambda \in [0, \lambda^0_k]$,

$$\mathcal{T}_{1,\lambda}(u^0_{1,\lambda}) < ... < \mathcal{T}_{1,\lambda}(u^0_{k,\lambda}) < 0 = \mathcal{T}_{1,\lambda}(0).$$

Based on these inequalities, it turns out that the elements $u^0_{1,\lambda}, ..., u^0_{k,\lambda}$ are distinct and non-trivial whenever $\lambda \in [0, \lambda^0_k]$.

Now, we are going to prove the estimate (2.13). We have for every $i \in \{1, ..., k\}$ and $\lambda \in [0, \lambda^0_k]$:

$$\mathcal{T}_{1,\lambda}(u^0_{i,\lambda}) = \mathcal{T}_{i,\lambda}(u^0_{i,\lambda}) < \theta_{i+1} < 0.$$

By Lebourg's mean value theorem and (2.31), we have for every $i \in \{1, ..., k\}$ and $\lambda \in [0, \lambda^0_k]$ that

$$\begin{aligned}\frac{1}{2}\|u^0_{i,\lambda}\|^2_{H^1_0} &< \int_\Omega F(u^0_{i,\lambda}(x))dx + \lambda \int_\Omega G(u^0_{i,\lambda}(x))dx \\ &\leq m(\Omega)\delta_i[\max_{s\in[0,1]} |\partial F(s)| + \max_{s\in[0,1]} |\partial G(s)|] \\ &\leq \frac{1}{2i^2}.\end{aligned}$$

This completes the proof of Theorem 2.3.3. □

2.3.3 Nonlinearities with oscillation at the infinity

Let assume:

(F^∞_0) $F(0) = 0$;

(F^∞_1) $-\infty < \liminf_{s\to\infty} \frac{F(s)}{s^2}$; $\limsup_{s\to\infty} \frac{F(s)}{s^2} = +\infty$;

(F^∞_2) $l_\infty := \liminf_{s\to\infty} \frac{\max\{\xi:\xi\in\partial F(s)\}}{s} < 0$.

(G^∞_0) $G(0) = 0$;

(G^∞_1) There exist $p > 0$ and $\underline{c}, \overline{c} \in \mathbb{R}$ such that

$$\underline{c} = \liminf_{s\to\infty} \frac{\min\{\xi : \xi \in \partial G(s)\}}{s^p} \leq \limsup_{s\to\infty} \frac{\max\{\xi : \xi \in \partial G(s)\}}{s^p} = \overline{c}.$$

Remark 2.3.2. Hypotheses (F_1^∞) and (F_2^∞) imply a strong oscillatory behavior of the set-valued map ∂F at infinity.

Example 2.3.2. We consider $F_\infty(s) = \int_0^s f_\infty(t)$, $s \geq 0$, where $f_\infty(t) = \sqrt{t}(\frac{1}{2} + \sin t)$, $t \geq 0$, or some of its jumping variants; one has that F_∞ verifies the assumptions $(F_0^\infty) - (F_2^\infty)$. For a fixed $p > 0$, let $G_\infty(s) = s^p \max\{0, \sin s\}$, $s \geq 0$; it is clear that G_∞ is a typically locally Lipschitz function on $[0, \infty)$ (not being of class C^1) and verifies (G_1^∞) with $\underline{c} = -1$ and $\overline{c} = 1$; see Figure 2.2 representing both f_∞ and G_∞ (for $p = 2$), respectively.

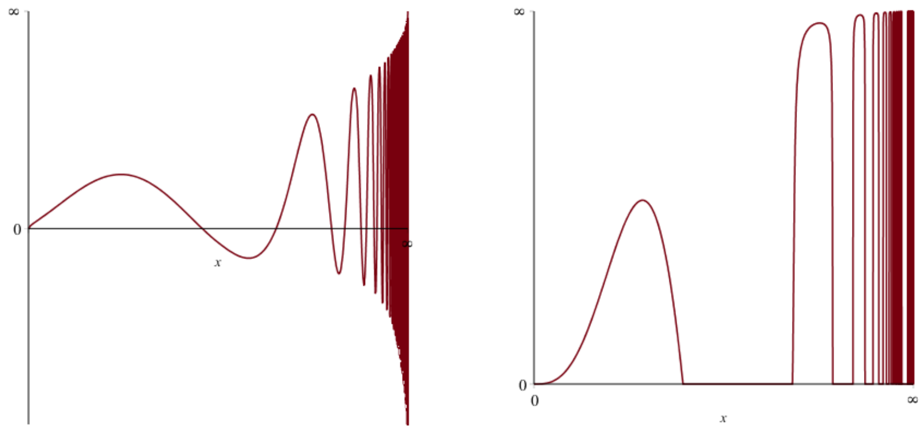

Figure 2.2: Graphs of f_∞ and G_∞ at infinity, respectively.

In the sequel, we investigate the competition at infinity concerning the terms $s \mapsto \partial F(s)$ and $s \mapsto \partial G(s)$, respectively. First, we show that when $p \leq 1$ then the 'leading' term is the oscillatory function F, i.e., the effect of $s \mapsto \partial G(s)$ is negligible. More precisely, we prove the following result:

Theorem 2.3.5. ([53]) (Case $p \leq 1$) *Assume that $p \leq 1$ and the locally Lipschitz functions $F, G : \mathbb{R}_+ \to \mathbb{R}$ satisfy $(F_0^\infty) - (F_2^\infty)$ and $(G_0^\infty) - (G_1^\infty)$. If*

(i) *either $p = 1$ and $\lambda\overline{c} \leq -l_0$ (with $\lambda \geq 0$),*

(ii) *or $p < 1$ and $\lambda \geq 0$ is arbitrary,*

then the differential inclusion $(\mathcal{D}_\lambda)$ admits a sequence $\{u_i\}_i \subset H_0^1(\Omega)$ of distinct weak solutions such that

$$\lim_{i\to\infty} \|u_i^\infty\|_{L^\infty} = \infty. \tag{2.40}$$

Remark 2.3.3. Let 2^* be the usual critical Sobolev exponent. In addition to (2.40), we also have $\lim_{i\to\infty} \|u_i^\infty\|_{H_0^1} = \infty$ whenever

$$\sup_{s\in[0,\infty)} \frac{\max\{|\xi| : \xi \in \partial F(s)\}}{1+s^{2^*-1}} < \infty. \tag{2.41}$$

In the case when $p > 1$, it turns out that the perturbation term ∂G may compete with the oscillatory function ∂F; more precisely, we have:

Theorem 2.3.6. ([53])(Case $p > 1$) *Assume that $p > 1$ and the locally Lipschitz functions $F, G : \mathbb{R}_+ \to \mathbb{R}$ satisfy $(F_0^\infty)-(F_2^\infty)$ and $(G_0^\infty)-(G_1^\infty)$. Then, for every $k \in \mathbb{N}$, there exists $\lambda_k^\infty > 0$ such that the differential inclusion $(\mathcal{D}_\lambda)$ has at least k distinct weak solutions $\{u_{1,\lambda}, ..., u_{k,\lambda}\} \subset H_0^1(\Omega)$ whenever $\lambda \in [0, \lambda_k^\infty]$. Moreover,*

$$\|u_{i,\lambda}\|_{L^\infty} > i-1 \quad \textit{for any } i = \overline{1,k};\ \lambda \in [0, \lambda_k^\infty]. \tag{2.42}$$

Remark 2.3.4. If (2.41) holds and $p \leq 2^*-1$ in Theorem 2.41, then we have in addition that

$$\|u_{i,\lambda}^\infty\|_{H_0^1} > i-1 \ \text{ for any } i = \overline{1,k};\ \lambda \in [0, \lambda_k^\infty].$$

We consider again the differential inclusion problem

$$\begin{cases} -\triangle u(x) + ku(x) \in \partial A(u(x)), \quad u(x) \geq 0 & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \qquad (\mathrm{D}_A^k),$$

where $k > 0$ and the locally Lipschitz function $A : \mathbb{R}_+ \to \mathbb{R}$ verifies

(H_0^∞): $A(0) = 0$;

(H_1^∞): $-\infty < \liminf_{s\to\infty} \frac{A(s)}{s^2}$ and $\limsup_{s\to\infty} \frac{A(s)}{s^2} = +\infty$;

(H_2^∞): there are two sequences $\{\delta_i\}$, $\{\eta_i\}$ with $0 < \delta_i < \eta_i < \delta_{i+1}$, $\lim_{i\to\infty} \delta_i = \infty$, and

$$\max\{\partial A(s)\} := \max\{\xi : \xi \in \partial A(s)\} \leq 0$$

for every $s \in [\delta_i, \eta_i]$, $i \in \mathbb{N}$.

The counterpart of Theorem 2.3.4 reads as follows.

Theorem 2.3.7. ([53]) *Let $k > 0$ and assume the hypotheses* (H_0^∞), (H_1^∞) *and* (H_2^∞) *hold. Then the differential inclusion problem* (D_A^k) *admits a sequence* $\{u_i^\infty\}_i$ *in* $H_0^1(\Omega)$ *of distinct weak solutions such that*

$$\lim_{i\to\infty} \|u_i^\infty\|_{L^\infty} = \infty. \tag{2.43}$$

Proof. The proof is similar to the one performed in Theorem 2.3.4; we shall show the differences only. We associate the energy functional $\mathcal{T}_i : H_0^1(\Omega) \to \mathbb{R}$ with problem $(\mathrm{D}_{A_i}^k)$, where $A_i : \mathbb{R} \to \mathbb{R}$ is given by

$$A_i(s) = A(\tau_{\eta_i}(s)), \tag{2.44}$$

with $A(s) = 0$ for $s \leq 0$. One can show that there exists $M_{A_i} > 0$ such that

$$\max|\partial A_i(s)| := \max\{|\xi| : \xi \in \partial A_i(s)\} \leq M_{A_i}$$

for all $s \geq 0$, i.e, hypothesis $(\mathrm{H}_{A_i}^1)$ holds. Moreover, $(\mathrm{H}_{A_i}^2)$ follows by (H_2^∞). Thus Theorem 2.3.4 can be applied for all $i \in \mathbb{N}$, i.e., we have an element $u_i^\infty \in W^{\eta_i}$ such that

$$u_i^\infty \text{ is the minimum point of the functional } \mathcal{T}_i \text{ on } W^{\eta_i}, \tag{2.45}$$

$$u_i^\infty(x) \in [0, \delta_i] \text{ for a.e. } x \in \Omega,$$

$$u_i^\infty \text{ is a weak solution of } (\mathrm{D}_{A_i}^k).$$

By (2.44), u_i^∞ turns to be a weak solution also for differential inclusion problem (D_A^k).

We shall prove that there are infinitely many distinct elements in the sequence $\{u_i^\infty\}_i$ by showing that

$$\lim_{i\to\infty} \mathcal{T}_i(u_i^\infty) = -\infty. \tag{2.46}$$

By the left part of (H_1^∞) we can find $l_\infty^A > 0$ and $\zeta > 0$ such that

$$A(s) \geq -l_\infty^A \text{ for all } s > \zeta. \tag{2.47}$$

Let us choose $L_\infty^A > 0$ large enough such that

$$\frac{1}{2}C(r,n) + \left(\frac{k}{2} + l_\infty^A\right) m(\Omega) < L_\infty^A (r/2)^n \omega_n. \tag{2.48}$$

On account of the right part of (H_1^∞), one can fix a sequence $\{\tilde{s}_i\}_i \subset (0,\infty)$ such that $\lim_{i\to\infty} \tilde{s}_i = \infty$ and

$$A(\tilde{s}_i) > L_\infty^A {\tilde{s}_i}^2 \text{ for every } i \in \mathbb{N}. \tag{2.49}$$

We know from (H_2^∞) that $\lim_{i\to\infty} \delta_i = \infty$, therefore one has a subsequence $\{\delta_{m_i}\}_i$ of $\{\delta_i\}_i$ such that $\tilde{s}_i \le \delta_{m_i}$ for all $i \in \mathbb{N}$. Let $i \in \mathbb{N}$, and recall $w_{si} \in H_0^1(\Omega)$ from (2.11) with $s_i := \tilde{s}_i > 0$. Then $w_{\tilde{s}_i} \in W^{\eta_{m_i}}$ and according to (2.12), (2.47) and (2.49) we have

$$\begin{aligned}
\mathcal{T}_{mi}(w_{\tilde{s}_i}) &= \frac{1}{2}\|w_{\tilde{s}_i}\|^2_{H_0^1} + \frac{k}{2}\int_\Omega w^2_{\tilde{s}_i} - \int_\Omega A_{m_i}(w_{\tilde{s}_i}(x))dx \\
&= \frac{1}{2}C(r,n)\tilde{s}_i^2 + \frac{k}{2}\int_\Omega w^2_{\tilde{s}_i} - \int_{B(x_0,r/2)} A(\tilde{s}_i)dx \\
&\quad - \int_{(B(x_0,r)\setminus B(x_0,r/2))\cap\{w_{\tilde{s}_i}>\zeta\}} A(w_{\tilde{s}_i}(x))dx \\
&\quad - \int_{(B(x_0,r)\setminus B(x_0,r/2))\cap\{w_{\tilde{s}_i}\le\zeta\}} A(w_{\tilde{s}_i}(x))dx \\
&\le \left[\frac{1}{2}C(r,n) + \frac{k}{2}m(\Omega) - L_\infty^A (r/2)^n\omega_n + l_\infty^A m(\Omega)\right]\tilde{s}_i^2 \\
&\quad + \tilde{M}_A m(\Omega)\zeta,
\end{aligned}$$

where $\tilde{M}_A = \max\{|A(s)| : s \in [0,\zeta]\}$ does not depend on $i \in \mathbb{N}$. This estimate combined by (2.48) and $\lim_{i\to\infty} \tilde{s}_i = \infty$ yields that

$$\lim_{i\to\infty} \mathcal{T}_{m_i}(w_{\tilde{s}_i}) = -\infty. \tag{2.50}$$

By equation (2.45), one has

$$\mathcal{T}_{m_i}(u^\infty_{m_i}) = \min_{W^{\eta_{m_i}}} \mathcal{T}_{m_i} \le \mathcal{T}_{m_i}(w_{\tilde{s}_i}).$$

It follows by (2.50) that $\lim_{i\to\infty} \mathcal{T}_{m_i}(u^\infty_{m_i}) = -\infty$.

We notice that the sequence $\{\mathcal{T}_i(u_i^\infty)\}_i$ is non-increasing. Indeed, let $i < k$; due to (2.44) one has that

$$\mathcal{T}_i(u_i^\infty) = \min_{W^{\eta_i}} \mathcal{T}_i = \min_{W^{\eta_i}} \mathcal{T}_k \ge \min_{W^{\eta_k}} \mathcal{T}_k = \mathcal{T}_k(u_k^\infty),$$

which completes the proof of (2.46).

The proof of (2.43) goes in a similar way as in [55]. $\square$

Proof of Theorem 2.3.5. We split the proof into two parts.
(i) *Case* $p = 1$. Let $\lambda \geq 0$ with $\lambda\overline{c} < -l_\infty$ and fix $\tilde{\lambda}_\infty \in \mathbb{R}$ such that $\lambda\overline{c} < \tilde{\lambda}_\infty < -l_\infty$. With these choices, we define

$$k := \tilde{\lambda}_\infty - \lambda\overline{c} > 0 \text{ and } A(s) := F(s) + \frac{\tilde{\lambda}_\infty}{2}s^2 + \lambda\left(G(s) - \frac{\overline{c}}{2}s^2\right) \text{ for every } s \in [0,\infty). \tag{2.51}$$

It is clear that $A(0) = 0$, i.e., (H_0^∞) is verified. A similar argument for the p-order perturbation ∂G as before shows that

$$\begin{aligned}\liminf_{s\to\infty}\frac{A(s)}{s^2} &\geq \liminf_{s\to\infty}\frac{F(s)}{s^2} + \frac{\tilde{\lambda}_\infty - \lambda\overline{c}}{2} + \lambda\liminf_{s\to\infty}\frac{G(s)}{s^2}\\ &\geq \liminf_{s\to\infty}\frac{F(s)}{s^2} + \frac{\tilde{\lambda}_\infty - \lambda\overline{c}}{2} + \lambda\underline{c} > -\infty,\end{aligned}$$

and

$$\limsup_{s\to\infty}\frac{A(s)}{s^2} \geq \limsup_{s\to\infty}\frac{F(s)}{s^2} + \frac{\tilde{\lambda}_\infty - \lambda\overline{c}}{2} + \lambda\liminf_{s\to\infty}\frac{G(s)}{s^2} = +\infty,$$

i.e., (H_1^∞) is verified.

Since

$$\partial A(s) \subseteq \partial F(s) + \tilde{\lambda}_\infty s + \lambda(\partial G(s) - \overline{c}s), \quad s \geq 0, \tag{2.52}$$

it turns out that

$$\begin{aligned}\liminf_{s\to\infty}\frac{\max\{\partial A(s)\}}{s} &\leq \liminf_{s\to\infty}\frac{\max\{\partial F(s)\}}{s} + \tilde{\lambda}_\infty - \lambda\overline{c} +\\ &\quad +\lambda\limsup_{s\to\infty}\frac{\max\{\partial G(s)\}}{s}\\ &= l_\infty + \tilde{\lambda}_\infty < 0.\end{aligned}$$

By using the upper semicontinuity of $s \mapsto \partial A(s)$, one may fix two sequences $\{\delta_i\}_i, \{\eta_i\}_i \subset (0,\infty)$ such that $0 < \delta_i < s_i < \eta_i < \delta_{i+1}$, $\lim_{i\to\infty}\delta_i = \infty$, and $\max\{\partial A(s)\} \leq 0$ for all $s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$. Thus, (H_2^∞) is verified as well. By applying the inclusion (2.52) and Theorem 2.3.4 with the choice (2.51), there exists a sequence $\{u_i\}_i \subset H_0^1(\Omega)$ of different elements such that

$$\begin{cases} -\triangle u_i(x) + (\tilde{\lambda}_\infty - \lambda\overline{c})u_i(x) \in \partial F(u_i(x)) + \tilde{\lambda}_\infty u_i(x) + & \\ \qquad\qquad +\lambda(\partial G(u_i(x)) - \overline{c}u_i(x)) & x \in \Omega,\\ u_i(x) \geq 0 & x \in \Omega,\\ u_i(x) = 0 & x \in \partial\Omega,\end{cases}$$

i.e., u_i solves problem $(\mathcal{D}_\lambda)$, $i \in \mathbb{N}$.

(ii) *Case* $p < 1$. Let $\lambda \geq 0$ be arbitrary fixed and choose a number $\lambda_\infty \in (0, -l_\infty)$. Let

$$k := \lambda_\infty > 0 \ \text{ and } A(s) := F(s) + \lambda G(s) + \lambda_\infty \frac{s^2}{2} \text{ for every } s \in [0, \infty). \tag{2.53}$$

Since $F(0) = G(0) = 0$, hypothesis (H_0^∞) clearly holds. Moreover, by (G_1^∞), for sufficiently small $\epsilon > 0$ there exists $s_0 > 0$, such that $(\underline{c} - \epsilon)s^{p+1} \leq G(s) \leq (\overline{c} + \epsilon)s^{p+1}$ for every $s > s_0$. Thus, since $p < 1$,

$$\lim_{s \to \infty} \frac{G(s)}{s^2} = \lim_{s \to \infty} \frac{G(s)}{s^{p+1}} s^{p-1} = 0.$$

Accordingly, by using (2.53) we obtain that hypothesis (H_1^∞) holds. A similar argument as above implies that

$$\liminf_{s \to \infty} \frac{\max\{\partial A(s)\}}{s} \leq l_0 + \lambda_\infty < 0,$$

and the upper semicontinuity of ∂A implies the existence of two sequences $\{\delta_i\}_i$ and $\{\eta_i\}_i \subset (0,1)$ such that $0 < \delta_i < s_i < \eta_i < \delta_{i+1}$, $\lim_{i \to \infty} \delta_i = \infty$, and $\max\{\partial A(s)\} \leq 0$ for all $s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$. Therefore, hypothesis (H_2^∞) holds. Now, we can apply Theorem 2.3.4, i.e., there is a sequence $\{u_i\}_i \subset H_0^1(\Omega)$ of different elements such that

$$\begin{cases} -\triangle u_i(x) + \lambda_\infty u_i(x) \in \partial A(u_i(x)) \subseteq \partial F(u_i(x)) + \\ \qquad\qquad\qquad\qquad + \lambda \partial G(u_i(x)) + \lambda_\infty u_i(x) & x \in \Omega, \\ u_i(x) \geq 0 & x \in \Omega, \\ u_i(x) = 0 & x \in \partial\Omega, \end{cases}$$

which means that u_i solves problem $(\mathcal{D}_\lambda)$, $i \in \mathbb{N}$, which completes the proof. $\square$

Proof of Theorem 2.3.6. The proof is done in two steps:

(i) Let $\lambda_\infty \in (0, -l_\infty), \lambda \geq 0$ and define

$$k := \lambda_\infty > 0 \ \text{ and } A^\lambda(s) := F(s) + \lambda G(s) + \lambda_\infty \frac{s^2}{2} \text{ for every } s \in [0, \infty).$$

One has clearly that $\partial A^\lambda(s) \subseteq \partial F(s) + \lambda_\infty s + \lambda \partial G(s)$ for every $s \in \mathbb{R}$. On account of (F_2^∞), there is a sequence $\{s_i\}_i \subset (0, \infty)$ converging to ∞ such that

$$\max\{\partial A^{\lambda=0}(s_i)\} \leq \max\{\partial F(s_i)\} + \lambda_\infty s_i < 0.$$

By the upper semicontinuity of $(s, \lambda) \mapsto \partial A^{\lambda}(s)$, we can choose the sequences $\{\delta_i\}_i, \{\eta_i\}_i, \{\lambda_i\}_i \subset (0, \infty)$ such that $0 < \delta_i < s_i < \eta_i < \delta_{i+1}, \lim_{i\to\infty} \delta_i = \infty$, and

$$\max\{\partial A^{\lambda}(s)\} \leq 0$$

for all $\lambda \in [0, \lambda_i], s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$.

For every $i \in \mathbb{N}$ and $\lambda \in [0, \lambda_i]$, let $A_i^{\lambda} : [0, \infty) \to \mathbb{R}$ be defined by

$$A_i^{\lambda}(s) = A^{\lambda}(\tau_{\eta_i}(s)), \tag{2.54}$$

and accordingly, the energy functional $\mathcal{T}_{i,\lambda} : H_0^1(\Omega) \to \mathbb{R}$ associated with the differential inclusion problem($\mathrm{D}^k_{A_i^{\lambda}}$) is

$$\mathcal{T}_{i,\lambda}(u) = \frac{1}{2}\|u\|^2_{H_0^1} + \frac{k}{2}\int_{\Omega} u^2 dx - \int_{\Omega} A_i^{\lambda}(u(x))dx.$$

Then for every $i \in \mathbb{N}$ and $\lambda \in [0, \lambda_i]$, the function A_i^{λ} clearly verifies the hypotheses of Theorem 2.3.1. Accordingly, for every $i \in \mathbb{N}$ and $\lambda \in [0, \lambda_i]$ there exists

$$\mathcal{T}_{i,\lambda} \text{ attains its infimum at some } \tilde{u}^{\infty}_{i,\lambda} \in W^{\eta_i}; \tag{2.55}$$

$$\tilde{u}^{\infty}_{i,\lambda} \in [0, \delta_i] \text{ for a.e. } x \in \Omega; \tag{2.56}$$

$$\tilde{u}^{\infty}_{i,\lambda}(x) \text{ is a weak solution of } (\mathrm{D}^k_{A_i^{\lambda}}). \tag{2.57}$$

Due to (2.54), $\tilde{u}^{\infty}_{i,\lambda}$ is not only a solution to ($\mathrm{D}^k_{A_i^{\lambda}}$) but also to the differential inclusion problem ($\mathrm{D}^k_{A^{\lambda}}$), so ($\mathcal{D}_{\lambda}$).

(ii) For $\lambda = 0$, the function $\partial A_i^{\lambda} = \partial A_i^0$ verifies the hypotheses of Theorem 2.3.4. Moreover, $\mathcal{T}_i := \mathcal{T}_{i,0}$ is the energy functional associated with problem ($\mathrm{D}^k_{A_i^0}$). Consequently, the elements $u_i^{\infty} := u^{\infty}_{i,0}$ verify not only (2.55)-(2.57) but also

$$\mathcal{T}_{m_i}(u^{\infty}_{m_i}) = \min_{W^{\eta_{m_i}}} (\mathcal{T}_{m_i}) \leq \mathcal{T}_{m_i}(w_{\tilde{s}_i}) \text{ for all } i \in \mathbb{N}, \tag{2.58}$$

where the subsequence $\{u^{\infty}_{m_i}\}_i$ of $\{u^{\infty}_i\}_i$ and $w_{\tilde{s}_i} \in W^{\eta_i}$ appear in the proof of Theorem 2.3.7.

Similarly to Kristály and Moroşanu [55], let $\{\theta_i\}_i$ be a sequence with negative terms such that $\lim_{i\to\infty} \theta_i = -\infty$. On account of (2.58) we may assume that

$$\theta_{i+1} < \mathcal{T}_{m_i}(u^{\infty}_{m_i}) \leq \mathcal{T}_{m_i}(w_{\tilde{s}_i}) < \theta_i. \tag{2.59}$$

Let

$$\lambda_i^{'} = \frac{\theta_i - \mathcal{T}_{m_i}(w_{\tilde{s}_i})}{m(\Omega)\max_{s\in[0,1]}|G(s)|+1} \text{ and } \lambda_i^{''} = \frac{\mathcal{T}_{m_i}(u^{\infty}_{m_i}) - \theta_{i+1}}{m(\Omega)\max_{s\in[0,1]}|G(s)|+1}, \, i \in \mathbb{N},$$

and for a fixed $k \in \mathbb{N}$, we set

$$\lambda_k^{\infty} = \min(1, \lambda_1, ..., \lambda_k, {\lambda_1}', ..., {\lambda_k}', {\lambda_1}'', ..., {\lambda_k}'') > 0.$$

Then, for every $i \in \{1, ..., k\}$ and $\lambda \in [0, \lambda_k^{\infty}]$, due to (2.59) we have that

$$\begin{aligned}
\mathcal{T}_{m_i,\lambda}(\tilde{u}^{\infty}_{m_i,\lambda}) &\leq \mathcal{T}_{m_i,\lambda}(w_{\tilde{s}_i}) \\
&= \frac{1}{2}\|w_{\tilde{s}_i}\|^2_{H^1_0} - \int_\Omega F(w_{\tilde{s}_i}(x))dx - \lambda\int_\Omega G(w_{\tilde{s}_i}(x))dx \\
&= \mathcal{T}_{m_i}(w_{\tilde{s}_i}) - \lambda\int_\Omega G(w_{\tilde{s}_i}(x))dx \\
&< \theta_i.
\end{aligned}$$

Similarly, since $\tilde{u}^{\infty}_{m_i,\lambda} \in W^{\eta_{m_i}}$ and $u^{\infty}_{m_i}$ is the minimum point of $\mathcal{T}_i$ on the set $W^{\eta_{m_i}}$, on account of (2.59) we have

$$\begin{aligned}
\mathcal{T}_{m_i,\lambda}(\tilde{u}^{\infty}_{m_i,\lambda}) &= \mathcal{T}_{m_i}(\tilde{u}^{\infty}_{m_i,\lambda}) - \lambda\int_\Omega G(\tilde{u}^{\infty}_{m_i,\lambda})dx \geq \mathcal{T}_{m_i}(u^{\infty}_{m_i}) - \lambda\int_\Omega G(\tilde{u}^{\infty}_{m_i,\lambda})dx \\
&> \theta_{i+1}.
\end{aligned}$$

Therefore, for every $i \in \{1, ..., k\}$ and $\lambda \in [0, \lambda_k^{\infty}]$,

$$\theta_{i+1} < \mathcal{T}_{m_i,\lambda}(\tilde{u}^{\infty}_{m_i,\lambda}) < \theta_i < 0,$$

thus

$$\mathcal{T}_{m_k,\lambda}(\tilde{u}^{\infty}_{m_k,\lambda}) < ... < \mathcal{T}_{m_1,\lambda}(\tilde{u}^{\infty}_{m_1,\lambda}) < 0.$$

Because of (2.54), we notice that $\tilde{u}^{\infty}_{m_i,\lambda} \in W^{\eta_{m_k}}$ for every $i \in \{1, ..., k\}$, thus $\mathcal{T}_{m_i,\lambda}(\tilde{u}^{\infty}_{m_i,\lambda}) = \mathcal{T}_{m_k,\lambda}(\tilde{u}^{\infty}_{i,\lambda})$. Therefore, for every $\lambda \in [0, \lambda_k^{\infty}]$,

$$\mathcal{T}_{m_k,\lambda}(\tilde{u}^{\infty}_{m_k,\lambda}) < ... < \mathcal{T}_{m_k,\lambda}(\tilde{u}^{\infty}_{m_1,\lambda}) < 0 = \mathcal{T}_{m_k,\lambda}(0),$$

i.e, the elements $\tilde{u}^{\infty}_{m_1,\lambda}, ..., \tilde{u}^{\infty}_{m_k,\lambda}$ are distinct and non-trivial whenever $\lambda \in [0, \lambda_k^{\infty}]$. The estimate (2.42) follows in a similar manner as in [55]. □

Chapter 3

Hemivariational inequalities, elliptic problems on unbounded domains

3.1 Solutions in weighted Sobolev spaces

3.1.1 Multiplicity results for a quasilinear elliptic problem in weighted Sobolev spaces

In this section we prove two multiplicity results for a double eigenvalue quasilinear problem on unbounded domain with nonlinear boundary conditions following Mezei [66] and Mezei and Varga [71]. The solutions of the problem will be found in a weighted Sobolev space.

Let $\Omega \subset \mathbb{R}^N$ be an unbounded domain with smooth boundary Γ. We assume throughout this section that m, p, q and α_1, α_2 are real numbers satisfying

$$1 < p < N, \quad p \leq q \leq \frac{pN}{N-p}, \quad -N < \alpha_1 \leq q \cdot \frac{N-p}{p} - N, \tag{3.1}$$

$$p \leq m \leq p \cdot \frac{N-1}{N-p} \quad \text{and} \quad -N < \alpha_2 \leq m \cdot \frac{N-p}{p} - N + 1. \tag{3.2}$$

We define the space E as the completion of $C_0^\infty(\Omega)$ in the norm

$$||u||_E = \left(\int_\Omega (|\nabla u(x)|^p + \frac{1}{(1+|x|)^p} |u(x)|^p) dx \right)^{\frac{1}{p}}.$$

We denote by $L^q(\Omega; w_1)$ and by $L^m(\Gamma; w_2)$ the weighted Lebesgue spaces with respect to

$$w_i(x) = (1 + |x|)^{\alpha_i}, i = 1, 2 \tag{3.3}$$

and norms

$$||u||^q_{q,w_1} = \int_\Omega w_1(x)|u(x)|^q dx \quad \text{and} \quad ||u||^m_{m,w_2} = \int_\Gamma w_2(x)|u(x)|^m d\Gamma.$$

We shall use the following embedding result.

Proposition 3.1.1. ([83]) *Assume that* (3.1) *holds. Then the embedding* $E \hookrightarrow L^q(\Omega; w_1)$ *is continuous. If the upper bound for* q *in* (3.1) *is strict, then the embedding is compact. Suppose that the inequalities in* (3.2) *are satisfied. Then the trace operator* $E \to L^m(\Gamma; w_2)$ *is continuous. If the upper bound for* m *in* (3.2) *is strict, then the trace operator is compact.*

Notations: The best embedding constant of $E \hookrightarrow L^q(\Omega; w_1)$ will be denoted by C_{q,w_1} and that of $E \hookrightarrow L^m(\Gamma; w_2)$ by C_{m,w_2}.

We assume throughout this section that $a \in L^\infty(\overline{\Omega})$ and $b \in L^\infty(\Gamma)$ such that

$$a(x) \geq a_0 > 0 \quad \text{for a.e. } x \in \Omega$$

and

$$\frac{c}{(1 + |x|)^{p-1}} \leq b(x) \leq \frac{C}{(1 + |x|)^{p-1}} \quad \text{for a.e. } x \in \Gamma,$$

where $0 < c < C$ are constants.

Remark 3.1.1. Note, that

$$||u||^p_b = \int_\Omega a(x)|\nabla u(x)|^p dx + \int_\Gamma b(x)|u(x)|^p d\Gamma$$

defines an equivalent norm on E, see [83].

For $\lambda > 0$ and $\mu \in \mathbb{R}$ we consider the problem

$$(P_{\lambda,\mu}) \qquad \begin{cases} -\text{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f(x, u) \text{ in } \Omega, \\ a(x)|\nabla u|^{p-2}\nabla u \cdot n + b(x)|u|^{p-2}u = \lambda\mu g(x, u) \text{ on } \Gamma \\ u \neq 0 \text{ in } \Omega, \end{cases}$$

where n denotes the unit outward normal on Γ, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions.

We consider the following **assumptions**:

(F1) $f(\cdot, 0) = 0$ and $|f(x,s)| \leq f_0(x) + f_1(x)|s|^{r-1}$, where $p \leq r < \dfrac{pN}{N-p}$, and f_0, f_1 are measurable functions which satisfy $0 < f_0(x) \leq C_f w_1(x)$, and $0 \leq f_1(x) \leq C_f w_1(x)$ a.e. $x \in \Omega$, $f_0 \in L^{\frac{r}{r-1}}(\Omega; w_1^{\frac{1}{1-r}})$.

(F2) $\displaystyle\lim_{s\to 0} \frac{f(x,s)}{f_0(x)|s|^{p-1}} = 0$, uniformly in $x \in \Omega$;

(F3) $\displaystyle\limsup_{s\to +\infty} \frac{F(x,s)}{f_0(x)|s|^{p}} \leq 0$, uniformly for all $x \in \Omega$ and $\displaystyle\max_{|s|\leq M} F(\cdot, s) \in L^1(\Omega)$ for all $M > 0$, where F denotes the primitive function of f with respect to the second variable, i.e. $F(x,u) = \displaystyle\int_0^u f(x,s)ds$;

(F4) there exist $x_0 \in \Omega$, $R_0 > 0$ and $s_0 \in \mathbb{R}$ such that $\displaystyle\inf_{|x-x_0|\leq R_0} F(x, s_0) > 0$.

(G1) $g(\cdot, 0) = 0$ and $|g(x,s)| \leq g_0(x) + g_1(x)|s|^{m-1}$, where $p \leq m < p \cdot \dfrac{N-1}{N-p}$, and g_0, g_1 are measurable functions satisfying $0 < g_0(x) \leq C_g w_2(x)$ and $0 \leq g_1(x) \leq C_g w_2(x)$, a.e. $x \in \Gamma$, $g_0 \in L^{\frac{m}{m-1}}(\Gamma; w_2^{\frac{1}{1-m}})$;

(G2) $\displaystyle\lim_{s\to 0} \frac{g(x,s)}{g_0(x)|s|^{p-1}} = 0$, uniformly in $x \in \Gamma$;

(G3) $\displaystyle\limsup_{s\to +\infty} \frac{G(x,s)}{g_0(x)|s|^{p}} \leq 0$, uniformly for all $x \in \Gamma$ and $\displaystyle\max_{|s|\leq M} G(\cdot, s) \in L^1(\Gamma)$ for all $M > 0$, where G is the primitive function of g with respect to the second variable, i.e. $G(x,u) = \displaystyle\int_0^u g(x,s)ds$.

The energy functional corresponding to $(P_{\lambda,\mu})$ is given by

$$\mathcal{E}_{\lambda,\mu} : E \to \mathbb{R},$$

$$\mathcal{E}_{\lambda,\mu}(u) = \frac{1}{p}\int_\Omega a(x)|\nabla u(x)|^p dx + \frac{1}{p}\int_\Gamma b(x)|u(x)|^p d\Gamma - \lambda J_\mu(u),$$

where $J_\mu : E \to \mathbb{R}$ is defined by $J_\mu(u) = J_F(u) + \mu J_G(u)$, with

$$J_F, J_G : E \to \mathbb{R}$$

$$J_F(u) = \int_\Omega F(x, u(x))dx \quad \text{and} \quad J_G(u) = \int_\Gamma G(x, u(x))d\Gamma.$$

Proposition 3.1.1 implies that $\mathcal{E}_{\lambda,\mu}$ is well defined. The solutions of problem $(P_{\lambda,\mu})$ will be found as critical points of $\mathcal{E}_{\lambda,\mu}$. Therefore, a function $u \in E$ is a solution of problem $(P_{\lambda,\mu})$ provided that, for any $v \in E$,

$$\begin{aligned}
&\int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx + \int_\Gamma b(x)|u(x)|^{p-2}u(x)v(x)d\Gamma \\
&= \lambda \int_\Omega f(x, u(x))v(x)dx + \lambda\mu \int_\Gamma g(x, u(x))v(x)d\Gamma.
\end{aligned}$$

The main results of this subsection are the following theorems.

Theorem 3.1.1. ([71]) *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ two functions, which satisfies the conditions $(F1) - (F4)$ and $(G1) - (G3)$ respectively. Then there exists $\mu_0 > 0$ such that to every $\mu \in [-\mu_0, \mu_0]$ it corresponds a nonempty open interval $\Gamma_\mu \in]0, +\infty[$ and a number $\sigma_\mu > 0$ for which $(P_{\lambda,\mu})$ has at least two distinct, nontrivial weak solutions: $u_{\lambda,\mu}, v_{\lambda,\mu} \in E$, with the property*

$$max\{||u_{\lambda,\mu}||_E, ||v_{\lambda,\mu}||_E\} \le \sigma_\mu,$$

whenever $\lambda \in \Gamma_\mu$.

Theorem 3.1.2. ([66]) *We suppose that the functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfy the conditions $(F1) - (F4)$ and $(G1) - (G3)$ respectively. Then there exists $\lambda_0 > 0$ such that to every $\lambda \in]\lambda_0, +\infty[$ it corresponds a nonempty open interval $I_\lambda \subset \mathbb{R}$ such that for every $\mu \in I_\lambda$ the problem $(P_{\lambda,\mu})$ has at least two distinct, nontrivial weak solutions $u_{\lambda,\mu}, v_{\lambda,\mu} \in E$, with the property*

$$\mathcal{E}_{\lambda,\mu}(u_{\lambda,\mu}) < 0 < \mathcal{E}_{\lambda,\mu}(v_{\lambda,\mu}).$$

Auxiliary results

First, we prove some results used in the proofs of the main theorems.

Lemma 3.1.1. *We assume that the conditions* (F1)-(F2) *and* (G1)-(G2) *are satisfied. Then the functional $J_\mu = J_F + \mu J_G$ is sequentially weakly continuous.*

Proof. First we observe that, from assumption (F1)-(F2) and (G1)-(G2), for every $\varepsilon > 0$, there exist $C_1(\varepsilon), C_2(\varepsilon) > 0$ such that

$$|F(x, u(x)| \leq \varepsilon f_0(x)|u(x)|^p + C_1(\varepsilon)(f_0(x) + f_1(x))|u(x)|^r \tag{3.4}$$

$$|G(x, u(x)| \leq \varepsilon g_0(x)|u(x)|^p + C_2(\varepsilon)(g_0(x) + g_1(x))|u(x)|^m, \tag{3.5}$$

where $r \in [p, p^*[$ and $m \in [p, p\frac{N-1}{N-p}[$.

Now we prove our assertion, arguing by contradiction. Let $\{u_n\}$ be a sequence in E, weakly convergent to $u \in E$, and $d > 0$ be such that

$$|J_\mu(u_n) - J_\mu(u)| \geq d \quad \text{for} \quad \text{all } n \in \mathbb{N}.$$

Without loss of generality, we can assume that there is a positive constant M such that

$$||u_n||_b \leq M, ||u_n - u||_b \leq M, \quad \text{for} \quad \text{all} \quad n \in \mathbb{N}.$$

Because the embeddings $E \hookrightarrow L^r(\Omega; w_1)$ and $E \hookrightarrow L^m(\Gamma; w_2)$ are compact, it follows that $||u_n - u||_{r,w_1} \to 0$ and $||u_n - u||_{m,w_2} \to 0$.

Using (3.4), Proposition 3.1.1 and the Hölder inequality we have

$$|J_\mu(u_n) - J_\mu(u)| \leq$$

$$\begin{aligned}
&\leq \int_\Omega |F(x, u_n(x)) - F(x, u(x))|dx + |\mu| \int_\Gamma |G(x, u_n(x)) - G(x, u(x))|d\Gamma \\
&\leq \varepsilon c' \int_\Omega f_0(x)|u_n(x) - u(x)|(|u_n(x)|^{p-1} + |u(x)|^{p-1})dx \\
&\quad + C_1(\varepsilon)c' \int_\Omega (f_0(x) + f_1(x))|u_n(x) - u(x)|(|u_n(x)|^{r-1} + |u(x)|^{r-1})dx \\
&\quad + |\mu|\varepsilon c'' \int_\Gamma g_0(x)|u_n(x) - u(x)|(|u_n(x)|^{p-1} + |u(x)|^{p-1})d\Gamma \\
&\quad + |\mu|C_2 c''(\varepsilon) \int_\Gamma (g_0(x) + g_1(x))|u_n(x) - u(x)|(|u_n(x)|^{m-1} + |u(x)|^{m-1})d\Gamma
\end{aligned}$$

$$\begin{aligned}
&\leq \varepsilon c' C_f \int_\Omega w_1(x)|u_n(x)-u(x)|(|u_n(x)|^{p-1}+|u(x)|^{p-1})dx \\
&\quad +2C_1(\varepsilon)c'C_f \int_\Omega w_1(x)|u_n(x)-u(x)|(|u_n(x)|^{r-1}+|u(x)|^{r-1})dx \\
&\quad +\varepsilon c''|\mu|C_g \int_\Omega w_2(x)|u_n(x)-u(x)|(|u_n(x)|^{p-1}+|u(x)|^{p-1})dx \\
&\quad +2|\mu|c''C_2(\varepsilon)C_g \int_\Omega w_2(x)|u_n(x)-u(x)|(|u_n(x)|^{m-1}+|u(x)|^{m-1})dx \\
&\leq \varepsilon c' C_f ||u_n-u||_{p,w_1}(||u_n||_{p,w_1}^{\frac{p}{p'}}+||u||_{p,w_1}^{\frac{p}{p'}}) \\
&\quad +2C_1c'(\varepsilon)C_f||u_n-u||_{r,w_1}(||u_n||_{r,w_1}^{\frac{r}{r'}}+||u||_{r,w_1}^{\frac{r}{r'}}) \\
&\quad +\varepsilon c''|\mu|C_g||u_n-u||_{p,w_2}(||u_n||_{p,w_2}^{\frac{p}{p'}}+||u||_{p,w_2}^{\frac{p}{p'}}) \\
&\quad +2|\mu|C_2(\varepsilon)c''C_g||u_n-u||_{m,w_2}(||u_n||_{m,w_2}^{\frac{m}{m'}}+||u||_{m,w_2}^{\frac{m}{m'}}),
\end{aligned}$$

where c' and c'' are some positive constants given by the Mean Value Theorem, $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{r}+\frac{1}{r'}=1$. By the embedding results from Proposition 3.1.1 it follows that

$$d \leq |J_\mu(u_n)-J_\mu(u)| \leq 2\varepsilon M^p\left(c'C_fC_{p,w_1}^p+|\mu|c''C_gC_{p,w_2}^p\right)+$$
$$+4c'C_1(\varepsilon)C_fC_{r,w_1}^{\frac{r}{r'}}M^{\frac{r}{r'}}||u_n-u||_{r,w_1}+4|\mu|c''C_2(\varepsilon)C_gC_{m,w_2}^{\frac{m}{m'}}M^{\frac{m}{m'}}||u_n-u||_{m,w_2}.$$

Therefore, if $\varepsilon>0$ is sufficiently small and $n\in\mathbb{N}$ is large enough, we have

$$d \leq |J_\mu(u_n)-J_\mu(u)| < d,$$

which is a contradiction. $\square$

Lemma 3.1.2. *Suppose that (F4) is fulfilled, then there exists $u_0\in E$ such that $J_F(u_0)>0$.*

Proof. Let $R_0>0$ and $s_0\in\mathbb{R}$ from (F4) and fix $\varepsilon\in(0,\frac{R_0}{2})$. We consider the function $u_\varepsilon\in\mathcal{C}_0^\infty(\Omega)$ such that

$$u_\varepsilon(x)=\begin{cases} 0, & \text{if } |x-x_0|\geq R_0 \\ s_0, & \text{if } |x-x_0|\leq R_0-\varepsilon \end{cases}$$

and $\|u_\varepsilon\|_\infty\leq|s_0|$. Then, we have $u_\varepsilon\in E$.

If we denote by $Vol(B_r(x_0))$, the volume of the ball $B_r(x_0) = \{x \in \mathbb{R}^N : |x - x_0| \leq r\}$, we have

$$J_F(u_\varepsilon) = \int_\Omega F(x, u_\varepsilon)dx = \int_{B_{R_0-\varepsilon}(x_0)\cap\Omega} F(x, u_\varepsilon(x))dx +$$

$$+ \int_{(\mathbb{R}^N \setminus B_{R_0}(x_0))\cap\Omega} F(x, u_\varepsilon(x))dx + \int_{(B_{R_0}(x_0)\setminus B_{R_0-\varepsilon}(x_0))\cap\Omega} F(x, u_\varepsilon(x))dx \geq$$

$$\geq A_0 Vol(B_{\frac{R_0}{2}}(x_0) \cap \Omega) + B_0[Vol(B_{R_0}(x_0) \cap \Omega) - Vol(B_{R_0-\varepsilon}(x_0) \cap \Omega)],$$

where $A_0 = \min\limits_{|x-x_0|\leq R_0} F(x, s_0) > 0$ and $B_0 = \min\limits_{\frac{R_0}{2}\leq|x-x_0|\leq R_0} \min\limits_{|s|\leq|s_0|} F(x, s)$. If ε goes to 0, then $[Vol(B_{R_0}(x_0) \cap \Omega) - Vol(B_{R_0-\varepsilon}(x_0) \cap \Omega)] \to 0$ as well, so we can choose an $\varepsilon = \varepsilon_0$, such that $J_F(u_{\varepsilon_0}) > 0$. Now, we choose $u_0 = u_{\varepsilon_0}$ and the proof is complete. $\square$

Lemma 3.1.3. *Suppose that the conditions* (F1)*,* (F3)*,* (G1) *and* (G3) *are satisfied. Then, for every $\lambda \geq 0$ and $\mu \in \mathbb{R}$ the functional $\mathcal{E}_{\lambda,\mu}$ is coercive on E.*

Proof. If $\lambda = 0$, then statement is trivial.

Now fix $\lambda > 0$, $\mu \in \mathbb{R}$ and $a, b > 0$ such that $\lambda a C_f C_{p,w_1}^p + \lambda|\mu| b C_g C_{p,w_2}^p < \dfrac{1}{p}$. There exist the positive functions $k_a \in L^1(\Omega; w_1)$, $k_b \in L^1(\Gamma; w_2)$ such that

$$F(x, s) \leq a f_0(x)|s|^p + k_a(x)w_1(x), \quad \text{for all} \quad (x, s) \in \Omega \times \mathbb{R}$$

$$G(x, s) \leq b g_0(x)|s|^p + k_b(x)w_2(x), \quad \text{for all} \quad (x, s) \in \Gamma \times \mathbb{R}.$$

Then, it follows that

$$\begin{aligned}
\mathcal{E}_{\lambda,\mu}(u) &= \frac{||u||_b^p}{p} - \lambda J_\mu(u) = \\
&= \frac{||u||_b^p}{p} - \lambda \left(\int_\Omega F(x, u(x))dx + \mu \int_\Gamma G(x, u(x))d\Gamma \right) \geq \\
&\geq \frac{||u||_b^p}{p} - \lambda a C_f \int_\Omega w_1(x)|u(x)|^p dx - \lambda \int_\Omega k_a(x)w_1(x)dx - \\
&\quad -\lambda|\mu| b C_g \int_\Gamma w_2(x)|u(x)|^p d\Gamma - \lambda|\mu| \int_\Gamma k_b(x)w_2(x)d\Gamma \geq \\
&\geq ||u||_b^p \left(\frac{1}{p} - \lambda a C_f C_{p,w_1}^p - \lambda|\mu| b C_g C_{p,w_2}^p \right) - \\
&\quad -\lambda||k_a||_{1,w_1} - \lambda|\mu|||k_b||_{1,w_2}
\end{aligned}$$

that goes to ∞ as $||u||_b \to \infty$. $\square$

Lemma 3.1.4. *For every $\lambda > 0$ and every $\mu \in \mathbb{R}$, $\mathcal{E}_{\lambda,\mu} : E \to \mathbb{R}$ satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\} \subset E$ be a (PS)-sequence for the function $\mathcal{E}_{\lambda,\mu}$, i.e.

(1) $\{\mathcal{E}_{\lambda,\mu}(u_n)\}$ is bounded;

(2) $\mathcal{E}'_{\lambda,\mu}(u_n) \to 0$, as $n \to \infty$.

Since $\mathcal{E}_{\lambda,\mu}$ is coercive, we have that $\{u_n\}$ is bounded. The reflexivity of the Banach space E implies the existence of a subsequence notated also by $\{u_n\}$ such that $\{u_n\}$ is weakly convergent to an element $u \in E$. Because the inclusion $E \hookrightarrow L^p(\Omega, w_1)$ is compact, we have that $u_n \to u$ strongly in $L^p(\Omega, w_1)$. We want to prove, that u_n converges strongly to u in E. For this end, we will use the following inequalities from ([32], lemma 4.10)

$$|\xi - \zeta|^p \leq M_1(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta), \quad \text{for } p \geq 2 \tag{3.6}$$

$$|\xi - \zeta|^2 \leq M_2(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta)(|\xi| + |\zeta|)^{2-p}, \quad \text{for } p \in]1, 2[, \tag{3.7}$$

where M_1 and M_2 are some positive constants. We separate the proof into two cases. In the first case let $p \geq 2$. Then we have:

$$\begin{aligned}
|| \quad & u_n - u||_b^p = \int_\Omega a(x)|\nabla u_n(x) - \nabla u(x)|^p dx + \int_\Gamma b(x)|u_n(x) - u(x)|^p d\Gamma \leq \\
\leq \quad & M_1 \int_\Omega a(x) \left[|\nabla u_n(x)|^{p-2}\nabla u_n(x) - |\nabla u(x)|^{p-2}\nabla u(x)\right] \cdot \\
& \qquad\qquad\qquad\qquad \cdot(\nabla u_n(x) - \nabla u(x))dx + \\
& + M_1 \int_\Gamma b(x) \left[|u_n(x)|^{p-2}u_n(x) - |u(x)|^{p-2}u(x)\right] (u_n(x) - u(x))d\Gamma = \\
= \quad & M_1(\langle \mathcal{E}'_{\lambda,\mu}(u_n), u_n - u\rangle - \langle \mathcal{E}'_{\lambda,\mu}(u), u_n - u\rangle + \lambda\langle J'_F(u_n) - J'_F(u), u_n - u\rangle \\
& + \lambda\mu\langle J'_G(u_n) - J'_G(u), u_n - u\rangle) \\
\leq \quad & M_1(||\mathcal{E}'_{\lambda,\mu}(u_n)|| + \lambda||J'_F(u_n) - J'_F(u)|| \\
& + \lambda\mu||J'_G(u_n) - J'_G(u)||) \cdot ||u_n - u||_b - M_1\langle \mathcal{E}'_{\lambda,\mu}(u), u_n - u\rangle.
\end{aligned}$$

Since $u_n \to u$ weakly in E and J'_F, J'_G are compact (see [83]), we have that $||J'_F(u_n) - J'_F(u)|| \to 0$ and $||J'_G(u_n) - J'_G(u)|| \to 0$. Moreover $\mathcal{E}'_{\lambda,\mu}(u) = 0$ and $||\mathcal{E}'_{\lambda,\mu}(u_n)|| \to 0$, so $||u_n - u||_b \to 0$, i.e. u_n converges strongly to u in E.

In the second case, when $1 < p < 2$, we use the following result: for all $s \in (0, \infty)$ there is a constant $C_s > 0$ such that

$$(x + y)^s \leq C_s(x^s + y^s), \quad \text{for any } x, y \in (0, \infty). \tag{3.8}$$

Then we obtain

$$
\begin{aligned}
||u_n \quad - \quad u||_b^2 = \\
&= \left(\int_\Omega a(x)|\nabla u_n(x) - \nabla u(x)|^p dx + \int_\Gamma b(x)|u_n(x) - u(x)|^p d\Gamma\right)^{\frac{2}{p}} \\
&\leq C_p \left(\int_\Omega a(x)|\nabla u_n(x) - \nabla u(x)|^p dx\right)^{\frac{2}{p}} \qquad (3.9) \\
&+ C_p \left(\int_\Gamma b(x)|u_n(x) - u(x)|^p d\Gamma\right)^{\frac{2}{p}}.
\end{aligned}
$$

Now, using the (3.7) and the Hölder inequalities we get

$$
\begin{aligned}
&\int_\Omega a(x)|\nabla u_n(x) - \nabla u(x)|^p dx = \int_\Omega a(x)(|\nabla u_n(x) - \nabla u(x)|^2)^{\frac{p}{2}} dx \\
\leq \quad & M_2 \cdot \int_\Omega a(x)\left(|\nabla u_n(x)|^{p-2}\nabla u_n(x) - |\nabla u(x)|^{p-2}\nabla u(x)\right)^{\frac{p}{2}} \cdot \\
& \cdot (\nabla u_n(x) - \nabla u(x))^{\frac{p}{2}} \cdot (|\nabla u_n(x)| + |\nabla u(x)|)^{\frac{p(2-p)}{2}} dx \\
= \quad & M_2 \cdot \int_\Omega \left[a(x)(|\nabla u_n(x)|^{p-2}\nabla u_n(x) - |\nabla u(x)|^{p-2}\nabla u(x)) \cdot \right. \\
& \left. \cdot (\nabla u_n(x) - \nabla u(x))\right]^{\frac{p}{2}} \left[a(x)(|\nabla u_n(x)| + |\nabla u(x)|)^p\right]^{\frac{2-p}{2}} dx \\
\leq \quad & \widetilde{M_2} \left(\int_\Omega a(x)|\nabla u_n(x)|^p dx + \int_\Omega a(x)|\nabla u(x)|^p dx\right)^{\frac{2-p}{2}} \cdot \\
& \left(\int_\Omega a(x)(|\nabla u_n(x)|^{p-2}\nabla u_n(x) - |\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x) - \nabla u(x))dx\right)^{\frac{p}{2}} \\
\leq \quad & \overline{M}_2 \left[\left(\int_\Omega a(x)|\nabla u_n(x)|^p dx\right)^{\frac{2-p}{2}} + \left(\int_\Omega a(x)|\nabla u(x)|^p dx\right)^{\frac{2-p}{2}}\right] \cdot \\
& \left(\int_\Omega a(x)(|\nabla u_n(x)|^{p-2}\nabla u_n(x) - |\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x) - \nabla u(x))dx\right)^{\frac{p}{2}} \\
\leq \quad & \widehat{M_2} \cdot \left(||u_n||_b^{\frac{(2-p)p}{2}} + ||u||_b^{\frac{(2-p)p}{2}}\right) \cdot \\
& \left(\int_\Omega a(x)(|\nabla u_n(x)|^{p-2}\nabla u_n(x) - |\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x) - \nabla u(x))dx\right)^{\frac{p}{2}}
\end{aligned}
$$

Then using again the relation (3.8) and the above inequality we have:

$$\left(\int_\Omega a(x)|\nabla u_n(x)-\nabla u(x)|^p dx\right)^{\frac{2}{p}} \leq M_2' \cdot \left(||u_n||_b^{2-p}+||u||_b^{2-p}\right)\cdot \tag{3.10}$$
$$\cdot \int_\Omega a(x)(|\nabla u_n(x)|^{p-2}\nabla u_n(x)-|\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x)-\nabla u(x))dx.$$

In a similar way we obtain the following estimate

$$\left(\int_\Gamma b(x)|u_n(x)-u(x)|^p d\Gamma\right)^{\frac{2}{p}} \leq M_2' \left(||u_n||_b^{2-p}+||u||_b^{2-p}\right)\cdot \tag{3.11}$$
$$\cdot \int_\Gamma b(x)(|u_n(x)|^{p-2}u_n(x)-|u(x)|^{p-2}u(x))(u_n(x)-u(x))d\Gamma.$$

We introduce the following notation: $\Phi(u)=\frac{1}{p}||u||_b^p$. As we used before, the directional derivative of Φ, in the direction $v\in E$ is

$$\langle \Phi'(u),v\rangle = \int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx + \int_\Gamma b(x)|u(x)|^{p-2}u(x)v(x)d\Gamma.$$

Using the inequalities (3.9), (3.10), (3.11) we have

$$||u_n-u||_b^2 < M_2'\cdot\langle\Phi'(u_n)-\Phi'(u),u_n-u\rangle\cdot(||u_n||_b^{p-2}+||u||_b^{2-p}).$$

Since u_n is bounded, the same argument as in the first case (when $p\geq 2$) shows that u_n converges to u strongly in E.

Thus $\mathcal{E}_{\lambda,\mu}$ satisfy the (PS) condition for all $\lambda\geq 0$ and $\mu\in\mathbb{R}$. □

Lemma 3.1.5. *For every $\mu\in\mathbb{R}$ we have*

$$\lim_{\rho\to 0}\frac{\sup\{J_\mu(u):||u||_b^p<\rho\}}{\rho}=0.$$

Proof. Fix arbitrarily $\varepsilon>0$ and $r\in\left]p,\frac{pN}{N-p}\right[$, $m\in\left]p,p\frac{N-1}{N-p}\right[$, then from (3.4) and (3.5) it follows that

$$\begin{aligned} J_\mu(u) &\leq \varepsilon C_f C_{p,w_1}^p ||u||_b^p + \varepsilon|\mu|C_g C_{p,w_2}^p||u||_b^p + 2C_1(\varepsilon)C_f C_{r,w_1}^r||u||_b^r + \\ &+ 2|\mu|C_2(\varepsilon)C_g C_{m,w_2}^m ||u||_b^m \leq \\ &\leq \varepsilon\left[C_f C_{p,w_1}^p + |\mu|C_g C_{p,w_2}^p\right]\rho + 2C_1(\varepsilon)C_f C_{r,w_1}^p \rho^{\frac{r}{p}} + \\ &+ 2|\mu|C_2(\varepsilon)C_g C_{m,w_2}^m \rho^{\frac{m}{p}}. \end{aligned}$$

Therefore,

$$\lim_{\rho\to 0}\frac{\sup\{J_\mu(u) : ||u||_b^p < \rho\}}{\rho} = 0.$$

□

In order to prove the Theorem 3.1.2, let us define $m = \int_\Gamma |G(x, u_0(x))| d\Gamma$,

$$\lambda_0 = \frac{\Phi(u_0)}{J_F(u_0)} > 0 \quad \text{and} \quad \mu_\lambda^* = \frac{1}{\lambda(1+m)} \cdot (\lambda - \lambda_0) J_F(u_0) > 0.$$

Lemma 3.1.6. *For $\lambda > \lambda_0$ and $|\mu| \in]0, \mu_\lambda^*]$ we have*

$$\inf_{u\in E} \mathcal{E}_{\lambda,\mu}(u) < 0.$$

Proof. It is sufficient to prove, that, for $\lambda > \lambda_0$ and $|\mu| \in]0, \mu_\lambda^*]$ we have $\mathcal{E}_{\lambda,\mu}(u_0) < 0$. Indeed,

$$\begin{aligned}
\mathcal{E}_{\lambda,\mu}(u_0) &= \Phi(u_0) - \lambda J_f(u_0) - \lambda\mu J_G(u_0) \leq \\
&\leq \lambda_0 J_F(u_0) - \lambda J_F(u_0) + \lambda|\mu| m = \\
&= (\lambda_0 - \lambda) J_F(u_0) + \lambda|\mu| m = \\
&= (\lambda_0 - \lambda)\frac{\lambda(1+m)\mu_\lambda^*}{\lambda - \lambda_0} + \lambda|\mu| m = \\
&= -(1+m)\lambda\mu_\lambda^* + \lambda|\mu| m = \\
&= -\lambda\mu_\lambda^* - m\lambda(\mu_\lambda^* - |\mu|) < 0.
\end{aligned}$$

for all $\lambda > \lambda_0$ and $|\mu| \in]0, \mu_\lambda^*]$. □

Lemma 3.1.7. *For every $\lambda > \lambda_0$ and $\mu \in]0, \mu_\lambda^*]$, the functional $\mathcal{E}_{\lambda,\mu}$ satisfies the Mountain Pass geometry, i.e. satisfies the hypotheses of the Mountain Pass Theorem 1.3.1.*

Proof. From the inequalities (3.4), (3.5), using (F1) and (G1) we get

$$|F(x, u(x))| \leq \varepsilon w_1(x) C_f |u(x)|^p + 2C_1(\varepsilon) C_f w_1(x))|u(x)|^r, \tag{3.12}$$

$$|G(x, u(x))| \leq \varepsilon w_2(x) C_g |u(x)|^p + 2C_2(\varepsilon) C_g w_2(x)|u(x)|^m. \tag{3.13}$$

Fix $\lambda > \lambda_0$ and $\mu \in]0, \mu_\lambda^*[$, then using the (3.12)and (3.13) inequalities for every $u \in E$ we have

$$
\begin{aligned}
\mathcal{E}_{\lambda,\mu}(u) &= \frac{1}{p}||u||_b^p - \lambda J_\mu(u) \\
&\geq \frac{1}{p}||u||_b^p - \lambda \int_\Omega |F(x,u(x))|dx - \lambda|\mu| \int_\Gamma |G(x,u(x))|d\Gamma \\
&\geq \frac{1}{p}||u||_b^p \\
&\quad -\lambda\varepsilon C_f \int_\Omega w_1(x)|u(x)|^p dx - 2\lambda C_1(\varepsilon)C_f \int_\Omega w_1(x)|u(x)|^r dx \\
&\quad -\lambda|\mu|\varepsilon C_g \int_\Gamma w_2(x)|u(x)|^p d\Gamma - 2\lambda|\mu|C_2(\varepsilon)C_g \int_\Gamma w_2(x)|u(x)|^m d\Gamma \\
&= \frac{1}{p}||u||_b^p - \lambda\varepsilon C_f||u||_{p,w_1}^p - 2\lambda C_1(\varepsilon)C_f||u||_{r,w_1}^r \\
&\quad -\lambda|\mu|\varepsilon C_g||u||_{p,w_2}^p - 2\lambda|\mu|C_2(\varepsilon)C_g||u||_{m,w_2}^m \\
&\geq \left(\frac{1}{p} - \lambda\varepsilon C_f C_{p,w_1}^p - \lambda|\mu|\varepsilon C_g C_{p,w_2}^p\right)||u||_b^p \\
&- \; 2\lambda C_1(\varepsilon)C_f C_{r,w_1}^r||u||_b^r - 2\lambda|\mu|C_2(\varepsilon)C_g C_{m,w_2}^m||u||_b^m.
\end{aligned}
$$

Using the notations

$$A = \left(\frac{1}{p} - \lambda\varepsilon C_f C_{p,w_1}^p - \lambda|\mu|\varepsilon C_g C_{p,w_2}^p\right),$$

$$B = 2\lambda C_1(\varepsilon)C_f C_{r,w_1}^r, \quad C = 2\lambda|\mu|C_2(\varepsilon)C_g C_{m,w_2}^m,$$

we get

$$\mathcal{E}_{\lambda,\mu}(u) \geq (A - B||u||_b^{r-p} - C||u||_b^{m-p})||u||_b^p.$$

We choose $\varepsilon \in \left]0, \frac{1}{2p}\frac{1}{\lambda(C_f C_{p,w_1}^p + |\mu|C_g C_{p,w_2}^p)}\right[$, so $A > 0$. Now, let $l : \mathbb{R}_+ \to \mathbb{R}$ be the function defined by $l(t) = A - Bt^{r-p} - Ct^{m-p}$. We can see, that $l(0) = A > 0$, so because l is continuous, there exists an $\varepsilon^* > 0$ such that $l(t) > 0$, for every $t \in]0, \varepsilon^*[$. Then for every $u \in E$, with $||u|| = \varepsilon^{**} < \min\{\varepsilon^*, ||u_0||\}$, we have $\mathcal{E}_{\lambda,\mu}(u) \geq \eta(\lambda, \mu, \varepsilon^*) > 0$. From Lemma 3.1.6 we have $\mathcal{E}_{\lambda,\mu}(u_0) < 0$.

Therefore the functional $\mathcal{E}_{\lambda,\mu}$ satisfies the hypotheses of the Mountain Pass Theorem 1.3.1. □

Proofs of the main results

The key tools used in the proofs of the main results are the Mountain Pass Theorem 1.3.1 and the Theorem 3.1.3 established by Bonanno in [12]. We recall here this later one.

Theorem 3.1.3. ([12]) *Let X be a separabile and reflexive real Banach space and let $\Phi, J : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that*

(i) *there exists $x_0 \in X$ such that $\Phi(x_0) = J(x_0) = 0$;*

(ii) *$\Phi(x) \geq 0$ for every $x \in X$;*

(iii) *there exists $x_1 \in X$, $\rho > 0$ such that $\rho < \Phi(x_1)$ and $\sup\{J(x) : \Phi(x) < \rho\} < \rho\frac{J(x_1)}{\Phi(x_1)}$.*

(iv) *the functional $\Phi - \lambda J$ is sequentially weakly lower semicontinuous, satisfies the Palais-Smale condition for every $\lambda > 0$ and it is coercive, for every $\lambda \in [0, \bar{a}]$, where $\bar{a} = \dfrac{\zeta\rho}{\rho\dfrac{J(x_1)}{\Phi(x_1)} - \sup\limits_{\Phi(x)<\rho} J(x)}$, with $\zeta > 1$.*

Then there is an open interval $\Lambda \subset [0, \bar{a}]$ and a number $\sigma > 0$ such that for each $\lambda \in \Lambda$, the equation $\Phi'(x) - \lambda J'(x) = 0$ admits at least three distinct solutions in X, having norm less than σ.

Proof of Theorem 3.1.1. Let $u_0 \in E$ from Lemma 3.1.2 and fix

$$\mu_0 = \frac{J_F(u_0)}{1 + |J_G(u_0)|}.$$

Now, we apply the Theorem 3.1.3 of Bonanno, by choosing $X = E$, $\Phi = \dfrac{1}{p}||u||_b^p$ and $J = J_\mu = J_F + \mu J_G$, for $\mu \in [-\mu_0, \mu_0]$. Note that $\mathcal{E}_{\lambda,\mu} = \Phi - \lambda J_\mu$. For every $\mu \in [-\mu_0, \mu_0]$ we have

$$J_\mu(u_0) = J_F(u_0) + \mu J_G(u_0) \geq \frac{J_F(u_0)}{1 + |J_G(u_0)|} > 0.$$

Taking account the Lemma 3.1.5 and the above inequality, for every $\mu \in [-\mu_0, \mu_0]$ one can choose $\rho_\mu > 0$ so small that

$$\rho_\mu < min\left\{1, \frac{||u_0||_b^p}{p}\right\} \tag{3.14}$$

$$\frac{\sup\{J_\mu(u) : ||u||_b^p < \rho_\mu\}}{\rho_\mu} < \frac{pJ_\mu(u_0)}{||u_0||_b^p} = \frac{J_\mu(u_0)}{\Phi(u_0)} \tag{3.15}$$

Since $\mathcal{E}_{\lambda,\mu}$ satisfies the PS-condition (see Lemma 3.1.4) , using the Lemma 3.1.1 and choosing $x_1 = u_0$, $x_0 = 0$, $\zeta = 1 + \rho_\mu$ and

$$a = \bar{a}_\mu = \frac{1 + \rho_\mu}{pJ_\mu(u_0)||u_0||_b^{-p} - sup\{J_\mu(u) : ||u||_b^p < \rho_\mu\}\rho_\mu^{-1}},$$

all the assumptions of Theorem 3.1.3 are verified. Then, there is an open interval $\Lambda_\mu \subset [0, \bar{a}_\mu]$ and a number $\sigma_\mu > 0$ such that for any $\lambda \in \Lambda_\mu$, the functional $\mathcal{E}_{\lambda,\mu} = \Phi - \lambda J_\mu$ admits at least three distinct critical points: $u^i_{\lambda,\mu} \in E$, $(i \in \{1, 2, 3\})$, having norm less than σ_μ, concluding the proof of Theorem 3.1.1.

Remark 3.1.2. Taking into account the equations (3.14)and (3.15), we have

$$\bar{a}_\mu < \frac{1 + \rho_\mu}{(p-1)J_\mu(u_0)||u_0||_b^{-p}} \leq \frac{||u||_b^p \left((1 + min\left\{1, \frac{||u_0||_b^p}{p}\right\}\right)}{(p-1)J_F(u_0)}(1 + |J_G(u_0)|).$$

Since the right side does not depend on $\mu \in \mathbb{R}$, we have an uniform estimation of Λ_μ, i.e. for every $\mu \in [-\mu_0, \mu_0]$ we have

$$\Lambda_\mu \subset \left[0, \frac{||u||_b^p \left((1 + min\left\{1, \frac{||u_0||_b^p}{p}\right\}\right)}{(p-1)J_F(u_0)}(1 + |J_G(u_0)|)\right].$$

Proof of Theorem 3.1.2.

Fix $\lambda > \lambda_0$ and $\mu \in]0, \mu^*_\lambda[= I_\lambda$.

The functional $\mathcal{E}_{\lambda,\mu}$ satisfies the (PS)-condition (Lemma 3.1.4), it is coercive (Lemma 3.1.3), therefore it attains its infimum (see [86]), i.e. there exists an element $u_{\lambda,\mu} \in E$ such that $\mathcal{E}_{\lambda,\mu}(u_{\lambda,\mu}) = \inf_{v\in E} \mathcal{E}_{\lambda,\mu}(v)$. So, using Lemma 3.1.6 $u_{\lambda,\mu}$ is a critical point of $\mathcal{E}_{\lambda,\mu}$, such that $\mathcal{E}_{\lambda,\mu}(u_{\lambda,\mu}) < 0$.

Then by Lemma 3.1.7 and the Mountain Pass Theorem 1.3.1, there exists an element $v_{\lambda,\mu} \in E$ such that $\mathcal{E}'_{\lambda,\mu}(v_{\lambda,\mu}) = 0$ and $\mathcal{E}_{\lambda,\mu}(v_{\lambda,\mu}) \geq \eta(\lambda, \mu, \varepsilon^*) > 0$, which completes the proof. $\square$

3.1.2 Semilinear elliptic problem in double weighted Sobolev spaces

In this section we study two semilinear elliptic problems on an unbounded domain $\Omega \subset \mathbb{R}^N$. The first one is an inhomogeneous semilinear double eigenvalue problem with nonlinear boundary conditions. To obtain existence and multiplicity results for this problem, we use the Mountain Pass Theorem 1.3.1 and the result of Bonanno (Theorem 3.1.3) as in Section 3.1.1. The second problem is a homogeneous semilinear elliptic problem. We obtain multiple solutions for this problem, using variational methods and some recent results from the theory of best approximation established by Ricceri in [88] and Tsar'kov in [96]. In both cases the solutions are from a double weighted Sobolev space. A main difference between the considered problems is that in the first one the nonlinearity function f is *superlinear at the origin* in the second variable, in the second one we assume that f is *sublinear* at the same point.

We will follow mainly the articles of Mezei [66] and of Mezei and Kovács [69].

Superlinear inhomogeneous problem

Let $\Omega \subset \mathbb{R}^N$, $(N \geq 3)$ be an unbounded domain with smooth boundary Γ.
For the positive measurable functions u and w, both defined in Ω, we define the weighted p-norm $(1 \leq p < \infty)$

$$||u||_{p,\Omega,w} = \left(\int_\Omega |u(x)|^p w(x) dx \right)^{\frac{1}{p}}$$

and denote by $L^q(\Omega; w)$ the space of all measurable functions u such that $||u||_{q,\Omega,w}$ is finite. The double weighted Sobolev space

$$W^{1,p}(\Omega; v_0, v_1)$$

is defined as the space of all functions $u \in L^p(\Omega; v_0)$ such that all derivatives $\frac{\partial u}{\partial x_i}$ belong to $L^p(\Omega; v_1)$. The corresponding norm is defined by

$$||u||_{p,\Omega,v_0,v_1} = \left(\int_\Omega |\nabla u(x)|^p v_1(x) + |u(x)|^p v_0(x) dx \right)^{\frac{1}{p}}.$$

We are choosing our weight functions from the so-called Muckenhoupt class A_p, which is defined as the set of all positive functions v in $\mathbb{R}^N$, which satisfy

$$\frac{1}{|Q|} \left(\int_\Omega v \, dx \right)^{\frac{1}{p}} \left(\int_\Omega v^{-\frac{1}{p-1}} \, dx \right)^{\frac{p-1}{p}} \leq \bar{C}, \text{ if } 1 < p < \infty$$

$$\frac{1}{|Q|}\int_\Omega v\ dx \leq \bar{C} \text{ ess} \inf_{x\in Q} v(x), \text{ if } p = 1,$$

for all cubes $Q \in \mathbb{R}^N$ and some $\bar{C} > 0$.

In this section we always assume that the weight functions v_0, v_1, w are defined in Ω, belong to A_p and are choosen such that the embeddings

$$W^{1,2}(\Omega; v_0, v_1) \hookrightarrow L^p(\Omega; w) \tag{3.16}$$

and the trace

$$W^{1,2}(\Omega; v_0, v_1) \hookrightarrow L^q(\Gamma; w) \tag{3.17}$$

are compact for $2 < p < 2N/(N-2)$, $2 < q < 2(N-1)/(N-2)$ and continuous for $2 \leq p \leq 2N/(N-2)$, $2 \leq q \leq 2(N-1)/(N-2)$ respectively. Such weight functions there exist, see for example [83], [84]. The best embedding constants are denoted by $C_{p,\Omega}$ and $C_{q,\Gamma}$, i.e. we have the inequalities

$$||u||_{p,\Omega,w} \leq C_{p,\Omega}||u||_{v_0,v_1}, \quad \text{for all } u \in W^{1,2}(\Omega; v_0, v_1) \tag{3.18}$$

$$||u||_{q,\Gamma,w} \leq C_{q,\Gamma}||u||_{v_0,v_1}, \quad \text{for all } u \in W^{1,2}(\Omega; v_0, v_1) \tag{3.19}$$

where we used the abbreviation $||u||_{v_0,v_1} = ||u||_{2,\Omega,v_0,v_1}$.

For $\lambda > 0$ and $\mu \in \mathbb{R}$ we consider the following semilinear elliptic double eigenvalue problem

$$(P'_{\lambda,\mu}) \qquad \begin{cases} -\Delta u + b(x)u = \lambda f(x,u) \text{ in } \Omega \\ \partial_n u = \lambda\mu g(x,u) \text{ on } \Gamma, \end{cases}$$

where b is a positive measurable function, ∂_n is the outer normal derivative on Γ, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions.
We define a bilinear form on $W^{1,2}(\Omega; v_0, v_1)$ associated with the operator $Au = -\Delta u + b(x)u$ by

$$\langle u, v\rangle_A = \int_\Omega (\nabla u \nabla v + b(x)uv)dx. \tag{3.20}$$

A weak solution of the problem $(P'_{\lambda,\mu})$ is a function $u \in W^{1,2}(\Omega; v_0, v_1)$, such that for every $v \in W^{1,2}(\Omega; v_0, v_1)$ we have

$$\langle u, v\rangle_A - \lambda \int_\Omega f(x, u(x))v(x)dx = \lambda\mu \int_\Gamma g(x, u(x))v(x)d\Gamma.$$

Furthermore we consider the following assumptions on A, f, g:

(A) the bilinear form $\langle\cdot,\cdot\rangle_A$ defined by (3.20) on $W^{1,2}(\Omega;v_0,v_1)$ is continuous and satisfies the ellipticity condition

$$\langle u,u\rangle_A \geq 2K||u||^2_{v_0,v_1} \text{ for every } u\in W^{1,2}(\Omega;v_0,v_1),$$

with some positive constant $K>0$;

(F1′) $f(\cdot,0)=0$ and $|f(x,s)|\leq f_0(x)+f_1(x)|s|^{p-1}$ for $x\in\Omega$, $s\in\mathbb{R}$, where $2<p<\dfrac{2N}{N-2}$, and f_0,f_1 are positive measurable functions satisfying $f_0\in L^{\frac{p}{p-1}}(\Omega;w^{\frac{1}{1-p}})$, $f_0(x)\leq C_f w(x)$ and $f_1(x)\leq C_f w(x)$ for a.e. $x\in\Omega$, with an appropiate constant C_f;

(F2′) $\displaystyle\lim_{s\to 0}\frac{f(x,s)}{f_0(x)|s|}=0$, uniformly in $x\in\Omega$;

(F3′) $\displaystyle\lim_{s\to\infty}\frac{F(x,s)}{f_0(x)|s|^2}=0$, uniformly in $x\in\Omega$,

$\displaystyle\max_{|s|\leq M}F(\cdot,s)\in L^1(\Omega)$, for all $M>0$, where $\displaystyle F(x,u)=\int_0^u f(x,s)ds$;

(F4′) there exist $x_0\in\Omega$, $s_0\in\mathbb{R}$ and $R_0>0$ such that $\displaystyle\min_{|x-x_0|<R}F(x,s_0)>0$.

(G1′) $g(\cdot,0)=0$ and $|g(x,s)|\leq g_0(x)+g_1(x)|s|^{q-1}$, for $x\in\Gamma$, $s\in\mathbb{R}$, where $2<q<\dfrac{2(N-1)}{N-2}$, and g_0,g_1 are positive measurable functions satisfying $g_0\in L^{\frac{q}{q-1}}(\Gamma;w^{\frac{1}{1-q}})$, $g_0(x)\leq C_g w(x)$ and $g_1(x)\leq C_g w(x)$, a.e. $x\in\Gamma$, with an appropiate constant C_g;

(G2′) $\displaystyle\lim_{s\to 0}\frac{g(x,s)}{g_0(x)|s|}=0$, uniformly in $x\in\Gamma$;

(G3′) $\displaystyle\lim_{s\to+\infty}\frac{G(x,s)}{g_0(x)|s|^2}=0$, uniformly for $x\in\Gamma$,

$\displaystyle\max_{|s|\leq M}G(\cdot,s)\in L^1(\Gamma)$, for every $M>0$, where $\displaystyle G(x,s)=\int_0^u g(x,s)ds$.

Next, we introduce the functionals $J_F,J_G,J_\mu : W^{1,2}(\Omega;v_0,v_1)\to\mathbb{R}$, defined by

$$J_F(u)=\int_\Omega F(x,u(x))dx,\qquad J_G(u)=\int_\Gamma G(x,u(x))d\Gamma,$$

$$J_\mu(u) = J_F(u) + \mu J_G(u)$$

and the energy functional $\mathcal{E}_{\lambda,\mu} : W^{1,2}(\Omega; v_0, v_1) \to \mathbb{R}$ associated to $(P'_{\lambda,\mu})$, defined by

$$\mathcal{E}_{\lambda,\mu}(u) = \frac{1}{2}\langle u, u\rangle_A - \lambda J_\mu(u).$$

The main result is the following

Theorem 3.1.4. *([67]) We suppose that the assumption (A) is satisfied and the functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfy the conditions $(F1') - (F4')$ and $(G1') - (G3')$ respectively. Then*

(a) there exists $\lambda_0 > 0$ such that to every $\lambda \in]\lambda_0, +\infty[$ it corresponds a nonempty open interval $I_\lambda \subset \mathbb{R}$ such that for every $\mu \in I_\lambda$ the problem $(P'_{\lambda,\mu})$ has at least two distinct, nontrivial weak solutions $u_{\lambda,\mu}$ and $v_{\lambda,\mu} \in W^{1,2}(\Omega; v_0, v_1)$, with the property

$$\mathcal{E}_{\lambda,\mu}(u_{\lambda,\mu}) < 0 < \mathcal{E}_{\lambda,\mu}(v_{\lambda,\mu});$$

(b) there exists $\mu_0 > 0$ such that to every $\mu \in [-\mu_0, \mu_0]$ it corresponds a nonempty open interval $\Gamma_\mu \in]0, +\infty[$ and a number $\sigma_\mu > 0$ for which $(P'_{\lambda,\mu})$ has at least two distinct, nontrivial weak solutions: $u^1_{\lambda,\mu}$ and $u^2_{\lambda,\mu} \in W^{1,2}(\Omega; v_0, v_1)$, with the property

$$max\{||u^1_{\lambda,\mu}||_{v_0,v_1}, ||u^2_{\lambda,\mu}||_{v_0,v_1}\} \le \sigma_\mu,$$

whenever $\lambda \in \Gamma_\mu$.

In what follows we present the auxiliary results needed for the proof of this theorem. We shall omit the proofs of those properties, which are similar to the corresponding proofs given in the previous chapter.

We recall here first a classical inequality that is widely used in our proofs. For all $s \in (0, \infty)$ there is a constant $C_s > 0$ such that

$$(x + y)^s \le C_s(x^s + y^s), \quad \text{for any } x, y \in (0, \infty). \tag{3.21}$$

We denote by p' and q' the dual indices of p, respectively q, i.e. $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$.

The next result deals with the Nemytskii operator of a Carathéodory function $h : \Omega \times \mathbb{R} \to \mathbb{R}$, which is the function defined by $N_h(u) = h(x, u(x))$.

Lemma 3.1.8. *Assume that the conditions* (F1′), (G1′) *are satisfied. Then, the Nemytskii operators* $N_f : L^p(\Omega; w) \to L^{\frac{p}{p-1}}(\Omega; w^{\frac{1}{1-p}})$, $N_F : L^p(\Omega; w) \to L^1(\Omega)$, $N_g : L^q(\Gamma; w) \to L^{\frac{q}{q-1}}(\Gamma; w^{\frac{1}{1-q}})$ *and* $N_F : L^q(\Gamma; w) \to L^1(\Gamma)$ *are bounded and continuous.*

Proof. To prove that N_f is bounded, we choose an arbitrary set $A \subseteq L^p(\Omega; w)$ and prove that $N_f(A)$ is bounded. For this, let $u \in A$ be an arbitrary element and we claim that $N_f(u)$ is bounded in $L^{\frac{p}{p-1}}(\Omega; w^{\frac{1}{1-p}})$. Using the (F1′) condition, the (3.21), the Hölder's inequalities, we have

$$
\begin{aligned}
||N_f(u)||^{\frac{1}{p'}}_{\frac{p}{p-1},\Omega,w^{\frac{1}{1-p}}} &= \int_\Omega |f(x,u(x))|^{p'} w(x)^{\frac{1}{1-p}} dx \leq \\
&\leq \int_\Omega \left(f_0(x) + f_1(x)|u(x)|^{p-1}\right)^{p'} w(x)^{\frac{1}{1-p}} dx \leq \\
&\leq C_{p'} \left(\int_\Omega f_0(x)^{p'} w(x)^{\frac{1}{1-p}} dx + \int_\Omega f_1(x)^{p'} |u(x)|^{(p-1)p'} w(x)^{\frac{1}{1-p}} dx\right) \leq \\
&\leq C_{p'} \left(C + \int_\Omega C_f^{p'} w(x)^{p'} w(x)^{\frac{1}{1-p}} |u(x)|^p dx\right) = \\
&= C_{p'}C + C_{p'}C_f^{p'} \int_\Omega |u(x)|^p w(x) dx = C_{p'}C + C_{p'}C_f^{p'} ||u||^p_{p,\Omega,w},
\end{aligned}
$$

where in the last inequality we used that $f_0 \in L^{\frac{p}{p-1}}(\Omega; w^{\frac{1}{1-p}})$, so there exists $C > 0$ such that $\int_\Omega f_0(x)^{\frac{p}{p-1}} w(x)^{\frac{1}{1-p}} dx \leq C$. Since $u \in A \subseteq L^p(\Omega; w)$, we have that $||u||^p_{p,\Omega,w}$ is finite, therefore N_f is bounded. Then the continuity follows from standard properties of the Nemytskii operators.

In the same way we obtain for $u \in L^p(\Omega; w)$

$$
\begin{aligned}
\int_\Omega |F(x,u(x))| dx &\leq \int_\Omega \left(f_0(x)|u(x)| + f_1(x)|u(x)|^p\right) dx = \\
&= \int_\Omega f_0(x) w(x)^{-\frac{1}{p}} |u(x)| w(x)^{\frac{1}{p}} dx + \int_\Omega f_1(x)|u(x)|^p dx \leq \\
&\leq \left(\int_\Omega f_0(x)^{p'} w(x)^{\frac{1}{1-p}} dx\right)^{\frac{1}{p'}} \left(\int_\Omega |u(x)|^p w(x) dx\right)^{\frac{1}{p}} + C_f \int_\Omega |u(x)|^p w(x) dx \leq \\
&\leq C^{\frac{1}{p'}} ||u||_{p,\Omega,w} + C_f ||u||^p_{p,\Omega,w},
\end{aligned}
$$

therefore N_F is bounded. For the operators N_g and N_G the arguments are identical, therefore we omit the details here. $\square$

Lemma 3.1.9. (Pflüger [84]) *The energy functional $\mathcal{E}_{\lambda,\mu}$ is Fréchet differentiable in $W^{1,2}(\Omega; v_0, v_1)$ and its derivative is given by*

$$\langle \mathcal{E}'_{\lambda,\mu}(u), v\rangle = \langle u, v\rangle_A - \lambda \int_\Omega f(x, u(x))v(x)dx - \lambda\mu \int_\Gamma g(x, u(x))v(x)d\Gamma.$$

for every $v \in W^{1,2}(\Omega; v_0, v_1)$.

Remark 3.1.3. Due to this result, we can see, that the critical points of $\mathcal{E}_{\lambda,\mu}$ are exactly the weak solutions of $(P'_{\lambda,\mu})$.

Lemma 3.1.10. *Suppose that the conditions* (F2′), (F3′), (G2′) *and* (G3′) *are satisfied. Then, for every $\lambda > 0$ and $\mu \in \mathbb{R}$ the functional $\mathcal{E}_{\lambda,\mu}$ is coercive and bounded from below on $W^{1,2}(\Omega; v_0, v_1)$.*

Proof. Let us fix $\lambda > 0$ and $\mu \in \mathbb{R}$ arbitrarily and $a,\ b > 0$ such that

$$\lambda a C_f C_{2,\Omega}^2 + \lambda|\mu| b C_g C_{2,\Gamma}^2 < K.$$

By the conditions (F2′),(F3′) and (G2′),(G3′) there exist the positive functions $k_a \in L^1(\Omega; w)$ and $k_b \in L^1(\Gamma; w)$ such that

$$|F(x, s)| \leq a f_0(x)|s|^2 + k_a(x)w(x), \quad \forall (x, s) \in \Omega \times \mathbb{R}$$

$$|G(x, s)| \leq b g_0(x)|s|^2 + k_b(x)w(x), \quad \forall (x, s) \in \Omega \times \mathbb{R}.$$

Thus, for every $u \in W^{1,2}(\Omega; v_0, v_1)$ we obtain

$$\begin{aligned}
\mathcal{E}_{\lambda,\mu}(u) &= \frac{1}{2}\langle u, u\rangle_A - \lambda \int_\Omega F(x, u(x))dx - \lambda\mu \int_\Gamma G(x, u(x)dx) \geq \\
&\geq K||u||_{v_0,v_1}^2 - \lambda \int_\Omega a f_0(x)|u(x)|^2 dx - \lambda \int_\Omega k_a(x)w(x)dx - \\
&\quad -\lambda|\mu| \int_\Gamma b g_0(x)|u(x)|^2 d\Gamma - \lambda|\mu| \int_\Gamma k_b(x)w(x)d\Gamma \geq \\
&\geq K||u||_{v_0,v_1}^2 - \lambda a C_f ||u||_{2,\Omega,w}^2 - \lambda||k_a||_{1,\Omega,w} - \\
&\quad -\lambda|\mu| b C_g ||u||_{2,\Gamma,w}^2 - \lambda|\mu|||k_b||_{1,\Gamma,w} \geq \\
&\geq \left(K - \lambda a C_f C_{2,\Omega}^2 - \lambda|\mu| b C_g C_{2,\Gamma}^2\right) ||u||_{v_0,v_1}^2 - \\
&\quad -\lambda||k_a||_{1,\Omega,w} - \lambda|\mu|||k_b||_{1,\Gamma,w}.
\end{aligned}$$

Since $k_a \in L^1(\Omega; w)$, $k_b \in L^1(\Gamma; w)$, we have that $||k_a||_{1,\Omega,w}$, $||k_b||_{1,\Gamma,w}$ are finite. Therefore $\mathcal{E}_{\lambda,\mu}$ is bounded from below on $W^{1,2}(\Omega; v_0, v_1)$ and $\mathcal{E}_{\lambda,\mu}(u) \to \infty$, whenever $||u||_{v_0,v_1} \to \infty$. Hence $\mathcal{E}_{\lambda,\mu}$ is coercive. □

Lemma 3.1.11. *$\mathcal{E}_{\lambda,\mu} : W^{1,2}(\Omega; v_0, v_1) \to \mathbb{R}$ satisfies the Palais-Smale condition on $W^{1,2}(\Omega; v_0, v_1)$, for every $\lambda > 0$ and $\mu \in \mathbb{R}$.*

Proof. Let $\{u_n\} \subset W^{1,2}(\Omega; v_0, v_1)$ be an arbitrary Palais-Smale sequence for $\mathcal{E}_{\lambda,\mu}$, i.e.

(a) $\{\mathcal{E}_{\lambda,\mu}(u_n)\}$ is bounded;

(b) $\mathcal{E}'_{\lambda,\mu}(u_n) \to 0$.

We will prove that $\{u_n\}$ contains a strongly convergent subsequence. Since $\mathcal{E}_{\lambda,\mu}$ is coercive, we have that $\{u_n\}$ is bounded. $W^{1,2}(\Omega; v_0, v_1)$ is a reflexive Banach space, so taking a subsequence if necessary (denoted in the same way), we get an element $u \in W^{1,2}(\Omega; v_0, v_1)$ such that $u_n \to u$ weakly in $W^{1,2}(\Omega; v_0, v_1)$. Because the embeddings (3.16) and (3.17) are compact for $2 < p < 2N/(N-2)$, $2 < q < 2(N-1)/(N-2)$, we have that $u_n \to u$ strongly in $L^p(\Omega; w)$ and $L^q(\Gamma; w)$.

From the condition (b) we have that $\left|\langle \mathcal{E}'_{\lambda,\mu}(u_n), \frac{u_n}{||u_n||_{v_0,v_1}}\rangle\right| \leq \varepsilon$, for every $\varepsilon > 0$ and for large $n \in \mathbb{N}$. So

$$-\langle u_n, u_n\rangle_A + \lambda \int_\Omega f(x, u_n(x))u_n(x)dx + \lambda\mu \int_\Gamma g(x, u_n(x))u_n(x)d\Gamma \leq$$

$$\leq \varepsilon ||u_n||_{v_0,v_1}.$$

Then we have

$$\begin{aligned} 2K||u_n - u||^2_{v_0,v_1} &\leq \langle u_n - u, u_n - u\rangle_A \leq |\langle u_n, u_n - u\rangle_A| + |\langle u, u_n - u\rangle_A| \\ &\leq 2\varepsilon||u_n - u||_{v_0,v_1} + \lambda \left|\int_\Omega f(x, u_n(x))(u_n(x) - u(x))dx\right| \\ &\quad +\lambda \left|\int_\Omega f(x, u(x))(u_n(x) - u(x))dx\right| \\ &\quad +\lambda|\mu| \left|\int_\Gamma g(x, u_n(x))(u_n(x) - u(x))d\Gamma\right| \\ &\quad +\lambda|\mu| \left|\int_\Gamma g(x, u(x))(u_n(x) - u(x))d\Gamma\right|. \end{aligned}$$

Using the Hölder's inequality we get

$$\left|\int_\Omega f(x, u_n(x))(u_n(x) - u(x))dx\right| \leq$$

$$\leq \int_\Omega \left|f(x, u_n(x))w(x)^{-\frac{1}{p}}\right| \left|(u_n(x) - u(x))w(x)^{\frac{1}{p}}\right| dx$$

$$\leq \left(\int_\Omega |f(x,u_n(x))|^{p'} w(x)^{-\frac{p'}{p}} dx\right)^{\frac{1}{p'}} \left(\int_\Omega |u_n(x)-u(x)|^p w(x) dx\right)^{\frac{1}{p}}$$

$$= \left(\int_\Omega |f(x,u_n(x))|^{p'} w(x)^{\frac{1}{1-p}} dx\right)^{\frac{1}{p'}} ||u_n - u||_{p,\Omega,w}$$

and arguing in the same way for g, we obtain

$$\left|\int_\Gamma g(x,u_n(x))(u_n(x)-u(x))dx\right| \leq$$

$$\leq \left(\int_\Gamma |g(x,u_n(x))|^{q'} w(x)^{\frac{1}{1-q}} d\Gamma\right)^{\frac{1}{q'}} ||u_n - u||_{q,\Gamma,w}.$$

Since $\varepsilon > 0$ is arbitrary, $||u_n - u||_{p,\Omega,w}$ and $||u_n - u||_{q,\Gamma,w}$ tend to zero and $\int_\Omega |f(x,u_n(x))|^{p'} w(x)^{\frac{1}{1-p}} dx$, $\int_\Gamma |g(x,u_n(x))|^{q'} w(x)^{\frac{1}{1-q}} d\Gamma$ are bounded (by Lemma 3.1.8, using that $\{u_n\}$ is bounded), it follows that $||u_n - u||_{v_0,v_1}$ tends to zero. □

The proofs of the next three lemmas can be made following the same ideas as in the corresponding proofs presented in the previous chapter, therefore we shall omit them.

Lemma 3.1.12. ([71, Lemma 3.2]) *Assume that (F4′) is satisfied. Then there exists a* $u_0 \in W^{1,2}(\Omega; v_0, v_1)$ *such that* $J_F(u_0) > 0$.

Let us define $m = \int_\Gamma |G(x,u_0(x))| d\Gamma$, $\lambda_0 = \dfrac{\frac{1}{2}\langle u_0, u_0\rangle_A}{J_F(u_0)} > 0$ and $\mu^*_\lambda = \frac{1}{\lambda(1+m)} \cdot (\lambda - \lambda_0) J_F(u_0) > 0$.

Lemma 3.1.13. ([67, Lemma 2.6]) *For* $\lambda > \lambda_0$ *and* $|\mu| \in]0, \mu^*_\lambda]$ *we have*

$$\inf_{u \in W^{1,2}(\Omega; v_0, v_1)} \mathcal{E}_{\lambda,\mu}(u) < 0.$$

Lemma 3.1.14. ([67, Lemma 2.7]) *For every* $\lambda > \lambda_0$ *and* $\mu \in]0, \mu^*_\lambda]$, *the functional* $\mathcal{E}_{\lambda,\mu}$ *satisfies the Mountain Pass geometry.*

Lemma 3.1.15. *For every* $\mu \in \mathbb{R}_+$, *we have*

$$\lim_{\rho \to 0} \frac{\sup\{J_\mu(u) : \frac{1}{2}\langle u, u\rangle_A < \rho\}}{\rho} = 0.$$

Proof. From the assumptions (F1′), (F2′), (G1′) and (G2′) results the existence of $\hat{c}_1(\varepsilon)$, $\hat{c}_2(\varepsilon) > 0$ such that, for every $\hat{\varepsilon} > 0$ we have

$$|f(x,s)| \le \hat{\varepsilon} f_0(x)|s| + \hat{c}_1(\varepsilon) f_1(x)|s|^{p-1}, \text{ for } 2 < p < \frac{2N}{N-2}, \tag{3.22}$$

$$|g(x,s)| \le \hat{\varepsilon} g_0(x)|s| + \hat{c}_2(\varepsilon) g_1(x)|s|^{q-1}, \text{ for } 2 < q < \frac{2(N-1)}{N-2}. \tag{3.23}$$

Then integrating with respect to the second variable, from 0 to $u(x)$, we get the existence of $c_1(\varepsilon)$, $c_2(\varepsilon) > 0$ such that, for every $\varepsilon > 0$ we have

$$|F(x,u(x))| \le \varepsilon f_0(x)|u(x)|^2 + c_1(\varepsilon) f_1(x)|u(x)|^p, \text{ for } 2 < p < \frac{2N}{N-2}, \tag{3.24}$$

$$|G(x,u(x))| \le \varepsilon g_0(x)|u(x)|^2 + c_2(\varepsilon) g_1(x)|u(x)|^q, \text{ for } 2 < q < \frac{2(N-1)}{N-2}. \tag{3.25}$$

Fix arbitrarily $\varepsilon > 0$ and $p \in \left]2, \frac{2N}{N-p}\right[$, $q \in \left]2, \frac{2(N-1)}{N-2}\right[$, then from (3.24) and (3.25) and the ellipticity condition (A), it follows that

$$\begin{aligned}
J_\mu(u) &= J_F(u) + \mu J_G(u) \\
&\le \varepsilon \left(C_f C_{2,\Omega}^2 + |\mu| C_g C_{2,\Gamma}^2\right) ||u||_{v_0,v_1}^2 + c_1(\varepsilon) C_f C_{p,\Omega}^p ||u||_{v_0,v_1}^p \\
&+ |\mu| c_2(\varepsilon) C_g C_{q,\Gamma}^q ||u||_{v_0,v_1}^q \\
&\le \varepsilon \left(C_f C_{2,\Omega}^2 + |\mu| C_g C_{2,\Gamma}^2\right) \frac{\langle u,u\rangle_A}{2K} + c_1(\varepsilon) C_f C_{p,\Omega}^p \left(\frac{\langle u,u\rangle_A}{2K}\right)^{\frac{p}{2}} \\
&+ |\mu| c_2(\varepsilon) C_g C_{q,\Gamma}^q \left(\frac{\langle u,u\rangle_A}{2K}\right)^{\frac{q}{2}}.
\end{aligned}$$

Therefore, we have

$$\sup\{J_\mu(u) : \frac{1}{2}\langle u,u\rangle_A < \rho\} \le$$

$$\le \varepsilon \frac{\left(C_f C_{2,\Omega}^2 + |\mu| C_g C_{2,\Gamma}^2\right)}{K}\rho + \frac{c_1(\varepsilon) C_f C_{p,\Omega}^p}{K^{\frac{p}{2}}}\rho^{\frac{p}{2}} + |\mu| \frac{c_2(\varepsilon) C_g C_{q,\Gamma}^q}{K^{\frac{q}{2}}}\rho^{\frac{q}{2}}.$$

Since $p > 2$, $q > 2$, dividing this last inequality with ρ and taking the limit whenever $\rho \to 0$, we have the required equality.

Lemma 3.1.16. *Assume that the conditions* (F1′)-(F3′) *and* (G1′)-(G3′) *are satisfied. Then, the functional* $J_\mu = J_F + \mu J_G$ *is sequentially weakly continuous.*

Proof. We argue by contradiction. Let u_n be a sequence from $W^{1,2}(\Omega; v_0, v_1)$ weakly convergent to some $u \in W^{1,2}(\Omega; v_0, v_1)$ and $d > 0$ such that

$$|J_\mu(u_n) - J_\mu(u)| \geq d, \qquad \text{for all } n \in \mathbb{N}.$$

At the same time we have

$$\begin{aligned} |J_\mu(u_n) - J_\mu(u)| &\leq \int_\Omega |F(x, u_n(x)) - F(x, u(x))| dx \\ &+ |\mu| \int_\Gamma |G(x, u_n(x)) - G(x, u(x))| d\Gamma. \end{aligned}$$

In the sequel, we will estimate the previous two integrals. For this end, first we use the Mean Value Theorem for the mapping F on the interval $(u_n(x), u(x))$, then we make use of the (3.18), (3.22) and the Hölder inequalities. So, there exists a $\theta \in]0, 1[$ such that

$$\begin{aligned} & \int_\Omega |F(x, u_n(x)) - F(x, u(x))| dx \\ = & \int_\Omega |f(x, (1-\theta)u_n(x) + \theta u(x))||u_n(x) - u(x)| dx \\ \leq & \hat{\varepsilon} \int_\Omega f_0(x)|(1-\theta)u_n(x) + \theta u(x)||u_n(x) - u(x)| dx \\ + & \hat{c}_1(\varepsilon) \int_\Omega f_1(x)|(1-\theta)u_n(x) + \theta u(x)|^{p-1}|u_n(x) - u(x)| dx \\ \leq & \hat{\varepsilon} \int_\Omega f_0(x)(|u_n(x)| + |u(x)|)|u_n(x) - u(x)| dx \\ & + \hat{c}_1(\varepsilon) \int_\Omega f_1(x)(|u_n(x)|^{p-1} + |u(x)|^{p-1})|u_n(x) - u(x)| dx \\ \leq & \hat{\varepsilon} C_f \int_\Omega |u_n(x) - u(x)| w(x)^{\frac{1}{2}} w(x)^{\frac{1}{2}} (|u_n(x)| + |u(x)|) dx \\ & + \hat{c}_1(\varepsilon) C_f \int_\Omega |u_n(x) - u(x)| w(x)^{\frac{1}{p}} w(x)^{\frac{1}{p'}} \left(|u_n(x)|^{p-1} + |u(x)|^{p-1}\right) dx \\ \leq & \hat{\varepsilon} C_f \left(\int_\Omega |u_n(x) - u(x)|^2 w(x) dx \right)^{\frac{1}{2}} \cdot \\ & \cdot \left[\left(\int_\Omega |u_n(x)|^2 w(x) dx \right)^{\frac{1}{2}} + \left(\int_\Omega |u(x)|^2 w(x) dx \right)^{\frac{1}{2}} \right. \end{aligned}$$

$$
\begin{aligned}
& +\hat{c}_1(\varepsilon)C_f\left(\int_\Omega |u_n(x)-u(x)|^p w(x)dx\right)^{\frac{1}{p}}\cdot \\
& \cdot\left[\left(\int_\Omega |u_n(x)|^{(p-1)p'}w(x)dx\right)^{\frac{1}{p'}}+\left(\int_\Omega |u(x)|^{(p-1)p'}w(x)dx\right)^{\frac{1}{p'}}\right] \\
\leq\ & \hat{\varepsilon}C_f||u_n-u||_{2,\Omega,w}(||u_n||_{2,\Omega,w}+||u_n||_{2,\Omega,w}) \\
& +\hat{c}_1(\varepsilon)C_f||u_n-u||_{p,\Omega,w}\left(||u_n||_{p,\Omega,w}^{\frac{p}{p'}}+||u||_{p,\Omega,w}^{\frac{p}{p'}}\right) \\
\leq\ & \hat{\varepsilon}C_fC_{2,\Omega}^2||u_n-u||_{v_0,v_1}(||u_n||_{v_0,v_1}+||u||_{v_0,v_1}) \\
& +\hat{c}_1(\varepsilon)C_fC_{p,\Omega}^{p-1}||u_n-u||_{p,\Omega,w}\left(||u_n||_{v_0,v_1}^{p-1}+||u||_{v_0,v_1}^{p-1}\right).
\end{aligned}
$$

Since u_n is weakly convergent to $u\in W^{1,2}(\Omega;v_0,v_1)$, we can assume without loss of generality, that there exist a constant $M>0$ such that

$$||u_n||_{v_0,v_1}\leq M \text{ and } ||u_n-u||_{v_0,v_1}\leq M, \text{ for all } n\in\mathbb{N}.$$

Then we have

$$|F(x,u_n(x))-F(x,u(x))|\leq 2\hat{\varepsilon}C_fC_{2,\Omega}^2M^2+2\hat{c}_1(\varepsilon)C_fC_{p,\Omega}^{p-1}M^{p-1}||u_n-u||_{p,\Omega,w}.$$

Arguing as above for the function G, we obtain

$$|G(x,u_n(x))-G(x,u(x))|\leq 2\hat{\varepsilon}C_gC_{2,\Gamma}^2M^2+2\hat{c}_2(\varepsilon)C_gC_{q,\Gamma}^{q-1}M^{q-1}||u_n-u||_{q,\Gamma,w}.$$

Therefore

$$
\begin{gathered}
d\leq|J_\mu(u_n)-J_\mu(u)|\leq 2\hat{\varepsilon}M^2(C_fC_{2,\Omega}^2+C_gC_{2,\Gamma}^2)+ \\
+2\hat{c}_1(\varepsilon)C_fC_{p,\Omega}^{p-1}M^{p-1}||u_n-u||_{p,\Omega,w}+2\hat{c}_2(\varepsilon)C_gC_{q,\Gamma}^{q-1}M^{q-1}||u_n-u||_{q,\Gamma,w}.
\end{gathered}
$$

Because the embeddings (3.16) and (3.17) are compact for $2<p<2N/(N-2)$, $2<q<2(N-1)/(N-2)$, it follows that $||u_n-u||_{p,\Omega,w}\to 0$ and $||u_n-u||_{q,\Gamma,w}\to 0$. Therefore, if $\hat{\varepsilon}>0$ is sufficiently small and $n\in\mathbb{N}$ is large enough, we have

$$d\leq|J_\mu(u_n)-J_\mu(u)|<d,$$

which is a contradiction. □

The **proof of Theorem 3.1.4** is similar to the proof of the Theorems 3.1.1 and 3.1.2. We will give the proof only for the completeness.

(a) Fix $\lambda>\lambda_0$ and $\mu\in]0,\mu_\lambda^*[=I_\lambda$. From the Lemma 3.1.10 and Lemma 3.1.11 we have that the functional $\mathcal{E}_{\lambda,\mu}$ is bounded from below and satisfies the

(PS)-condition. Then $\mathcal{E}_{\lambda,\mu}$ achieves its infimum, i.e. there exists an element $u_{\lambda,\mu} \in W^{1,2}(\Omega; v_0, v_1)$ such that $\mathcal{E}_{\lambda,\mu}(u_{\lambda,\mu}) = \inf_{v \in W^{1,2}(\Omega; v_0, v_1)} \mathcal{E}_{\lambda,\mu}(v)$ (see[86, Theorem 2.7]). So $\mathcal{E}'_{\lambda,\mu}(u_{\lambda,\mu}) = 0$ and by Lemma 3.1.13, we have $\mathcal{E}_{\lambda,\mu}(u_{\lambda,\mu}) < 0$.

On the other hand, there exists an element $v_{\lambda,\mu} \in W^{1,2}(\Omega; v_0, v_1)$ such that $\mathcal{E}'_{\lambda,\mu}(v_{\lambda,\mu}) = 0$ and $\mathcal{E}_{\lambda,\mu}(v_{\lambda,\mu}) \geq \eta(\lambda, \mu, \varepsilon^*) > 0$ (by the Lemma 3.1.14 and the Mountain Pass Theorem 1.3.1), which completes the proof.

(b) Let $u_0 \in W^{1,2}(\Omega; v_0, v_1)$ be the function from Lemma 3.1.12 and fix $\mu_0 = \dfrac{J_F(u_0)}{1 + |J_G(u_0)|}$. Then for every $\mu \in [-\mu_0, \mu_0]$ we have

$$J_\mu(u_0) = J_F(u_0) + \mu J_G(u_0) \geq \frac{J_F(u_0)}{1 + |J_G(u_0)|} > 0.$$

Now, we apply the Theorem 3.1.3 of Bonanno, by choosing

$$X = W^{1,2}(\Omega; v_0, v_1),\ \ \Phi(u) = \frac{1}{2}\langle u, u\rangle_A$$

and

$$J = J_\mu, \text{for } \mu \in [-\mu_0, \mu_0].$$

Taking account the lema 3.1.15 and the inequalities $J_\mu(u_0) > 0$, $\Phi(u_0) > 0$, we can choose for every $\mu \in [-\mu_0, \mu_0]$ a $\rho_\mu > 0$ so small that $\rho_\mu < \frac{1}{2}\langle u_0, u_0\rangle_A = \Phi(u_0)$ and $\dfrac{\sup\{J_\mu(u) : \frac{1}{2}\langle u, u\rangle_A < \rho_\mu\}}{\rho_\mu} < \dfrac{J_\mu(u_0)}{\Phi(u_0)}$

Now, choosing $x_1 = u_0$, $x_0 = 0$, $\zeta = 1 + \rho_\mu$ and

$$a = \bar{a}_\mu = \frac{1 + \rho_\mu}{\frac{J_\mu(u_0)}{\Phi(u_0)} - \frac{\sup\{J_\mu(u) : \frac{1}{2}\langle u, u\rangle_A < \rho_\mu\}}{\rho_\mu}},$$

all the assumptions of the Theorem 3.1.3 are verified. Then, there is an open interval $\Lambda_\mu \subset [0, \bar{a}_\mu]$ and a number $\sigma_\mu > 0$ such that for any $\lambda \in \Lambda_\mu$, the functional $\mathcal{E}_{\lambda,\mu} = \Phi - \lambda J_\mu$ admits at least three distinct critical points: $u^i_{\lambda,\mu} \in W^{1,2}(\Omega; v_0, v_1)$, $(i \in \{1, 2, 3\})$, having norms less than σ_μ.

We can see, that $u = 0$ is a solution of the problem $(P'_{\lambda,\mu})$. So if we are looking for nontrivial solutions, we can affirm that $(P'_{\lambda,\mu})$ has at least two distinct, nontrivial solutions in $W^{1,2}(\Omega; v_0, v_1)$, having norms less than σ_μ, concluding the proof of the Theorem 3.1.4. $\square$

Example 4.1 As an example, we consider the weight functions

$$v_0(x) = w(x) = \begin{cases} ||x||^{-2}, \text{ if } x \in \mathbb{R}^N \setminus B_1 \\ 1, \text{ if } x \in B_1 \end{cases},$$

$$v_1(x) = 1, \forall x \in \mathbb{R}^N,$$

where $B_1 = \{x \in \mathbb{R}^N : ||x|| \leq 1\}$. For these functions the embeddings

$$W^{1,2}(\Omega; v_0, 1) \hookrightarrow L^p(\Omega; w) \text{ and } W^{1,2}(\Omega; v_0, 1) \hookrightarrow L^q(\Gamma; w)$$

are compact, if $2 < p < 2N/(N-2)$, $2 < q < 2(N-1)/(N-2)$ (see [84]). Assuming that f and g satisfy the conditions (F1′)-(F4′), (G1′)-(G3′) respectively and A defines a bilinear form with (A), we can apply the Theorem 3.1.4.

Sublinear homogeneous problem

Let $\Omega \subset \mathbb{R}^N$ be an unbounded domain with smooth boundary Γ.

We consider the double weighted Sobolev space $W^{1,2}(\Omega; v_0, v_1)$ with the weight functions v_0, v_1, w, which are defined on Ω, belong to A_p and are choosen such that the following condition holds:

(E) for $p \in]2, 2^*[$ the embedding $W^{1,2}(\Omega; v_0, v_1) \hookrightarrow L^p(\Omega; w)$ is compact.

The best embedding constant is denoted also by $C_{p,\Omega}$ as in the previous section, i.e. the inequality (3.18) is satisfied.

We recall the definition of the following Sobolev space

$$\begin{aligned} L^\infty(\Omega) \quad = \quad & \{u : \Omega \to \mathbb{R} \mid u \text{ is measurable}, \\ & \qquad \text{and } \exists C > 0 \text{ such that } |u(x)| \leq C \text{ a.e. in } \Omega\} \end{aligned}$$

endowed with the norm

$$||u||_\infty = \inf\{C : |u(x)| \leq C \text{ for a.e. } x \in \Omega\}.$$

We consider on $W^{1,2}(\Omega; v_0, v_1)$ the continuous bilinear form (3.20) associated with the operator $Au = -\Delta u + b(x)u$ and the corresponding norm with

$$||u||_A^2 = \langle u, u\rangle_A = \int_\Omega (|\nabla u(x)|^2 + b(x)|u(x)|^2)dx.$$

Now, we define the Banach space

$$X_A = \{u \in W^{1,2}(\Omega; v_0, v_1) : ||u||_A < \infty\} \tag{3.26}$$

endowed with the norm $||\cdot||_A$.

In this section we will prove a multiplicity result for the following problem:

For a given $u_0 \in X_A$ and $\lambda > 0$ find $u \in X_A$ such that:

$$(P_\lambda) \qquad \begin{cases} -\Delta(u-u_0) + b(x)(u-u_0) = \lambda\alpha(x)f(u) \text{ in } \Omega \\ \\ u = 0 \text{ on } \Gamma, \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\alpha : \Omega \to \mathbb{R}$ and $b : \Omega \to \mathbb{R}$ are a positive and measurable functions.

By the weak solution to this problem we mean a function $u \in X_A$, such that for every $v \in X_A$ we have

$$\langle u - u_0, v\rangle_A - \lambda \int_\Omega \alpha(x)f(u(x))v(x)dx = 0.$$

In the previous problem $(P'_{\lambda,\mu})$ the function f was *superlinear at the origin* in the second variable. Here, we study the problem (P_λ) assuming that f is *sublinear* at the same point, that is

- (f) $f(0) = 0$ and there is a positive measurable function $f_0 : \Omega \to \mathbb{R}$ satisfying $f_0 \in L^{\frac{p}{p-1}}(\Omega, w^{\frac{1}{1-p}})$, $f_0(x) \le C_f w(x)$ for a.e. $x \in \Omega$, where C_f is a positive constant and there exists $q \in]0,1[$ such that

$$|f(s)| \le f_0(x)|s|^q, \text{ for every } s \in \mathbb{R} \text{ and every } x \in \Omega;$$

Furthermore we consider the following assumptions:

- (K) ellipticity condition: there is a positive constant K, such that $||u||_A^2 \ge 2K||u||^2_{v_0,v_1}$ for every $u \in W^{1,2}(\Omega; v_0, v_1)$.
- (α) $\alpha \in L^1(\Omega, w) \cap L^\infty(\Omega)$.

We define the funtional $J : X_A \to \mathbb{R}$ by

$$J(u) = \int_\Omega \alpha(x) F(u(x))dx,$$

where $F(t) = \int_0^t f(s)ds$.

Now, we can state the main result of this subsection.

Theorem 3.1.5. ([69]) *Let $\Omega \subseteq \mathbb{R}^N$ be an unbounded domain with smooth boundary Γ or $\Omega = \mathbb{R}^N (N \geq 2)$.*

Suppose that $W^{1,2}(\Omega; v_0, v_1)$ satisfies the embedding property (E) and X_A is the subspace defined by (3.26). Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (f) and let $\alpha : \Omega \to \mathbb{R}$ be a strictly positive function satisfying (α).

Then for every $\sigma \in]\inf_{X_A} J, \sup_{X_A} J[$ and every $u_0 \in J^{-1}(]-\infty, \sigma[)$, one of the following assertions is true:

(B1) there exists $\lambda > 0$ such that the problem (P_λ) has at least three solutions in X_A;

(B2) there exists $v \in J^{-1}(\sigma)$ such that for all $u \in J^{-1}([\sigma, \infty[)$, $u \neq v$,

$$||u - u_0||_A > ||v - u_0||_A.$$

In the proof of this theorem we need some auxiliary results, proved below.

Lemma 3.1.17. *$L^1(\Omega; w) \cap L^\infty(\Omega) \subseteq L^r(\Omega; w)$, for every $r \in [1, \infty[$.*

Proof. Let $u \in L^1(\Omega; w) \cap L^\infty(\Omega)$. Then for every $r \geq 1$ we have

$$\begin{aligned} ||u||_{r,\Omega,w} &= \left(\int_\Omega |u(x)|^r w(x)dx\right)^{\frac{1}{r}} = \left(\int_\Omega |u(x)|^{r-1}|u(x)|w(x)dx\right)^{\frac{1}{r}} \\ &\leq \left(\int_\Omega ||u||_\infty^{r-1}|u(x)|w(x)dx\right)^{\frac{1}{r}} = ||u||_\infty^{\frac{r-1}{r}} ||u||_{1,\Omega,w}^{\frac{1}{r}}, \end{aligned}$$

which means that $||u||_{r,\Omega,w}$ is finite, so $u \in L^r(\Omega; w)$. $\square$

Notation. Let $\nu = \dfrac{p}{p-(q+1)}$ and we denote by $\nu' = \dfrac{p}{q+1}$ its dual index, that is $\frac{1}{\nu} + \frac{1}{\nu'} = 1$. Using the Lemma 3.1.17 we have that

$$L^1(\Omega; w) \cap L^\infty(\Omega) \subseteq L^\nu(\Omega; w),$$

so $\alpha \in L^\nu(\Omega; w)$.

The next lemma summarize the properties of the functional J.

Lemma 3.1.18. *Let condition (f) be satisfied. Then, the functional J is well defined and it is sequentially weakly continuous.*

Proof. From the assumption (f) we have

$$|F(u(x))| \leq \int_0^{u(x)} |f(s)|ds \leq f_0(x) \int_0^{u(x)} |s|^q ds \leq f_0(x)|u(x)|^{q+1}.$$

Then, using the conditions (f), (E) and the Hölder's inequality, we get

$$\begin{aligned}
|J(u)| = \left| \int_\Omega \alpha(x) F(u(x)) dx \right| &\leq \int_\Omega \alpha(x) f_0(x) |u(x)|^{q+1} dx \\
\leq\ & C_f \int_\Omega \alpha(x)|u(x)|^{q+1} w(x) dx \leq C_f \int_\Omega \alpha(x) w(x)^{\frac{1}{\nu}} |u(x)|^{q+1} w(x)^{\frac{1}{\nu'}} dx \\
\leq\ & C_f \left(\int_\Omega \alpha(x)^\nu w(x) dx \right)^{\frac{1}{\nu}} \left(\int_\Omega |u(x)|^{\nu'(q+1)} w(x) dx \right)^{\frac{1}{\nu'}} \\
=\ & C_f ||\alpha||_{\nu,\Omega,w} \left(\int_\Omega |u(x)|^p w(x) dx \right)^{\frac{q+1}{p}} = C_f ||\alpha||_{\nu,\Omega,w} ||u||_{p,\Omega,w}^{q+1} \\
\leq\ & C_f ||\alpha||_{\nu,\Omega,w} C_{p,w}^{q+1} ||u||_{v_0,v_1}^{q+1} \leq C_f ||\alpha||_{\nu,\Omega,w} C_{p,w}^{q+1} (2K)^{-\frac{q+1}{2}} ||u||_A^{q+1} \\
=\ & C||u||_A^{q+1},
\end{aligned}$$

which means that the functional J is well defined over X_A.

We prove now, that J is sequentially weakly continuous. Let $\{u_n\}$ be a sequence in X_A, weakly convergent to some $u \in X_A$. Then, by the embedding (E), it follows that $||u_n - u||_{p,\Omega,w} \to 0$.

Applying the (3.21), the Hölder inequalities and the Mean Value Theorem, we obtain

$$\begin{aligned}
|J(u_n) - J(u)| &= \left| \int_\Omega \alpha(x) F(u_n(x)) dx - \int_\Omega \alpha(x) F(u(x)) dx \right| \\
\leq\ & \int_\Omega \alpha(x) |F(u_n(x)) - F(u(x))| dx \\
=\ & \int_\Omega \alpha(x) |f((1-\theta)u_n(x) + \theta u(x))| |u_n(x) - u(x)| dx \\
\leq\ & \int_\Omega \alpha(x) f_0(x) \left| (1-\theta)u_n(x) + \theta u(x) \right|^q |u_n(x) - u(x)| dx \\
\leq\ & \int_\Omega \alpha(x) f_0(x) \left((1-\theta)|u_n(x)|^q + \theta |u(x)|^q \right) |u_n(x) - u(x)| dx \\
\leq\ & C_f \int_\Omega \alpha(x) w(x)^{\frac{1}{\nu}} \left(|u_n(x)|^q + |u(x)|^q \right) |u_n(x) - u(x)| w(x)^{\frac{1}{\nu'}} dx
\end{aligned}$$

$$
\begin{aligned}
&\leq C_f||\alpha||_{\nu,\Omega,w}\left(\int_\Omega (|u_n(x)|^q+|u(x)|^q)^{\nu'}|u_n(x)-u(x)|^{\nu'}w(x)dx\right)^{\frac{1}{\nu'}}\\
&= C_f||\alpha||_{\nu,\Omega,w}C_1\cdot\Bigg[\int_\Omega\left(|u_n(x)|^{\frac{pq}{q+1}}+|u(x)|^{\frac{pq}{q+1}}\right)w(x)^{\frac{q}{q+1}}\cdot\\
&\qquad\qquad (|u_n(x)-u(x)|^p)^{\frac{1}{q+1}}w(x)^{\frac{1}{q+1}}dx\Bigg]^{\frac{1}{\nu'}}\\
&\leq C_f||\alpha||_{\nu,\Omega,w}C_1\left[\left(\int_\Omega|u_n(x)|^pw(x)dx\right)^{\frac{q}{q+1}}+\left(\int_\Omega|u(x)|^pw(x)dx\right)^{\frac{q}{q+1}}\right]^{\frac{1}{\nu'}}\\
&\quad\cdot\left(\int_\Omega|u_n(x)-u(x)|^pw(x)dx\right)^{\frac{1}{q+1}\cdot\frac{1}{\nu'}}\\
&= C_f||\alpha||_{\nu,\Omega,w}C_1\left(||u_n||_{p,\Omega,w}^{\frac{pq}{q+1}}+||u||_{p,\Omega,w}^{\frac{pq}{q+1}}\right)^{\frac{q+1}{p}}||u_n-u||_{p,\Omega,w}\\
&\leq C_f||\alpha||_{\nu,\Omega,w}C_1C_2\left(||u_n||_{p,\Omega,w}^q+||u||_{p,\Omega,w}^q\right)||u_n-u||_{p,\Omega,w}\\
&\leq C_f||\alpha||_{\nu,\Omega,w}C_1C_2C_{p,w}^q\left(||u_n||_{v_0,v_1}^q+||u||_{v_0,v_1}^q\right)||u_n-u||_{p,\Omega,w}\leq\\
&\leq C_f||\alpha||_{\nu,\Omega,w}C_1C_2C_{p,w}^q(2K)^{-\frac{q}{2}}\left(||u_n||_A^q+||u||_A^q\right)||u_n-u||_{p,\Omega,w},
\end{aligned}
$$

where $\theta\in]0,1[$ is the constant from the Mean Value Theorem, C_1, C_2 are the constants from the inequality (3.21) and K is the constant from the ellipticity condition (K).

Since u_n is weakly convergent to $u\in X_A$, we can assume without loss of generality that there exist a constant $M>0$ such that

$$
||u_n||_A\leq M \text{ and } ||u_n-u||_A\leq M, \text{ for all } n\in\mathbb{N}.
$$

Then we have

$$
|J(u_n)-J(u)|\leq||\alpha||_{\nu,\Omega,w}C_fC_1C_2C_{p,w}^q(2K)^{-\frac{q}{2}}2M^q\cdot||u_n-u||_{p,\Omega,w},
$$

concluding that $J(u_n)\to J(u)$, whenever $n\to\infty$. □

Now, for a given $u_0\in X_A$ and for $\lambda>0$, we can define the energy functional $\mathcal{E}_\lambda:X_A\to\mathbb{R}$ related to the problem (P_λ) by

$$
\mathcal{E}_\lambda(u)=\frac{1}{2}||u-u_0||_A^2-\lambda J(u).
$$

We observe, that for every $v\in X_A$, we have

$$
\langle\mathcal{E}_\lambda'(u),v\rangle_A=\langle u-u_0,v\rangle_A-\lambda\int_\Omega\alpha(x)f(u(x))v(x)dx. \tag{3.27}
$$

Hence the critical points of the energy functional $\mathcal{E}_\lambda$ are exactly the weak solutions of the problem (P_λ). Therefore, instead of looking for solutions of the problem (P_λ), we are seeking for the critical points of $\mathcal{E}_\lambda$.

In the next lemmas we prove two properties of the energy functional, namely that $\mathcal{E}_\lambda$ is coercive and it satisfies the Palais-Smale condition.

Lemma 3.1.19. *Let the condition (f) be satisfied. Then the functional $\mathcal{E}_\lambda$ is coercive, for every $\lambda > 0$.*

Proof. Using again the Hölder's inequality combined with the conditions (f) and (E), we obtain

$$
\begin{aligned}
\mathcal{E}_\lambda(u) &= \frac{1}{2}||u-u_0||_A^2 - \lambda \int_\Omega \alpha(x) F(u(x)) v(x) dx \\
&\geq \frac{1}{2}||u-u_0||_A^2 - \lambda \int_\Omega \alpha(x) f_0(x) |u(x)|^{q+1} dx \\
&\geq \frac{1}{2}||u-u_0||_A^2 - \lambda C_f \int_\Omega \alpha(x) w(x)^{\frac{1}{\nu}} |u(x)|^{q+1} w(x)^{\frac{1}{\nu'}} dx \\
&\geq \frac{1}{2}||u-u_0||_A^2 - \lambda C_f \left(\int_\Omega \alpha(x)^\nu w(x) dx \right)^{\frac{1}{\nu}} \left(\int_\Omega |u(x)|^{(q+1)\nu'} w(x) dx \right)^{\frac{1}{\nu'}} \\
&= \frac{1}{2}||u-u_0||_A^2 - \lambda C_f ||\alpha||_{\nu,\Omega,w} ||u||_{p,\Omega,w}^{q+1} \\
&\geq \frac{1}{2}||u-u_0||_A^2 - \lambda C_f C_{p,w}^{q+1} ||\alpha||_{\nu,\Omega,w} ||u||_{v_0,v_1}^{q+1} \\
&\geq \frac{1}{2}||u-u_0||_A^2 - \lambda C_f C_{p,w}^{q+1} ||\alpha||_{\nu,\Omega,w} (2K)^{-\frac{q+1}{2}} ||u||_A^{q+1}.
\end{aligned}
$$

Therefore $\mathcal{E}_\lambda(u) \to \infty$, whenever $||u||_A \to \infty$, since $q+1 < 2$. □

Lemma 3.1.20. *Assume that (f) is satisfied. Then $\mathcal{E}_\lambda$ satisfies the Palais-Smale condition for every $\lambda > 0$.*

Proof. Let $\{u_n\} \subset X_A$ be an arbitrary Palais-Smale sequence for $\mathcal{E}_\lambda$, i.e.

(a) $\{\mathcal{E}_\lambda(u_n)\}$ is bounded;

(b) $\mathcal{E}'_\lambda(u_n) \to 0$, as $n \to \infty$.

We will prove that $\{u_n\}$ contains a strongly convergent subsequence in X_A. From the coercivity of $\mathcal{E}_\lambda$, it follows that $\{u_n\}$ is bounded, hence we can find a subsequence, which we still denote by $\{u_n\}$, weakly convergent to a point $u \in X_A$. Then by the embedding condition (E), u_n tends strongly to u in $L^p(\Omega; w)$, so $||u_n - u||_{p,\Omega,w} \to 0$, as $n \to \infty$.

Since the sequence from (b) tends to 0, for $n \in \mathbb{N}$ big enough, we have

$$\left| \langle \mathcal{E}'_\lambda(u_n), \frac{u_n}{||u_n||_A} \rangle_A \right| \leq \varepsilon,$$

or equivalently

$$\left| \langle \mathcal{E}'_\lambda(u_n), u_n \rangle_A \right| \leq \varepsilon ||u_n||_A.$$

Then, by (3.27) we get

$$\langle u_n - u_0, u_n \rangle_A - \lambda \int_\Omega \alpha(x) f(u_n(x)) u_n(x) dx \leq \varepsilon ||u_n||_A.$$

Rearranging the inequality and taking the absolute value, we obtain

$$|\langle u_n - u_0, u_n \rangle_A| \leq \varepsilon ||u_n||_A + \lambda \int_\Omega \alpha(x) |f(u_n(x)) u_n(x)| dx.$$

After simple computations this inequality gives us

$$||u_n - u||_A^2 \leq |\langle u_n - u_0, u_n - u \rangle_A| + |\langle u_0 - u, u_n - u \rangle_A| \leq$$

$$\leq |\langle u_n, u_n - u \rangle_A| + |\langle u, u_n - u \rangle_A| + 2|\langle u_0, u_n - u \rangle_A| \leq$$

$$\leq 4\varepsilon ||u_n - u||_A + \lambda \int_\Omega \alpha(x) |f(u_n(x))(u_n(x) - u(x))| dx +$$

$$+\lambda \int_\Omega \alpha(x) |f(u(x))(u_n(x) - u(x))| dx + \lambda \int_\Omega \alpha(x) |f(u_0(x))(u_n(x) - u(x))| dx.$$

Now, we will estimate the integrals from the above inequality using the inequalities of Hölder, the ellipticity condition (K) and the embedding (E). The first integral can be estimated as follows

$$\begin{aligned}
&\int_\Omega \alpha(x) |f(u_n(x))(u_n(x) - u(x))| dx \leq \\
\leq\ & C_f \int_\Omega \alpha(x) w(x)^{\frac{1}{\nu}} |u_n(x)|^q |u_n(x) - u(x)| w(x)^{\frac{1}{\nu'}} dx \\
\leq\ & C_f ||\alpha||_{\nu,\Omega,w} \left(\int_\Omega |u_n(x)|^{q\nu'} |u_n(x) - u(x)|^{\nu'} w(x) dx \right)^{\frac{1}{\nu'}} \\
=\ & C_f ||\alpha||_{\nu,\Omega,w} \left(\int_\Omega (|u_n(x)|^p w(x))^{\frac{q}{q+1}} (|u_n(x) - u(x)|^p w(x))^{\frac{1}{q+1}} dx \right)^{\frac{1}{\nu'}} \leq
\end{aligned}$$

$$\begin{aligned}
&\leq C_f||\alpha||_{\nu,\Omega,w}\left[\left(\int_\Omega |u_n(x)|^p w(x)dx\right)^{\frac{q}{q+1}}\cdot\right.\\
&\qquad\qquad\qquad\left.\left(\int_\Omega |u_n(x)-u(x)|^p w(x)dx\right)^{\frac{1}{q+1}}\right]^{\frac{q+1}{p}}\\
&= C_f||\alpha||_{\nu,\Omega,w}||u_n||^q_{p,\Omega,w}||u_n-u||_{p,\Omega,w}\\
&\leq C_f||\alpha||_{\nu,\Omega,w}C^q_{p,w}||u_n||^q_{v_0,v_1}||u_n-u||_{p,\Omega,w}\\
&\leq C_f||\alpha||_{\nu,\Omega,w}C^q_{p,w}(2K)^{-\frac{q}{2}}M^q||u_n-u||_{p,\Omega,w}
\end{aligned}$$

where in the last inequality we used that $\{u_n\}$ is bounded, hence there is a constant $M > 0$ such that $||u_n||_A < M$, $||u||_A < M$.

Proceeding in the same manner for the other two integrals, we obtain:

$$\int_\Omega \alpha(x)|f(u(x))(u_n(x)-u(x))|dx \leq$$
$$C_f||\alpha||_{\nu,\Omega,w}C^q_{p,w}(2K)^{\frac{-q}{2}}M^q||u_n-u||_{p,\Omega,w}$$

$$\int_\Omega \alpha(x)|f(u_0(x))(u_n(x)-u(x))|dx \leq$$
$$\leq C_f||\alpha||_{\nu,\Omega,w}C^q_{p,w}(2K)^{\frac{-q}{2}}||u_0||^q_A||u_n-u||_{p,\Omega,w}.$$

Then, we have

$$\begin{aligned}
||u_n-u||^2_A &\leq 4\varepsilon||u_n-u||_A+\\
&+\lambda C_f||\alpha||_{\nu,\Omega,w}C^q_{p,w}(2K)^{\frac{-q}{2}}(2M^q+||u_0||^q_A)||u_n-u||_{p,\Omega,w}.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrarily choosen, $||u_n-u||_A$ is bounded, $||u_0||_{p,\Omega,w}$ is finite (u_0 being given) and $||u_n-u||_{p,\Omega,w}$ tends to 0 as $n \to \infty$, we conclude that $||u_n-u||_A \to 0$, whenever $n \to \infty$. □

Now, we are in position to prove our main result, where a key tool is the following result of Ricceri [87].

Theorem 3.1.6. ([87, Theorem 1 and Remark 1]) *Let X be a topological space, Γ a real interval, and $f : X \times \Gamma \to \mathbb{R}$ a function satisfying the following conditions:*

$(A1)$ *for every $x \in X$, the function $f(x,\cdot)$ is quasi-concave and continuous;*

$(A2)$ *for every* $\lambda \in \Gamma$, *the function* $f(\cdot, \lambda)$ *is lower semicontinuous and each of its local minima is a global minimum;*

$(A3)$ *there exist* $\rho_0 > \sup_\Gamma \inf_X f$ *and* $\lambda_0 \in \Gamma$ *such that* $\{x \in X : f(x, \lambda_0) \leq \rho_0\}$ *is compact.*

Then,

$$\sup_\Gamma \inf_X f = \inf_X \sup_\Gamma f.$$

Proof of Theorem 3.1.5. Fix λ and u_0 as in the statement of the theorem and assume that (B1) does not hold. We shall prove that (B2) is true.

Choosing $\Lambda = [0, \infty)$ and endowing X_A with the weak topology, we define the function $g : X_A \times \Lambda \to \mathbb{R}$ by

$$g(u, \lambda) = \frac{1}{2}||u - u_0||_A^2 + \lambda(\sigma - J(u)).$$

We show that all the hypotheses of Theorem 3.1.6 are satisfied.

(A1): It is trivial.

(A2): Let $\lambda > 0$ be fixed. By Lemma 3.1.18, the functional $g(\cdot, \lambda)$ is sequentially weakly continuous. Moreover, $g(\cdot, \lambda)$ is coercive. Indeed, using Lemma 3.1.19, we have the following inequality for all $u \in X_A$

$$g(u, \lambda) \geq \frac{1}{2}||u - u_0||_A^2 - \lambda C_f C_{p,w}^{q+1} (2K)^{-\frac{q+1}{2}} ||\alpha||_{\nu,\Omega,w} ||u||_A^{q+1} + \lambda\sigma,$$

which goes to ∞ as $||u||_A \to \infty$, since $q + 1 < 2$.

Then, as a consequence of the Eberlain-Smulian theorem, $g(\cdot, \lambda)$ is weakly continuous.

It remains to show that every local minimum of $g(\cdot, \lambda)$ is a global one. Assume by contradiction that $g(\cdot, \lambda)$ has a local minimum which is not global. Since $g(\cdot, \lambda)$ is coercive and satisfies the Palais–Smale condition (Lemma 3.1.20), it admits a global minimum, and by the Eberlein-Smulian theorem it has two strong local minima. By the Mountain Pass Theorem (see [85]), $g(\cdot, \lambda)$ (equivalently the energy functional $\mathcal{E}_\lambda$) then admits a third critical point. Hence (P_λ) would have at least three solutions in X_A, contradicting the assumption that $(B1)$ fails. Thus, condition $(A2)$ holds.

(A3): We observe that there exists some $u_1 \in X_A$ such that $J(u_1) > \sigma$, so

$$\sup_{\lambda \in \Lambda} \inf_{u \in X_A} g(u, \lambda) \leq \sup_{\lambda \in \Lambda} g(u_1, \lambda) = \frac{1}{2}||u_1 - u_0||_A < \infty,$$

hence (A3) is satisfied.

Now, the Theorem 3.1.6 assures that

$$\sup_{\lambda\in\Lambda}\inf_{u\in X_A} g(u,\lambda) = \inf_{u\in X_A}\sup_{\lambda\in\Lambda} g(u,\lambda) := \alpha. \tag{3.28}$$

We observe, that the function $\lambda \mapsto \inf_{u\in X_A} g(u,\lambda)$ tends to $-\infty$ as $\lambda \to \infty$ (since $\sigma < \sup_{u\in X_A} J(u)$) and it is upper semicontinuous in Λ. Hence, it attains its supremum in some $\bar\lambda \in \Lambda$, that is,

$$\alpha = \inf_{u\in X_A} g(u,\bar\lambda) = \inf_{u\in X_A}\left(\frac{||u-u_0||_A^2}{2} + \bar\lambda(\sigma - J(u))\right). \tag{3.29}$$

We shall determine the infimum in the right-hand side of (3.28). Since for any $u \in J^{-1}(]-\infty,\sigma[)$ we have $\sup_{\lambda\in\Lambda} g(u,\lambda) = \infty$, it follows that

$$\alpha = \inf_{u\in J^{-1}([\sigma,\infty[)} \frac{||u-u_0||_A^2}{2}.$$

Then, since the functional $u \mapsto \frac{||u-u_0||_A^2}{2}$ is coercive and sequentially weakly lower semicontinuous while the set $J^{-1}([\sigma,\infty[)$ is sequentially weakly closed, there exists $v \in J^{-1}([\sigma,\infty[)$ such that it attains its infimum in v, that is

$$\alpha = \frac{||v-u_0||_A^2}{2}.$$

We can observe that v is actually belonging to $J^{-1}(\sigma)$, so we can write

$$\alpha = \inf_{u\in J^{-1}(\sigma)} \frac{||u-u_0||_A^2}{2} > 0, \tag{3.30}$$

where the inequality is motivated by the choice of u_0 in the assertion.

Combining (3.29) and (3.30) yields that

$$\inf_{u\in X_A}\left(\frac{||u-u_0||_A^2}{2} + \bar\lambda(\sigma - J(u))\right) = \inf_{u\in J^{-1}(\sigma)} \frac{||u-u_0||_A^2}{2},$$

which became after a rearrangment of the equation

$$\inf_{u\in X_A}\left(\frac{||u-u_0||_A^2}{2} - J(u)\right) = \inf_{u\in J^{-1}(\sigma)}\left(\frac{||u-u_0||_A^2}{2} - \bar\lambda\sigma\right). \tag{3.31}$$

Now, we prove that $\bar\lambda > 0$. Arguing by contradiction, we suppose that $\bar\lambda = 0$. Then by (3.29) we get, that $\alpha = 0$, against (3.30).

Finally, we prove $(B2)$, namely we prove that v defined above is the only point of $J^{-1}([\sigma,+\infty[)$ minimizing the distance from u_0. We argue by contradiction.

Let $w \in J^{-1}([\sigma,+\infty[)$ be such that $||w-u_0||_A = ||v-u_0||_A$ and w is different from v. As above, we have that $w \in J^{-1}(\sigma)$, so w and v are global minima of the functional $\mathcal{E}_\lambda$ over $J^{-1}(\sigma)$ for $\lambda = \bar{\lambda}$. Hence, by (3.31) both w and v are global minima for $\mathcal{E}_\lambda$ over the all space X_A. Thus, applying the mountain pass theorem again (see [85]), we obtain that $\mathcal{E}_\lambda$ has at least three critical points, against the assumption that (B1) does not hold (recall $\bar{\lambda}$ is positive). This concludes the proof. $\square$

In the next corollary the alternative of Theorem 3.1.5 is resolved, so we obtain a multiplicity result for the problem (P_λ). Here we shall use a result of Tsar'kov [96], which we recall here for completeness.

Theorem 3.1.7. ([96, Theorem 2]) *Let X be an uniformly convex Banach space, with strictly convex topological dual, M a sequentially weakly closed, non-convex subset of X. Then, for any convex, dense subset S of X, there exists $x_0 \in S$ such that the set*

$$\{y \in M : ||y-x_0|| = d(x_0, M)\}$$

contains at least two points.

Corollary 3.1.1. ([69])*Let Ω, f, α, X_A be as in the Theorem 3.1.5 and let S be a convex, dense subset of X_A. Moreover, let $J^{-1}([\sigma,+\infty[)$ be not convex for some $\sigma \in]\inf_{X_A} J, \sup_{X_A} J[$.*

Then there exist $u_0 \in J^{-1}(]-\infty,\sigma[) \cap S$ and $\lambda > 0$ such that the problem (P_λ) admits at least tree solutions.

Proof. From Lemma 3.1.18, it follows that J is sequentially weakly continuous, hence the set $M = J^{-1}(]\sigma,+\infty[)$ is sequentially weakly closed. Since M is not convex, we can apply the Theorem 3.1.7 , which assures the existence of some $u_0 \in S$, such that the set $\{y \in M : ||y-u_0||_A = d(u_0, M)\}$ contains at least two points. So, there exist two different points $v_1, v_2 \in M$ such that

$$||v_1 - u_0||_A = ||v_2 - u_0||_A = d(u_0, M).$$

Clearly $u_0 \notin M$, so $u_0 \in J^{-1}(]-\infty,\sigma[)$. Then the condition (B2) in Theorem 3.1.5 is false, so (B1) must be true, which means that there exist $\lambda > 0$ such that (P_λ) has at least three solutions in X_A. $\square$

3.2 Symmetric solutions on strip-like domains

3.2.1 Group-invariant multiple solutions for quasilinear elliptic problems

In this section following Farkas and Mezei [36], we consider a quasilinear equation coupled with the homogeneous Neumann boundary condition

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda\alpha(x,y)f(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mathcal{N}_\lambda)$$

where $\Omega = \omega \times \mathbb{R}^{N-m}$, $\omega \subset \mathbb{R}^m$ being bounded and open with smooth boundary, $p > N$, $N - m \geq 2$, Δ_p is the p-Laplacian operator, λ is a positive parameter, $\alpha \in L^\infty(\Omega)$ is a non-zero potential with compact support, n is the outward normal vector, and $f : [0, \infty) \to \mathbb{R}$ is a continuous function with $f(0) = 0$.

Our purpose is to ensure the existence of multiple solutions for the problem $(\mathcal{N}_\lambda)$ where the natural functional space is $W^{1,p}(\Omega)$.

The main difficulty to treat this problem comes from the unboundedness of the domain $\Omega = \omega \times \mathbb{R}^{N-m}$; indeed, no compact embedding is available for $W^{1,p}(\Omega)$ into any Lebesgue space. Since $p > N$, due to Morrey's theorem (see Brézis [18]), the embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ is continuous, without being compact. The latter fact is due to the lack of concentration compactness along certain directions inside of the strip-like domain $\Omega = \omega \times \mathbb{R}^{N-m}$. On the other hand, Faraci, Iannizzotto and Kristály (see [34]) proved a compact embedding result for cylindrically symmetric functions on strip-like domains in low dimension; namely, if $p > N$, the subspace of cylindrically symmetric functions of $W^{1,p}(\Omega)$, denoted in the sequel by $W_c^{1,p}(\Omega)$ is compactly embedded into $L^\infty(\Omega)$. More precisely, the (closed) subspace of $W^{1,p}(\Omega)$ consisting of the *cylindrically symmetric* functions is defined by

$$W_c^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : u(x, \cdot) \text{ is radially symmetric for all } x \in \omega\}.$$

It is clear that such a compactness result, combined with a suitable variational method, figures out to be a hopeful argument to handle the problem $(\mathcal{N}_\lambda)$. Indeed, in [34], by assuming suitable oscillatory behavior on the nonlinear term f (at zero and infinity), the authors guaranteed the existence of a whole sequence of weak solutions for the problem $(\mathcal{N}_\lambda)$, by exploiting the aforementioned compactness result, the principle of symmetric criticality and a recent variational principle of Ricceri.

In this subsection we guarantee two or three non-zero weak solutions to problem $(\mathcal{N}_\lambda)$ which are invariant with respect to certain groups, whenever the nonlinear term verifies unusual growth assumptions, similar to the ones assumed by Faraci and Kristály in [33] (for a Dirichlet problem studied on bounded domains). In order to present our main result, we first recall that $u \in W^{1,p}(\Omega)$ is a *weak solution* to problem $(\mathcal{N}_\lambda)$ if for all $v \in W^{1,p}(\Omega)$

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv)dxdy = \lambda \int_\Omega \alpha(x,y)f(u)vdxdy. \tag{3.32}$$

Assume that $N - m \geq 2$, and let

$$\mathcal{G}_{N,m} = \{G = \mathrm{id} \times \mathbf{O}(k_1) \times ... \times \mathbf{O}(k_l) : k_1, ..., k_l \geq 2, k_1 + ... + k_l = N - m\}.$$

Let $G \in \mathcal{G}_{N,m}$. We say that a function $h : \Omega \to \mathbb{R}$ is *$G-$invariant*, if $h(\tilde{g}(x,y)) = h(x,y)$ for all $\tilde{g} = \mathrm{id} \times g \in G$ and $(x,y) \in \Omega$. The function h is called *cylindrically symmetric*, if it is $\mathrm{id} \times \mathbf{O}(N-m)-$invariant.

For later use, if K is a subset of $\mathbb{R}^N$ and $\mu > 0$, we introduce the notation

$$K_\mu = \{x \in \Omega : \mathrm{dist}(x, K) \leq \mu\},$$

$m(\cdot)$ is the Lebesgue measure, while $c_\infty > 0$ is the embedding constant in the embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$.

Remark 3.2.1. Based on [18], after elementary estimation, one certainly has $c_\infty \leq \frac{2p}{p-N}$.

Now, we are in the position to state the main result of this subsection.

Theorem 3.2.1. ([36]) *Let $p > N \geq m + 2$, $\Omega = \omega \times \mathbb{R}^{N-m}$, $\alpha \in L^\infty(\Omega)$ be a cylindrically symmetric, non-negative, non-zero function with compact support K. Let $\mu = \mathrm{dist}(K, \partial\Omega)$. Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function with $f(0) = 0$ and assume that*

i) $M_F = \sup\limits_{[0,\infty[} F < \infty$, *where* $F(s) = \int_0^s f(t)dt$;

ii) $\lim\limits_{s \to 0^+} \dfrac{f(s)}{s^{p-1}} = 0$.

Moreover, assume that there exists $r > 0$ such that *iii)* $F(r) = \max\limits_{[0, c_\infty (pM_F \|\alpha\|_{L^1})^{\frac{1}{p}}]} F < M_F$;

iv) $\dfrac{F(r)}{r^p} > \dfrac{1}{p\|\alpha\|_1}\left[\dfrac{m(K_\mu \setminus K)}{\mu^p} + m(K_\mu)\right]$.

Then the following statements hold:

(a) For every $G \in \mathcal{G}_{N,m}$ and $\lambda > 1$, problem $(\mathcal{N}_\lambda)$ has at least two non-zero, non-negative $G-$invariant weak solutions.

(b) For every $G \in \mathcal{G}_{N,m}$, there exists $\lambda_G > 1$ such that $(\mathcal{N}_{\lambda_G})$ has at least three non-zero, non-negative $G-$invariant weak solutions.

Remark 3.2.2. (a) Since $f(0) = 0$, without loss of generality we can assume that f is defined on the whole real axis, putting $f(s) = 0$ for all $s \leq 0$. If we still denote by F the primitive of f, then $F(s) = 0$ for all $s \leq 0$.

(b) Note that if $u \in W^{1,p}(\Omega)$ is a weak solution of $(\mathcal{N}_\lambda)$, then $u \geq 0$. Indeed, if we put $v = u_- = \min\{0, u\}$ as a test function in (3.32), then from

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \nabla u_- + |u|^{p-2}uu_-)dxdy = \lambda \int_\Omega \alpha(x,y) f(u) u_- dxdy = 0,$$

it follows that $u_- = 0$.

(c) Note that beside of the above assumptions, if

$$\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = 0,$$

then for small values of $\lambda > 0$, problem $(\mathcal{N}_\lambda)$ has only the trivial solution. Indeed, if $u \in W^{1,p}(\Omega)$ is a weak solution of $(\mathcal{N}_\lambda)$, and we put as the test function $v = u$ in relation (3.32), one obtains

$$\int_\Omega (|\nabla u|^p + |u|^p)dxdy = \lambda \int_\Omega \alpha(x,y) f(u) u dxdy \leq \lambda c_f \|\alpha\|_{L^\infty} \int_\Omega |u|^p dxdy,$$

where $c_f = \max\limits_{s>0} \dfrac{|f(s)|}{s^{p-1}} > 0$. Therefore, if $\lambda < (c_f \|\alpha\|_{L^\infty})^{-1}$, then $u = 0$.

We conclude with a simple consequence of Theorem 3.2.1.

Corollary 3.2.1. *Let $p > N \geq m+2$, $\Omega = \omega \times \mathbb{R}^{N-m}$, $\alpha \in L^\infty(\Omega)$ be a cylindrically symmetric, non-negative, non-zero function with compact support K. Let $\mu = \mathrm{dist}(K, \partial\Omega)$, and besides of i) and ii) from Theorem 3.2.1, we assume that there exists constant c, δ and $c_\infty < \delta$ and $0 < r < \min\{1, \mu, c_\infty\}$, such that $0 < F(r) = \max_{[0,\delta]} F < M_F$. Then, there exists $p_0 > n$ such that for each $p \geq p_0$, both conclusions of Theorem 3.2.1 hold.*

Auxiliary results

This subsection is devoted to some preparatory results.

Let $m \geq 1$, and $n = N - m \geq 2$ be integers and $\omega \subset \mathbb{R}^m$ be a bounded domain with Lipschitz boundary $\partial\omega$, and let $p > N$ be a real number. We set $\Omega = \omega \times \mathbb{R}^{N-m}$, and consider the Sobolev space $W^{1,p}(\Omega)$, endowed with the norm

$$\|u\| = \left(\|\nabla u\|_{L^p}^p + \|u\|_{L^p}^p\right)^{1/p}.$$

We note that $\partial\Omega = \partial\omega \times \mathbb{R}^{N-m}$ is Lipschitz. Then, by Morrey's theorem, $W^{1,p}(\Omega)$ is continuously embedded into $L^\infty(\Omega)$, endowed with the sup-norm

$$\|u\|_{L^\infty} = \text{esssup}_{(x,y)\in\Omega}|u(x,y)|.$$

We consider the energy functional $\mathcal{E}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ associated with problem $(\mathcal{N}_\lambda)$, which is defined by

$$\mathcal{E}_\lambda(u) = \frac{1}{p}\|u\|^p - \lambda\mathcal{L}(u),$$

where

$$\mathcal{L}(u) = \int_\Omega \alpha(x,y)F(u(x,y))dxdy.$$

In a standard manner, we can prove that $\mathcal{E}_\lambda$ is of class C^1, while the critical points for $\mathcal{E}_\lambda$ are precisely the weak solutions for the problem $(\mathcal{N}_\lambda)$. We recall the notation

$$\mathcal{G}_{N,m} = \{G = \text{id} \times \mathbf{O}(k_1) \times ... \times \mathbf{O}(k_l) : k_1, ..., k_l \geq 2, k_1 + ... + k_l = N - m\}.$$

Let $G \in \mathcal{G}_{N,m}$ be fixed. We introduce the action of the group G on the Sobolev space $W^{1,p}(\Omega)$ by

$$\tilde{g}u(x,y) = u(x, g^{-1}y) \text{ for all } \tilde{g} = \text{id} \times g \in G,\ (x,y) \in \Omega,\ u \in W^{1,p}(\Omega).$$

Note that G acts linearly, continuously and isometrically on $W^{1,p}(\Omega)$, i.e., $\|\tilde{g}u\| = \|u\|$ for every $\tilde{g} \in G$ and $u \in W^{1,p}(\Omega)$. Let

$$W_G^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : \tilde{g}u = u \text{ for all } \tilde{g} \in G\}$$

be the subspace of $G-$invariant functions in $W^{1,p}(\Omega)$. In particular, one has that

$$W^{1,p}_{\text{id}\times\mathbf{O}(N-m)}(\Omega) = W_c^{1,p}(\Omega).$$

Theorem 3.2.2. *Let m, N, Ω, ω and p be as above, and $G \in \mathcal{G}_{N,m}$. Then the embedding $W_G^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ is compact.*

Remark 3.2.3. Note that in [34], the authors proved that $W_c^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ is compact, which is a particular case of Theorem 3.2.2. In fact, the proof of Theorem 3.2.2 works similarly, since the set $\Omega = \omega \times \mathbb{R}^{N-m}$ is *compatible* with every group $G \in \mathcal{G}_{N,m}$, see Willem [99, Definition 1.22], and Kobayashi and Otani [45].

Due to the Principle of Symmetric Criticality 1.4.1, the critical points of $\mathcal{E}_\lambda^G = \mathcal{E}_\lambda|_{W_G^{1,p}(\Omega)}$ become critical points of $\mathcal{E}_\lambda$ as well, so $G-$invariant, weak solutions of the problem $(\mathcal{N}_\lambda)$. Therefore, it is enough to guarantee critical points for $\mathcal{E}_\lambda^G = \mathcal{E}_\lambda|_{W_G^{1,p}(\Omega)}$ where the compactness of the embedding $W_G^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ resulted in Theorem 3.2.2 will be deeply exploited.

We use the following notation

$$\mathcal{L}^G = \mathcal{L}|_{W_G^{1,p}(\Omega)}.$$

We mention that under the assumptions of Theorem 3.2.1 on α, we have that $\alpha \in L^1(\Omega)$, since $\alpha \in L^\infty(\Omega)$ has a compact support.

Lemma 3.2.1. *We assume that all assumptions of Theorem 3.2.1 are satisfied. Then*

(a) The functional $\mathcal{L}^G$ is weakly lower semicontinuous.

(b) For every $\lambda \geq 0$, $\mathcal{E}_\lambda^G$ satisfies the Palais-Smale condition.

Proof. (a) Let $\{u_n\}$ be a sequence in $W_G^{1,p}(\Omega)$ which converges weakly to $u \in W_G^{1,p}(\Omega)$ and suppose that the sequence $\{\mathcal{L}^G(u_n)\}$ does not converge to $\mathcal{L}^G(u)$ as $n \to \infty$. Thus, there exist $\varepsilon > 0$ such that

$$0 < \varepsilon \leq |\mathcal{L}^G(u_n) - \mathcal{L}^G(u)| \text{ for every } n \in \mathbb{N}.$$

Thanks to the compact embedding $W_G^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, up to a subsequence, one has $u_n \to u$ strongly in $L^\infty(\Omega)$. By the mean value theorem, we can conclude that for some $\theta_n \in (0,1)$ with n large enough, one has

$$0 < \varepsilon \leq |\langle(\mathcal{L}^G)'(u + \theta_n(u_n - u)), u_n - u\rangle| \leq$$

$$\leq \int_{\Omega} \alpha(x,y)|f(u(x,y)+\theta_n(u_n(x,y)-u(x,y)))||u_n(x,y)-u(x,y)|dxdy \leq$$

$$\leq \|\alpha\|_{L^1} \max\{|f(s)| : |s| \leq \|u\|_{L^\infty}+1\}\|u_n-u\|_{L^\infty}.$$

But the last term tends to 0, which is a contradiction.

(b) First of all, due to i), one has for every $u \in W^{1,p}_G(\Omega)$ that

$$\mathcal{E}^G_\lambda(u) \geq \frac{1}{p}\|u\|^p - \lambda\|\alpha\|_{L^1} M_F.$$

In particular, $\mathcal{E}^G_\lambda$ is coercive of every $\lambda \geq 0$.

Now, we consider a Palais-Smale sequence $\{u_n\} \subset W^{1,p}_G(\Omega)$, i.e., $\{\mathcal{E}^G_\lambda(u_n)\}$ is bounded and

$$\|(\mathcal{E}^G_\lambda)'(u_n)\|_{W^{1,p}_G(\Omega)^*} \to 0 \text{ as } n \to +\infty.$$

Since $\mathcal{E}^G_\lambda$ is coercive, the sequence $\{u_n\}$ is bounded. Therefore, taking a subsequence if necessary, due to Theorem 3.2.2, we may assume that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } W^{1,p}_G(\Omega); \\ u_n \to u & \text{strongly in } L^\infty(\Omega). \end{cases}$$

For simplicity, let $I(u) = \frac{1}{p}\|u\|^p$. Then,

$$\langle I'(u_n), u_n - u\rangle = \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n\right)(\nabla u_n - \nabla u) + \int_\Omega |u_n|^{p-2}u_n(u_n-u),$$

and

$$\langle I'(u), u_n - u\rangle = \int_\Omega \left(|\nabla u|^{p-2}\nabla u\right)(\nabla u_n - \nabla u) + \int_\Omega |u|^{p-2}u(u_n-u).$$

On the other hand,

$$\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u_n - \nabla u) + \int_\Omega \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)(u_n-u) =$$

$$\langle(\mathcal{E}^G_\lambda)'(u_n), u_n-u\rangle + \langle(\mathcal{E}^G_\lambda)'(u), u-u_n\rangle + \lambda\int_\Omega \alpha(x,y)[f(u_n(x,y))-f(u(x,y))](u_n-u).$$

Since $\|(\mathcal{E}^G_\lambda)'(u_n)\|^{1,p}_{W_G}(\Omega)^* \to 0$ and $u_n \rightharpoonup u$ weakly in $W^{1,p}_G(\Omega)$, the first two terms at the right hand side tend to 0. Since $u_n \to u$ strongly in $L^\infty(\Omega)$, for $n \in \mathbb{N}$ large enough, one has

$$\left|\int_\Omega \alpha(x,y)[f(u_n)-f(u)](u_n-u)\right| \leq$$

$$\leq 2\|\alpha\|_1\|u-u_n\|_{L^\infty}\max\{|f(z)|:|z|\leq \|u\|_{L^\infty}+1\}\to 0.$$

Therefore,

$$\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u_n-\nabla u)+$$
$$+\int_\Omega \left(|u_n|^{p-2}u_n-|u|^{p-2}u\right)(u_n-u)\to 0. \tag{3.33}$$

Using the fact that $|v-w|^p\leq(|v|^{p-2}v-|w|^{p-2}w)(v-w)$, one has

$$\|u_n-u\|^p\leq \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u_n-\nabla u)+$$
$$+\int_\Omega \left(|u_n|^{p-2}u_n-|u|^{p-2}u\right)(u_n-u)$$

Consequently, the last inequality combined with (3.33) leads to the fact that $u_n\to u$ strongly in $W^{1,p}_G(\Omega)$, which proves the claim. □

Proof of main results

Besides of the well known Mountain Pass Theorem1.3.1, our main tool is the Theorem 3.2.3 of Ricceri combined with the Principle of Symmetric Criticality 1.4.1 and a suitable compactness result which extends the embedding result from [34].

Theorem 3.2.3. ([90, Theorem 1]) *Let $(X,\|\cdot\|)$ be a reflexive real Banach space, $p>1$, $\mathcal{L}:X\to\mathbb{R}$ a bounded C^1 functional, with compact derivative such that $\mathcal{L}(0)=0$ and*

$$\max\left\{\limsup_{u\to 0}\frac{\mathcal{L}(u)}{\|u\|^p},\ \limsup_{\|u\|\to\infty}\frac{\mathcal{L}(u)}{\|u\|^p}\right\}\leq 0.$$

Moreover, assume that $u_1\in X$ such that $\|u_1\|^p<p\mathcal{L}(u_1)$ and

$$\mathcal{L}(u_1)=\sup_{\|u\|^p\leq p\sup_X\mathcal{L}}\mathcal{L}(u)<\sup_X\mathcal{L}.$$

Then there exists $\hat\lambda>1$ such that the functional $\mathcal{E}_\lambda:X\to\mathbb{R}$ defined by

$$\mathcal{E}_{\hat\lambda}(u)=\frac{1}{p}\|u\|^p-\hat\lambda\mathcal{L}(u)$$

has at least three non-zero critical points.

We recall that $\mu = \mathrm{dist}(K, \partial\Omega)$.

Proof of Theorem 3.2.1.

(a) Define $u_r : \Omega \to \mathbb{R}$ as

$$u_r(x,y) = r\left(1 - \frac{\mathrm{dist}((x,y),K)}{\mu}\right)_+,$$

where $t_+ = \max\{0,t\}$, while the number r is from assumption iii). A simple estimate shows that $||u_r||^p \geq r^p m(K) > 0$. Since α is cylindrically symmetric, and $K = \mathrm{supp}(\alpha)$, then K is a $\mathrm{id} \times \mathbf{O}(N-m)-$invariant set, i.e. $\tilde{g}K = K$ for every $\tilde{g} = \mathrm{id} \times g \in \mathrm{id} \times \mathbf{O}(N-m)$. Consequently, $u_r \in W_c^{1,p}(\Omega)$ and it is Lipshitzian; in particular, $u_r \in W_G^{1,p}(\Omega)$.

From the definition of $\mathcal{L}^G$ and u_r, one has

$$\begin{aligned}\mathcal{L}^G(u_r) &= \int_\Omega \alpha(x,y)F(u_r(x,y))dxdy \\ &= \int_K \alpha(x,y)F(u_r(x,y))dxdy \\ &= F(r)\|\alpha\|_{L^1}.\end{aligned} \tag{3.34}$$

Consequently, there exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$, we have

$$\mathcal{E}_\lambda^G(u_r) = \frac{1}{p}||u_r||^p - \lambda\mathcal{L}^G(u_r) = \frac{1}{p}||u_r||^p - \lambda F(r)||\alpha||_1 < 0. \tag{3.35}$$

In fact, we may choose

$$\lambda_0 = \frac{1}{p}\inf\left\{\frac{||u||^p}{\mathcal{L}^G(u)} : u \in W_G^{1,p}(\Omega), \mathcal{L}^G(u) > 0\right\}. \tag{3.36}$$

Fix $\lambda > \lambda_0$. From ii) it follows that for fixed $\frac{1}{p||\alpha||_1\lambda c_\infty^p} > \varepsilon > 0$, there exists $s_0 > 0$ such that $|F(s)| \leq \varepsilon|s|^p$ for all $s \in [-s_0, s_0]$. Then, if $u \in W_G^{1,p}(\Omega)$ and $\|u\| \leq \frac{s_0}{c_\infty}$, one has

$$F(u(x,y)) \leq \varepsilon|u(x,y)|^p \text{ for every } (x,y) \in \Omega.$$

Thus, if $u \in W_G^{1,p}(\Omega)$ with $||u|| = \rho < \min\{\frac{s_0}{c_\infty}, \|u_r\|\}$, then

$$\begin{aligned}
\mathcal{E}_\lambda^G(u) &= \frac{1}{p}||u||^p - \lambda\mathcal{L}^G(u) \\
&\geq \frac{1}{p}||u||^p - \varepsilon\lambda\|\alpha\|_{L^1}c_\infty^p\|u\|^p \\
&= ||u||^p\left(\frac{1}{p} - \varepsilon\lambda\|\alpha\|_{L^1}c_\infty^p\right) \\
&= \rho^p\left(\frac{1}{p} - \varepsilon\lambda\|\alpha\|_{L^1}c_\infty^p\right) > 0.
\end{aligned}$$

Since

$$\inf_{\|u\|=\rho} \mathcal{E}_\lambda^G(u) > 0 = \mathcal{E}_\lambda^G(0) > \mathcal{E}_\lambda^G(u_r)$$

and $\mathcal{E}_\lambda^G$ satisfies the Palais-Smale condition, we are in the position to apply the Mountain Pass Theorem 1.3.1. Consequently, for $\lambda > \lambda_0$, the functional $\mathcal{E}_\lambda^G$ has a critical point with positive energy level. On the other hand, it is clear that the functional $\mathcal{E}_\lambda^G$ achieves its global minimum; on account of (3.35), the energy level of this minimum point is negative. Summing up the above facts, for every $\lambda > \lambda_0$, the energy function $\mathcal{E}_\lambda^G$ has two distinct critical points. Due to the Principle of Symmetric Criticality 1.4.1, these critical points of $\mathcal{E}_\lambda^G$ are also critical points of $\mathcal{E}_\lambda$; thus, they are non-zero, non-negative $G-$invariant weak solutions of the problem $(\mathcal{N}_\lambda)$.

We now show that $\lambda_0 < 1$, which will conclude the proof of the first part. To see this, a direct calculation provides

$$\begin{aligned}
||u_r||^p &= \int_\Omega (|\nabla u_r|^p + |u_r|^p) = \\
&= \int_{K_\mu\setminus K}\left(\frac{r}{\mu}\right)^p + \int_K r^p + \int_{K_\mu\setminus K} r^p\left(1 - \frac{\operatorname{dist}(x,K)}{\mu}\right)^p = \\
&= \left(\frac{r}{\mu}\right)^p m(K_\mu\setminus K) + r^p m(K) + r^p \int_{K_\mu\setminus K}\left(1 - \frac{\operatorname{dist}(x,K)}{\mu}\right)^p = \\
&\leq \left(\frac{r}{\mu}\right)^p m(K_\mu\setminus K) + r^p m(K) + r^p m(K_\mu\setminus K) =
\end{aligned}$$

$$= \left(\frac{r}{\mu}\right)^p m(K_\mu \setminus K) + r^p m(K_\mu) =$$
$$= r^p \left[\frac{m(K_\mu \setminus K)}{\mu^p} + m(K_\mu)\right].$$

By relations (3.36), (3.34) and assumption iv), we have

$$\lambda_0 \leq \frac{||u_r||^p}{p\mathcal{L}^G(u_r)} \leq \frac{r^p}{pF(r)\|\alpha\|_{L^1}} \left[\frac{m(K_\mu \setminus K)}{\mu^p} + m(K_\mu)\right] < 1.$$

(b) The functional $\mathcal{L}^G$ is of class C^1, has compact derivative and $\mathcal{L}^G(0) = 0$. Moreover,

$$\sup_{W_G^{1,p}(\Omega)} \mathcal{L}^G \leq \|\alpha\|_{L^1} \sup_{[0,\infty[} F = \|\alpha\|_{L^1} M_F. \tag{3.37}$$

In particular,

$$\limsup_{\|u\|\to\infty} \frac{\mathcal{L}^G(u)}{\|u\|^p} \leq 0. \tag{3.38}$$

As above, if $u \in W_G^{1,p}(\Omega)$ and $\|u\| \leq \frac{s_0}{c_\infty}$, then

$$F(u(x,y)) \leq \varepsilon |u(x,y)|^p \ \text{ for every } (x,y) \in \Omega.$$

Therefore, when $\|u\| \leq \frac{s_0}{c_\infty}$, one has

$$\mathcal{L}^G(u) \leq \varepsilon \|\alpha\|_{L^1} \|u\|_\infty^p \leq \varepsilon \|\alpha\|_{L^1} c_\infty^p \|u\|^p,$$

which implies at once

$$\limsup_{u\to 0} \frac{\mathcal{L}^G(u)}{\|u\|^p} \leq 0. \tag{3.39}$$

We claim to apply Theorem 3.2.3 to the space $W_G^{1,p}(\Omega)$ and functional $\mathcal{L}^G$ defined above which, in view of (3.38) and (3.39), verifies

$$\max\left\{\limsup_{u\to 0} \frac{\mathcal{L}^G(u)}{\|u\|^p}, \limsup_{\|u\|\to\infty} \frac{\mathcal{L}^G(u)}{\|u\|^p}\right\} \leq 0.$$

Consider $u_r : \Omega \to \mathbb{R}$ as above; we have that

$$\|u_r\|^p \leq r^p \left[\frac{m(K_\mu \setminus K)}{\mu^p} + m(K_\mu)\right].$$

The assumption iv) implies at once that

$$\|u_r\|^p < p\mathcal{L}^G(u_r). \tag{3.40}$$

From assumption iii) we also have that u_r is not a global maximum of $\mathcal{L}^G$. If $u \in W_G^{1,p}(\Omega)$ and $\|u\| \leq (pM_F\|\alpha\|_{L^1})^{\frac{1}{p}}$, then $\|u\|_{L^\infty} \leq c_\infty(pM_F\|\alpha\|_{L^1})^{\frac{1}{p}}$; thus, due to assumption iii), one has

$$\mathcal{L}^G(u) \leq \mathcal{L}^G(u_r),\ u \in W_G^{1,p}(\Omega) \text{ and } \|u\| \leq (pM_F\|\alpha\|_{L^1})^{\frac{1}{p}}.$$

The above inequality together with the fact that $\|u_r\| \leq (pM_F\|\alpha\|_{L^1})^{\frac{1}{p}}$ (cf. (3.40)), gives that

$$\mathcal{L}^G(u_r) = \max\left\{\mathcal{L}^G(u) : u \in W_G^{1,p}(\Omega), \|u\| \leq (pM_F\|\alpha\|_{L^1})^{\frac{1}{p}}\right\}$$

By choosing $u_1 = u_r$, the conclusion of Theorem 3.2.3 implies that there exists $\lambda_G > 1$ such that the functional $\mathcal{E}^G_{\lambda_G}$ has at least three non-zero critical points. By applying again the Principle of Symmetric Criticality 1.4.1, the problem $(\mathcal{N}_{\lambda_G})$ has at least three non-zero, non-negative $G-$invariant weak solutions. □

Example 3.2.1.

Let us choose $\Omega = [-1,1] \times \mathbb{R}^2$ and

$$K = \left\{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq \frac{1}{4}\right\}.$$

Then, we can see that $\mu = \text{dist}(K, \partial\Omega) = \frac{1}{2}$, and $m(K) = \frac{\pi}{6}$, $m(K_\mu) = \frac{4}{3}\pi$.

We assume that

$$||\alpha||_{L^1} > \frac{75\pi^2}{2^7}. \tag{3.41}$$

Let us consider the following function $g : [0, +\infty[\to \mathbb{R}, g(x) = \sin^5(2\pi x)$ and a positive integer $k_0 \in \mathbb{N}$, such that $k_0 \geq c_\infty \sqrt[4]{4\|\alpha\|_{L^1}}$ and a function $h : [0, +\infty[\to \mathbb{R}$ such that $h(x) = \frac{3}{2} - 6\left(x - k_0 + \frac{1}{2}\right)^2$. Now we are in the position to the construct the following function:

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k}(1 - \delta_{k,k_0}) \cdot \chi_{[k-1,k]}(x) \cdot g(x) + \delta_{k,k_0} \cdot \chi_{[k_0-1,k_0]}(x) \cdot h(x),\ (\forall) x \geq 0,$$

where $\chi_{[k-1,k]}$is the characteristic function of the interval $[k-1,k]$, and δ_{k,k_0} is the Kronecker's delta, i.e $\delta_{k,k_0} = \begin{cases} 0, \text{if } k \neq k_0 \\ 1,\ \text{if } k = k_0 \end{cases}$, more precisely

$$f(x) = \begin{cases} \frac{1}{k}\sin^5(2\pi x), & \text{if } x \in [k-1,k], k \neq k_0 \\ \frac{3}{2} - 6\left(x - k_0 + \frac{1}{2}\right)^2, & \text{if } x \in [k_0 - 1, k_0]. \end{cases}$$

First of all, we show that $F(x) = \int\limits_0^x f(t)dt$ is bounded from above. Indeed, from the construction yields that

$$\max_{x\in\mathbb{R}_+\setminus[k_0-1,k_0]} F(x) = \max_{x\in[0,1]} F(x) = \int_0^{\frac{1}{2}} \sin^5(2\pi x)dx = \frac{8}{15\pi}.$$

On the other hand

$$\max_{x\in[k_0-1,k_0]} F(x) = \int_{k_0-1}^{k_0} \left[\frac{3}{2} - 6\left(x - k_0 + \frac{1}{2}\right)^2\right] dx = 1,$$

which means, that F, is bounded from above and $M_F = 1$.

On the other hand, one has

$$\lim_{s\to 0^+} \frac{f(s)}{s^3} = \lim_{s\to 0^+} \frac{\sin^5(2\pi s)}{s^3} = 0. \tag{3.42}$$

By choosing $p = 4$, $N = 3$ and taking into account (3.42) and $M_F = 1$ it is easy to see that F satisfies the assumptions i) and ii).

In our case, the estimation described in Remark 3.2.1 reduce to $c_\infty \leq 8$

Inequalities (3.41) and $k_0 \geq 8\cdot\sqrt[4]{4\|\alpha\|_{L^1}}$ guarantees assumptions iii) and iv) of Theorem 3.2.1. In conclusion, we can apply Theorem 3.2.1,

3.2.2 Axially symmetric multiple solutions for a perturbed system

We prove a multiplicity result for a perturbed gradient-type system defined on strip-like domains. The approach is based on a Ricceri-type three critical point theorem in [89]. We follow mainly the paper of Kristály and Mezei [52].

Let us consider the following system

$$\begin{cases} -\Delta_p u = \lambda\alpha(x)F_u(u,v) + \mu\beta(x)G_u(u,v) & \text{in} \quad \Omega, \\ -\Delta_q v = \lambda\alpha(x)F_v(u,v) + \mu\beta(x)G_v(u,v) & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{$D^{p,q}_{\lambda,\mu}$}$$

where $1 < p, q < N$; $\Omega = \omega \times \mathbb{R}^{N-m} \subset \mathbb{R}^N$ is a strip-like domain, where $\omega \subset \mathbb{R}^m$ is a bounded domain with smooth boundary, $N - m \geq 2$; $\lambda, \mu \geq 0$ are parameters;

$\alpha, \beta \in L^1(\Omega) \cap L^\infty(\Omega)$; and F_u (resp. G_u) and F_v (resp. G_v) denote the partial derivatives of $F \in C^1(\mathbb{R}^2, \mathbb{R})$ (resp. $G \in C^1(\mathbb{R}^2, \mathbb{R})$) with respect to the first and second variables.

Systems of the type $(D^{p,q}_{\lambda,0})$ have been the object of intensive investigations on bounded domains; see for instance Afrouzi and Heidarkhani [1], [2], Boccardo and de Figueiredo [11], Heidarkhani and Tian [40], Li and Tang [60], etc. Carrião and Miyagaki [24] studied the existence of positive non-trivial solutions of a related system to $(D^{p,q}_{\lambda,\mu})$ on strip-like domains and on domains which are situated between two infinite cylinders when $p = q$. They assumed that the nonlinear terms F and G have some sort of homogeneity. Their approach is based on a suitable version of the concentration compactness principle. Here, we allow the case $p \neq q$ and we do not assume any homogeneity properties either on F or on G.

Before to present our main result, we notice that the unperturbed problem $(D^{p,q}_{\lambda,0})$ on strip-like domains has been considered by Kristály [50], where the existence of an open interval $\Lambda \subset (0, \infty)$ is guaranteed such that the system $(D^{p,q}_{\lambda,0})$ has at least two non-trivial solutions whenever $\lambda \in \Lambda$. In the present paper, not only the perturbed case can be treated (with any subcritical perturbation) but also a half-interval (λ_0, ∞) can be constructed such that $(D^{p,q}_{\lambda,\mu})$ has at least three distinct solutions whenever $\lambda \in (\lambda_0, \infty)$ and μ is small enough.

We assume that the nonlinear term $F \in C^1(\mathbb{R}^2, \mathbb{R})$ satisfies the following properties:

(F) There exist $c_1 > 0$, $r \in (p, p^*)$, $l \in (q, q^*)$ such that

$$|F_s(s,t)| \leq c_1(|s|^{p-1} + |t|^{\frac{p-1}{p}q} + |s|^{r-1}),$$

$$|F_t(s,t)| \leq c_1(|t|^{q-1} + |s|^{\frac{q-1}{q}p} + |t|^{l-1}),$$

for every $(s,t) \in \mathbb{R}^2$;

$(\mathbf{F}_+)$ $F(s,t) \geq 0$ for every $(s,t) \in \mathbb{R}^2$, $F(0,0) = 0$, and $F \not\equiv 0$;

$(\mathbf{F_0})$ $\displaystyle\lim_{(s,t)\to(0,0)} \frac{F(s,t)}{|s|^p + |t|^q} = 0;$

$(\mathbf{F}_\infty)$ $\displaystyle\lim_{|s|+|t|\to\infty} \frac{F(s,t)}{|s|^p + |t|^q} = 0.$

On the perturbation term $G \in C^1(\mathbb{R}^2, \mathbb{R})$, we assume that it is of subcritical type (as F in the $(\mathbf{F})$), i.e.

(**G**) There exist $c_2 > 0$, $r \in (p, p^*)$, $l \in (q, q^*)$ such that

$$|G_s(s,t)| \leq c_2(|s|^{p-1} + |t|^{\frac{p-1}{p}q} + |s|^{r-1}),$$

$$|G_t(s,t)| \leq c_2(|t|^{q-1} + |s|^{\frac{q-1}{q}p} + |t|^{l-1}),$$

for every $(s,t) \in \mathbb{R}^2$.

Finally, we assume on $\alpha : \Omega \to \mathbb{R}$ that

(α_+) $\alpha \geq 0$ and $\sup_{R>0} \inf_{(x,y)\in\omega\times B_R} \alpha(x,y) > 0$,

where B_r denotes the $(N-m)-$dimensional open ball with radius $r > 0$ and center 0.

Our main result reads as follows.

Theorem 3.2.4. ([52]) *Let $F \in C^1(\mathbb{R}^2, \mathbb{R})$ be a function which satisfies* (**F**), (**F**$_+$), (**F**$_\mathbf{0}$) *and* (**F**$_\infty$), *$\alpha, \beta \in L^1(\Omega) \cap L^\infty(\Omega)$ axially symmetric functions such that (α_+) holds. Then there exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$, and every $G \in C^1(\mathbb{R}^2, \mathbb{R})$ which verifies* (**G**), *there exists $\delta > 0$ such that for every $\mu \in [0, \delta]$, problem $(D^{p,q}_{\lambda,\mu})$ has at least three distinct, axially symmetric weak solutions $(u_i, v_i) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)$, $i \in \{1, 2, 3\}$.*

We use the notation $C_{a,b}$ for the best embedding constant of the continuous embedding $W^{1,a}_0(\Omega) \hookrightarrow L^b(\Omega)$, where $b \in [a, a^*]$, $a^* = \frac{aN}{N-a}$, $a \in \{p, q\}$, i.e.

$$||u||_{L^b} \leq C_{a,b}||u||_a,$$

where $\|\cdot\|_{L^b}$ denotes the usual norm of the Lebesgue space $L^b(\Omega)$ and $||\cdot||_a$ the norm on $W^{1,a}_0(\Omega)$, i.e.

$$\|u\|_a = \left(\int_\Omega |\nabla u|^a\right)^{1/a}.$$

A solution for $(D^{p,q}_{\lambda,\mu})$ is a pair $(u, v) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)$ such that

$$\begin{cases} \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi dx = \lambda \int_\Omega \alpha(x) F_u(u,v)\phi dx + \mu \int_\Omega \beta(x) G_u(u,v)\phi dx, \\ \int_\Omega |\nabla v|^{q-2}\nabla v \nabla\psi dx = \lambda \int_\Omega \alpha(x) F_v(u,v)\psi dx + \mu \int_\Omega \beta(x) G_v(u,v)\psi dx, \end{cases} \tag{3.43}$$

for all $\phi \in W^{1,p}_0(\Omega)$ and $\psi \in W^{1,q}_0(\Omega)$.

Remark 3.2.4. Note that we cannot expect usually the existence of nontrivial solutions for small values of $\lambda > 0$ and $\mu > 0$. Indeed, let $K = \max\{C_{p,p}^p, C_{q,q}^q\}$.

If the numbers

$$S_F = K||\alpha||_{L^\infty} \sup_{(s,t)\neq(0,0)} \frac{|sF_s(s,t) + tF_t(s,t)|}{|s|^p + |t|^q}$$

$$S_G = K||\beta||_{L^\infty} \sup_{(s,t)\neq(0,0)} \frac{|sG_s(s,t) + tG_t(s,t)|}{|s|^p + |t|^q}$$

are finite, problem $(D_{\lambda,\mu}^{p,q})$ has only the trivial pair of solutions whenever

$$0 \leq \lambda S_F + \mu S_G < 1.$$

Proof. Let $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ be a solution of $(D_{\lambda,\mu}^{p,q})$.

Choosing $\phi = u$ and $\psi = v$ in (3.43), we obtain that

$$\begin{aligned}
\|u\|_p^p + \|v\|_q^q &= \int_\Omega (|\nabla u|^p + |\nabla v|^q) dx \\
&= \int_\Omega [\lambda\alpha(x)F_u(u,v)u + \mu\beta(x)G_u(u,v)u + \lambda\alpha(x)F_v(u,v)v \\
&\qquad\qquad + \mu\beta(x)G_v(u,v)v]dx \\
&= \lambda \int_\Omega \alpha(x)\left(F_u(u,v)u + F_v(u,v)v\right) dx \\
&\quad + \mu \int_\Omega \beta(x)\left(G_u(u,v)u + G_v(u,v)v\right) dx \\
&\leq \lambda\frac{S_F}{K}\int_\Omega (|u|^p + |v|^q)dx + \mu\frac{S_G}{K}\int_\Omega (|u|^p + |v|^q)dx \\
&= \left(\lambda\frac{S_F}{K} + \mu\frac{S_G}{K}\right)\left(\|u\|_{L^p}^p + \|v\|_{L^q}^q\right) \\
&\leq \frac{1}{K}(\lambda S_F + \mu S_G)\left(C_{p,p}^p\|u\|_p^p + C_{q,q}^q\|v\|_q^q\right) \\
&\leq (\lambda S_F + \mu S_G)\left(\|u\|_p^p + \|v\|_q^q\right).
\end{aligned}$$

Now, if $0 \leq \lambda S_F + \mu S_G < 1$, we necessarily have that $(u,v) = (0,0)$, which concludes the proof. □

Example 3.2.2. The function $F(s,t) = \ln(1 + |s|^p|t|^q)$ fulfills hypotheses $(\mathbf{F})$, $(\mathbf{F}_+)$, $(\mathbf{F_0})$ and $(\mathbf{F}_\infty)$. Even more, if $p = q$ then $S_F = \frac{p}{2}K||\alpha||_{L^\infty}$.

We associate to the system $(D^{p,q}_{\lambda,\mu})$ the energy functional

$$\mathcal{I}_{\lambda,\mu} : W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega) \to \mathbb{R}$$

defined by

$$\mathcal{I}_{\lambda,\mu}(u,v) = \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q - \lambda\mathcal{F}(u,v) - \mu\mathcal{G}(u,v),$$

where

$$\mathcal{F}(u,v) = \int_\Omega \alpha(x)F(u,v) \ \text{ and } \ \mathcal{G}(u,v) = \int_\Omega \beta(x)G(u,v).$$

Note that if $F, G \in C^1(\mathbb{R}^2, \mathbb{R})$ verify the hypotheses $(\mathbf{F})$ and $(\mathbf{G})$, the functional $\mathcal{I}_{\lambda,\mu}$ is well-defined, of class C^1 on $W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)$ and its critical points are exactly the solutions for $(D^{p,q}_{\lambda,\mu})$.

Taking into account the unboundness of Ω (which causes, among other things, the non-compactness of the Sobolev embeddings $W^{1,a}_0(\Omega) \hookrightarrow L^b(\Omega)$, $b \in [a, a^*]$, $a \in \{p,q\}$), we construct a subspace of $W^{1,a}_0(\Omega), a \in \{p,q\}$, which can be embedded compactly in $L^b(\Omega), b \in (a, a^*)$. The compactness of this embedding will be useful in order to obtain critical points for $\mathcal{I}_{\lambda,\mu}$. This construction can be described as follows.

The action of the compact group $G = id^m \times \mathbf{O}(N-m)$ on $W^{1,a}_0(\Omega)$ is defined by

$$gu(\tilde{x}, \overline{x}) = u(\tilde{x}, g_0^{-1}\overline{x})$$

for every $(\tilde{x}, \overline{x}) \in \omega \times \mathbb{R}^{N-m}$, $g = id^m \times g_0 \in G$ and $u \in W^{1,a}_0(\Omega), a \in \{p,q\}$.

It is clear that the action G on $W^{1,a}_0(\Omega)$ is isometric. Indeed,

$$||gu||_a = ||u||_a, \text{ for every } g \in G, u \in W^{1,a}_0(\Omega), a \in \{p,q\}.$$

The space $W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)$ will be endowed with the norm

$$||(u,v)||_{p,q} = ||u||_p + ||v||_q,$$

while the group G acts on it by

$$g(u,v) = (gu, gv), \text{for every } g \in G, (u,v) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega).$$

Let $W^{1,a}_{0,G}(\Omega)$ be the following subset of $W^{1,a}_0(\Omega)$

$$W^{1,a}_{0,G}(\Omega) = Fix_G(W^{1,a}_0(\Omega)) = \{u \in W^{1,a}_0(\Omega) : gu = u, \text{ for every} g \in G\}.$$

Since $N-m \geq 2$, the embedding $W^{1,a}_{0,G}(\Omega) \hookrightarrow L^b(\Omega)$ with $b \in (a, a^*)$, $a \in \{p,q\}$ is compact (see Lions [61, Théorème III.2]).

We have that

$$
\begin{aligned}
&Fix_G(W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)) = \\
=&\{(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) : g(u,v) = (u,v) \text{ for every} g \in G\} \\
=&W^{1,p}_{0,G}(\Omega) \times W^{1,q}_{0,G}(\Omega).
\end{aligned}
$$

We say that a function $h : \Omega \to \mathbb{R}$ is *axially symmetric* if $h(\tilde{x}, \bar{x}) = h(\tilde{x}, g\bar{x})$, for every $\tilde{x} \in \omega, \bar{x} \in \mathbb{R}^{N-m}$ and $g \in \mathbf{O}(N-m)$. In particular, the elements of $W^{1,a}_{0,G}(\Omega)$ are exactly the axially symmetric functions of $W_0^{1,a}(\Omega)$.

In order to prove Theorem 3.2.4, we must find critical points for $\mathcal{I}_{\lambda,\mu}$. Let us denote by $\mathcal{I}^G_{\lambda,\mu}$ the restriction of $\mathcal{I}_{\lambda,\mu}$ to the space $W^{1,p}_{0,G}(\Omega) \times W^{1,q}_{0,G}(\Omega)$. Due to the principle of symmetric criticality 1.4.1 of Palais, the critical points of the $\mathcal{I}^G_{\lambda,\mu}$ will be critical points of $\mathcal{I}_{\lambda,\mu}$ as well. Therefore, it is enough to guarantee the existence of critical points for $\mathcal{I}^G_{\lambda,\mu}$.

To do this, we recall the following Ricceri-type three critical point theorem. First, we need the following notion: if X is a Banach space, we denote by $\mathcal{W}_X$ the class of those functionals $E : X \to \mathbb{R}$ that possess the property that if $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ and $\liminf_n E(u_n) \leq E(u)$ then $\{u_n\}$ has a subsequence strongly converging to u.

Theorem 3.2.5. ([89, Theorem 2] *Let X be a separable and reflexive real Banach space, let $E_1 : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1 functional belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*; and $E_2 : X \to \mathbb{R}$ a C^1 functional with a compact derivative. Assume that E_1 has a strict local minimum u_0 with $E_1(u_0) = E_2(u_0) = 0$. Setting the numbers*

$$
\tau = \max\left\{0, \limsup_{\|u\|\to\infty} \frac{E_2(u)}{E_1(u)}, \limsup_{u\to u_0} \frac{E_2(u)}{E_1(u)}\right\}, \tag{3.44}
$$

$$
\chi = \sup_{E_1(u)>0} \frac{E_2(u)}{E_1(u)}, \tag{3.45}
$$

assume that $\tau < \chi$.

Then, for each compact interval $[a,b] \subset (1/\chi, 1/\tau)$ (with the conventions $1/0 = \infty$ and $1/\infty = 0$) there exists $\kappa > 0$ with the following property: for every $\lambda \in [a,b]$

and every C^1 functional $E_3 : X \to \mathbb{R}$ with a compact derivative, there exists $\delta > 0$ such that for each $\theta \in [0, \delta]$, the equation

$$E_1'(u) - \lambda E_2'(u) - \theta E_3'(u) = 0$$

admits at least three solutions in X having norm less than κ.

Proof of Theorem 3.2.4

Proof. In Theorem 3.2.5 we choose $X = W^{1,p}_{0,G}(\Omega) \times W^{1,q}_{0,G}(\Omega)$ endowed with the norm $\|(u, v)\|_{p,q} = \|u\|_p + \|v\|_q$, and $E_1, E_2, E_3 : X \to \mathbb{R}$ defined by

$$E_1(u, v) = \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q, \quad E_2(u, v) = \mathcal{F}(u, v) \text{ and } E_3(u, v) = \mathcal{G}(u, v).$$

It is clear that both E_i, $i = \{1, 2, 3\}$ are C^1 functionals and $\mathcal{I}^G_{\lambda,\mu} = E_1 - \lambda E_2 - \mu E_3$. It is also a standard fact that E_1 is a coercive, sequentially weakly lower semicontinuous functional which belongs to $\mathcal{W}_X$, bounded on each bounded subset of X, and its derivative admits a continuous inverse on X^*. Moreover, due to $(\mathbf{F})$ and $(\mathbf{G})$ both E_2 and E_3 have a compact derivatives since $W^{1,a}_{0,G}(\Omega) \hookrightarrow L^b(\Omega)$ are compact embeddings for every $b \in (a, a^*), a \in \{p, q\}$.

Now, we prove that the functional $(u, v) \mapsto \dfrac{E_2(u, v)}{E_1(u, v)}$ has the following property

$$\lim_{\|(u,v)\|_{p,q}\to 0} \frac{E_2(u, v)}{E_1(u, v)} = \lim_{\|(u,v)\|_{p,q}\to\infty} \frac{E_2(u, v)}{E_1(u, v)} = 0. \tag{3.46}$$

First, relations $(\mathbf{F_0})$ and $(\mathbf{F}_\infty)$ imply that for every $\varepsilon > 0$ there exists $\delta_\varepsilon \in (0, 1)$ such that for every $(s, t) \in \mathbb{R}^2$ with $|s| + |t| \in (0, \delta_\varepsilon) \cup (\delta_\varepsilon^{-1}, +\infty)$, one has

$$0 \le \frac{F(s, t)}{|s|^p + |t|^q} < \varepsilon. \tag{3.47}$$

Fix $\gamma \in \left(1, \min\left\{\frac{p^*}{p}, \frac{q^*}{q}\right\}\right)$. Note that the continuous function $(s, t) \mapsto \frac{F(s,t)}{|s|^{p\gamma}+|t|^{q\gamma}}$ is bounded on the set $\{(s, t) \in \mathbb{R}^2 : |s| + |t| \in [\delta_\varepsilon, \delta_\varepsilon^{-1}]\}$. Therefore, for some $m_\varepsilon > 0$, we have that in particular

$$\begin{aligned} 0 \le F(s, t) \le & \frac{1}{||\alpha||_{L^\infty}} \cdot \left(\frac{\varepsilon}{\max\{pC^p_{p,p}, qC^q_{q,q}\}}(|s|^p + |t|^q) \right. \\ & \left. + \frac{m_\varepsilon}{(\max\{p, q\})^\gamma}(|s|^{p\gamma} + |t|^{q\gamma}) \right), \text{for all } (s, t) \in \mathbb{R}^2. \end{aligned}$$

Therefore, for each $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ we get

$$\begin{aligned}
0 \leq E_2(u,v) = & \int_\Omega \alpha(x) F(u,v) \\
\leq & \frac{1}{||\alpha||_{L^\infty}} \cdot \frac{\varepsilon}{\max\{pC_{p,p}^p, qC_{q,q}^q\}} \int_\Omega \alpha(x)(|u|^p + |v|^q) \\
& + \frac{1}{||\alpha||_{L^\infty}} \cdot \frac{m_\varepsilon}{(\max\{p,q\})^\gamma} \int_\Omega \alpha(x)(|u|^{p\gamma} + |v|^{q\gamma}) \\
\leq & \frac{\varepsilon}{\max\{pC_{p,p}^p, qC_{q,q}^q\}} (\|u\|_{L^p}^p + \|v\|_{L^q}^q) \\
& + \frac{m_\varepsilon}{(\max\{p,q\})^\gamma} (\|u\|_{L^{p\gamma}}^{p\gamma} + \|v\|_{L^{q\gamma}}^{q\gamma}) \\
\leq & \frac{\varepsilon}{\max\{pC_{p,p}^p, qC_{q,q}^q\}} (C_{p,p}^p \|u\|_p^p + C_{q,q}^q \|v\|_q^q) + \\
& + \frac{m_\varepsilon}{(\max\{p,q\})^\gamma} (C_{p,p\gamma}^{p\gamma} \|u\|_p^{p\gamma} + C_{q,q\gamma}^{q\gamma} \|v\|_q^{q\gamma}) \\
\leq & \varepsilon \left(\frac{1}{p} \|u\|_p^p + \frac{1}{q} \|v\|_q^q \right) + m_\varepsilon C_\gamma \left(\frac{1}{p^\gamma} \|u\|_p^{p\gamma} + \frac{1}{q^\gamma} \|v\|_q^{q\gamma} \right) \\
= & \varepsilon \left(\frac{1}{p} \|u\|_p^p + \frac{1}{q} \|v\|_q^q \right) + m_\varepsilon C_\gamma \left[\left(\frac{1}{p} \|u\|_p^p \right)^\gamma + \left(\frac{1}{q} \|v\|_q^q \right)^\gamma \right] \\
\leq & \varepsilon \left(\frac{1}{p} \|u\|_p^p + \frac{1}{q} \|v\|_q^q \right) + m_\varepsilon C_\gamma \left(\frac{1}{p} \|u\|_p^p + \frac{1}{q} \|v\|_q^q \right)^\gamma,
\end{aligned}$$

where $C_\gamma = \max\{C_{p,p\gamma}^{p\gamma}, C_{q,q\gamma}^{q\gamma}\}$ and in the last inequality we used the fact that the function $\gamma \mapsto (s^\gamma + t^\gamma)^{\frac{1}{\gamma}}$ is decreasing, $s,t \geq 0$. Consequently, for every $(u,v) \neq (0,0)$, we obtain

$$0 \leq \frac{E_2(u,v)}{E_1(u,v)} \leq \varepsilon + m_\varepsilon C_\gamma \left(\frac{1}{p} \|u\|_p^p + \frac{1}{q} \|v\|_q^q \right)^{\gamma - 1}.$$

Since $\gamma > 1$ and $\varepsilon > 0$ is arbitrarily small, we obtain the first limit from (3.46), whenever $\|(u,v)\|_{p,q} \to 0$.

Now, we fix $0 < \eta < 1$. Since $\alpha \in L^1(\Omega) \cap L^\infty(\Omega)$ and it verifies (α_+), then $0 < ||\alpha||_{L^{\frac{1}{1-\eta}}} < \infty$. Using the relation (3.47) and the fact that the continuous function $(s,t) \mapsto \dfrac{F(s,t)}{|s|^{p\eta} + |t|^{q\eta}}$ is bounded on the set $\{(s,t) \in \mathbb{R}^2 : |s| + |t| \in [\delta_\varepsilon, \delta_\varepsilon^{-1}]\}$, for some $M_\varepsilon > 0$, we have that

$$0 \leq F(s,t) \leq \frac{1}{||\alpha||_{L^\infty}} \frac{\varepsilon}{\max\{pC_{p,p}^p, qC_{q,q}^q\}}(|s|^p + |t|^q) + \\ + \frac{1}{||\alpha||_{L^{\frac{1}{1-\eta}}}} \frac{M_\varepsilon}{(\max\{p,q\})^\eta}(|s|^{p\eta} + |t|^{q\eta}), \text{for all } (s,t) \in \mathbb{R}^2.$$

Therefore, we have

$$\begin{aligned} 0 \leq E_2(u,v) = & \int_\Omega \alpha(x) F(u,v) \\ \leq & \frac{1}{||\alpha||_{L^\infty}} \cdot \frac{\varepsilon}{\max\{pC_{p,p}^p, qC_{q,q}^q\}} \int_\Omega \alpha(x)(|u|^p + |v|^q) \\ + & \frac{1}{||\alpha||_{L^{\frac{1}{1-\eta}}}} \cdot \frac{M_\varepsilon}{(\max\{p,q\})^\eta} \int_\Omega \alpha(x)(|u|^{p\eta} + |v|^{q\eta}). \end{aligned}$$

A similar calculation as above on the first term in this inequality shows that

$$\frac{1}{||\alpha||_{L^\infty}} \cdot \frac{\varepsilon}{\max\{pC_{p,p}^p, qC_{q,q}^q\}} \int_\Omega \alpha(x)(|u|^p + |v|^q) \leq \varepsilon \left(\frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q \right).$$

To estimate the second term we use the Hölder's inequality and we obtain

$$\begin{aligned} & \frac{1}{||\alpha||_{L^{\frac{1}{1-\eta}}}} \cdot \frac{M_\varepsilon}{(\max\{p,q\})^\eta} \int_\Omega \alpha(x)(|u(x)|^{p\gamma} + |v(x)|^{q\gamma}) dx \\ & \leq \frac{1}{||\alpha||_{L^{\frac{1}{1-\eta}}}} \cdot \frac{M_\varepsilon}{(\max\{p,q\})^\eta} \left(\int_\Omega \alpha^{\frac{1}{1-\eta}}(x) dx \right)^{1-\eta} \cdot \\ & \qquad \cdot \left[\left(\int_\Omega |u(x)|^p dx \right)^\eta + \left(\int_\Omega |v(x)|^q dx \right)^\eta \right] \leq \\ & \leq \frac{M_\varepsilon}{(\max\{p,q\})^\eta} (||u||_{L^p}^{p\eta} + ||v||_{L^q}^{q\eta}) \leq M_\varepsilon C_\eta \left[\left(\frac{1}{p}\|u\|_p^p \right)^\eta + \left(\frac{1}{q}\|v\|_q^q \right)^\eta \right], \end{aligned}$$

where $C_\eta = \max\{C_{p,p}^{p\eta}, C_{q,q}^{q\eta}\}$. Therefore, we have the following estimate

$$0 \leq E_2(u,v) \leq \varepsilon \left(\frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q \right) + M_\varepsilon C_\eta \left[\left(\frac{1}{p}\|u\|_p^p \right)^\eta + \left(\frac{1}{q}\|v\|_q^q \right)^\eta \right].$$

Here we use the concavity of the function $s \mapsto s^\eta$, with $s > 0$ and $\eta < 1$, which yields that

$$2^{\eta-1}(s^\eta + t^\eta) \leq (s+t)^\eta \ \text{ for every } s,t \geq 0.$$

Therefore,

$$0 \leq E_2(u,v) \leq \varepsilon\left(\frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q\right) + M_\varepsilon C_\eta 2^{1-\eta}\left(\frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q\right)^\eta.$$

For every $(u,v) \neq (0,0)$, we have that

$$0 \leq \frac{E_2(u,v)}{E_1(u,v)} \quad \leq \quad \varepsilon + M_\varepsilon C_\eta 2^{1-\eta}\left(\frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q\right)^{\eta-1}.$$

Due to the arbitrariness of $\varepsilon > 0$ and $\eta < 1$, by letting the limit $\|(u,v)\|_{p,q} \to \infty$, we obtain the second relation from (3.46).

Note that E_1 has a strict global minimum $(u_0,v_0) = (0,0)$, and $E_1(0,0) = E_2(0,0) = 0$. The definition of the number τ in Theorem 3.2.5, see (3.44), and the limits in (3.46) imply that $\tau = 0$.

In the sequel we will prove that there exists $(u_1,v_1) \in X$ such that $E_2(u_1,v_1) > 0$. From the assumption $(\mathbf{F}_+)$ follows the existence of the numbers $(s_0,t_0) \in \mathbb{R}^2$ such that

$$F(s_0,t_0) > 0.$$

Due to the condition (α_+) there exists an $R > 0$ such that $\alpha \not\equiv 0$ on $\omega \times B_R$.

We define the functions $u_1, v_1 : \omega \times \mathbb{R}^{N-m} \to \mathbb{R}$ by

$$u_1(\tilde{x},\bar{x}) = \begin{cases} 0, & \text{if} \quad \bar{x} \notin B_R \\ s_0, & \text{if} \quad \bar{x} \in B_{R/2} \\ \dfrac{2s_0}{R}(R - ||\bar{x}||_{\mathbb{R}^{N-m}}), & \text{if} \quad \bar{x} \in B_R \setminus B_{R/2} \end{cases}$$

$$v_1(\tilde{x},\bar{x}) = \begin{cases} 0, & \text{if} \quad \bar{x} \notin B_R \\ t_0, & \text{if} \quad \bar{x} \in B_{R/2} \\ \dfrac{2t_0}{R}(R - ||\bar{x}||_{\mathbb{R}^{N-m}}), & \text{if} \quad \bar{x} \in B_R \setminus B_{R/2} \end{cases}$$

It is clear that (u_1,v_1) belongs to X, so using the conditions $(\mathbf{F}_+)$, (α_+), we get

$$\begin{aligned} E_2(u_1,v_1) =& \int_\Omega \alpha(\tilde{x},\bar{x})F(u_1(\tilde{x},\bar{x}),v_1(\tilde{x},\bar{x}))d\tilde{x}d\bar{x} \\ =& \int_{\omega\times B_{R/2}} \alpha(\tilde{x},\bar{x})F(s_0,t_0)d\tilde{x}d\bar{x} + \\ & \qquad + \int_{\omega\times(\mathbb{R}^{N-m}\setminus B_{R/2})} \alpha(\tilde{x},\bar{x})F(u_1(\tilde{x},\bar{x}),v_1(\tilde{x},\bar{x}))d\tilde{x}d\bar{x} \\ \geq& F(s_0,t_0)\int_{\omega\times B_{R/2}} \alpha(\tilde{x},\bar{x})d\tilde{x}d\bar{x} > 0. \end{aligned}$$

Furthermore, keeping the notation from (3.45), we obtain that

$$\frac{1}{\chi} = \inf_{E_2(u,v)>0} \frac{E_1(u,v)}{E_2(u,v)}$$

is well-defined. Therefore, applying Theorem 3.2.5, we obtain that in particular for every $\lambda > \frac{1}{\chi}$ and every $G \in C^1(\mathbb{R}^2, \mathbb{R})$ which verifies (**G**), there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the equation $(\mathcal{I}^G_{\lambda,\mu})'(u,v) \equiv E_1'(u,v) - \lambda E_2'(u,v) - \mu E_3'(u,v) = 0$ admits at least three distinct pairs of solutions in X, which are also critical points to $\mathcal{I}_{\lambda,\mu}$ by the principle of symmetric criticality (see [79]). Consequently, for every $\lambda > \frac{1}{\chi} =: \lambda_0$ and every $G \in C^1(\mathbb{R}^2, \mathbb{R})$ which verifies (**G**), there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the system $(D^{p,q}_{\lambda,\mu})$ has at least three distinct, axially symmetric weak solutions $(u_i, v_i) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, $i \in \{1, 2, 3\}$. This concludes the proof. $\square$

3.3 Spherical cap symmetric solutions for hemivariational inequalities on $\mathbb{R}^N$

In this section, following Mezei, Molnár and Vas [70], we obtain multiplicity results for hemivariational inequalities on the whole space. Our approach relies on variational methods, combining the symmetric version of Ekeland's variational principle [92] with a nonsmooth symmetric minimax principle due to Van Schaftingen [91], and shows that solutions are invariant under spherical cap symmetrization.

Let $\Omega = \mathbb{R}^N$. Consider a real, separable, reflexive Banach space $(X, \|\cdot\|_X)$ and its topological dual $(X^\star, \|\cdot\|_{X^\star})$. Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ a locally Lipschitz function. In addition, let p be such that $2 \le p < N$, while $p^\star = \dfrac{Np}{N-p}$ denotes the Sobolev critical exponent.

Our problem is formulated as follows:

Find $u \in X$ such that

$$\langle Au, v\rangle + \int_{\mathbb{R}^N} F_y^0(x; u(x); -v(x))dx \ge 0, \ \forall v \in X, \tag{$\mathcal{P}^2_\lambda$}$$

where F_y^0 denotes the generalized directional derivative of F in the second variable.

In order to derive our existence result, we need to impose the following hypotheses:

(CT) Suppose that for $r \in [p, p^\star]$, the inclusion $X \hookrightarrow L^r(\mathbb{R}^N)$ is continuous with the embedding constant C_r.

(**CP**) Assume that for $r \in (p, p^\star)$, the embedding $X \hookrightarrow L^r(\mathbb{R}^N)$ is compact.

Notice that $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $X^\star$ and X and $|| \cdot ||_r$ is the norm of $L^r(\mathbb{R}^N)$.

Let $A : X \to X^\star$ be a potential operator with the potential $a : X \to \mathbb{R}$, that is, a is Gâteaux differentiable and for every $u, v \in X$ we have

$$\lim_{t \to 0} \frac{a(u + tv) - a(u)}{t} = \langle A(u), v \rangle.$$

For a potential we always assume that $a(0) = 0$. In addition, we suppose that $A : X \to X^\star$ satisfies the following properties:

($\mathbf{A_1}$) A is hemicontinuous, i.e. A is continuous on line segments in X and $X^\star$ equipped with the weak topology.

($\mathbf{A_2}$) A is homogeneous of degree $p - 1$, i.e. for every $u \in X$ and $t > 0$ we have $A(tu) = t^{p-1}A(u)$.

($\mathbf{A_3}$) $A : X \to X^\star$ is a strongly monotone operator, i.e. there exists a continuous function $\tau : [0, \infty) \to [0, \infty)$ which is strictly positive on $(0, \infty)$, $\tau(0) = 0$, $\lim\limits_{t \to \infty} \tau(t) = \infty$ and

$$\langle A(u) - A(v), u - v \rangle \geq \tau(||u - v||_X)||u - v||_X,$$

for all $u, v \in X$.

($\mathbf{A_4}$) $a(u) \geq c\|u\|_X^p$, for all $u \in X$, where c is a positive constant.

($\mathbf{A_5}$) $a(u^H) \leq a(u)$, for all $u \in X$, where u^H denotes the polarization of u (for details, see Section 2.).

Remark 3.3.1. By conditions ($\mathbf{A_1}$) and ($\mathbf{A_2}$), we have $a(u) = \frac{1}{p}\langle A(u), u \rangle$.

Furthermore, we suppose that the following additional condition holds: there exists $c > 0$ and $r \in (p, p^\star)$ such that

($\mathbf{F_1}$) $|\xi| \leq c(|s|^{p-1} + |s|^{r-1}), \forall s \in \mathbb{R}, \xi \in F(x, s)$ and a.e. $x \in \mathbb{R}^N$.

$(\mathbf{F_2})$ there exists $q \in (0,p)$, $\nu \in (p,p^\star)$, $\alpha \in L^{\frac{\nu}{\nu-q}}(\mathbb{R}^N)$, $\beta \in L^1(\mathbb{R}^N)$ such that

$$F(z,s) \leq \alpha(z)|s|^q + \beta(z)$$

for all $s \in \mathbb{R}$ and a.e. $z \in \mathbb{R}^N$.

$(\mathbf{F_3})$ $F(x,s) = F(y,s)$ for a.e. $x, y \in \Omega$, with $|x| = |y|$ and all $s \in \mathbb{R}$;

$(\mathbf{F_4})$ $F(x,s) \leq F(x,-s)$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^-$.

Our main result reads as follow

Theorem 3.3.1. ([70]) *Assume that $2 \leq p < N$ and let $\Omega = \mathbb{R}^N$. Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function, $A : X \to X^\star$ be a potential operator such that the conditions $(\mathbf{A_1}) - (\mathbf{A_5})$ and $(\mathbf{CT})$, $(\mathbf{CP})$, $(\mathbf{F_1})$, $(\mathbf{F_1})$, $(\mathbf{F_2})$, $(\mathbf{F_3})$, $(\mathbf{F_4})$ are fulfilled. Then, there exists $\lambda_2 > 0$ such that for every $\lambda > \lambda_2$ the problem $(\mathcal{P}_\lambda^2)$ has two nontrivial solutions, which are invariants by spherical cap symmetrization.*

The energy functional related to the problem $(\mathcal{P}_\lambda^2)$ is defined as follows:

$$\mathcal{A}_\lambda(u) = a(u) - \lambda \tilde{\mathcal{F}}(u),$$

where $\tilde{\mathcal{F}} : X \to \mathbb{R}$ is a function defined by $\tilde{\mathcal{F}}(u) = \displaystyle\int_{\mathbb{R}^N} F(x,u(x))dx.$

Remark 3.3.2. We observe that, using Proposition 5.1.2. from Varga and Kristály [59], due to condition $(\mathbf{F_1})$, we have that

$$\tilde{\mathcal{F}}^0(u;v) \leq \int_{\mathbb{R}^N} F_y^0(x,u(x);v(x))dx.$$

Therefore, it follows that the critical points of the energy functional $\mathcal{A}_\lambda$ are the (weak) solutions of the problem $(\mathcal{P}_\lambda^2)$.

Lemma 3.3.1. *Let the conditions $(\mathbf{F_2})$ and $(\mathbf{A_4})$ be satisfied. Then the functional $\mathcal{A}_\lambda : X \to \mathbb{R}$ is coercive for each $\lambda > 0$, that is, $\mathcal{A}_\lambda(u) \to \infty$ as $\|u\|_X \to \infty$, for all $u \in X$.*

Proof. Due to $(\mathbf{F_2})$, for all $u \in X$ we have:

$$F(x,u(x)) \leq \alpha(x)|u(x)|^q + \beta(x).$$

Hence, by using Hölder's inequality, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^N} F(x,u(x))dx &\leq \int_{\mathbb{R}^N} \alpha(x)|u(x)|^q dx + \int_{\mathbb{R}^N} \beta(x)dx \\
&\leq \left[\int_{\mathbb{R}^N} \alpha(x)^{\frac{\nu}{\nu-q}}\right]^{\frac{\nu-q}{\nu}} \cdot \left[\int_{\mathbb{R}^N} [|u(x)|^q]^{\frac{\nu}{q}}\right]^{\frac{q}{\nu}} dx + \int_{\mathbb{R}^N} \beta(x)dx \\
&\leq \|\alpha\|_{\frac{\nu}{\nu-q}} \cdot \|u\|_\nu^q + \|\beta\|_1. \qquad (3.48)
\end{aligned}$$

Since $X \hookrightarrow L^\nu(\mathbb{R}^N)$, when $\nu \in [p,p^*]$, one can find a number $C_\nu \geq 0$ such that

$$\|u\|_\nu^q \leq C_\nu^q \|u\|_X^q. \qquad (3.49)$$

Combining the relations (3.48) and (3.49), we obtain that for all $\lambda > 0$, we have

$$-\lambda \int_{\mathbb{R}^N} F(x,u(x))dx \geq -\lambda\|\alpha\|_{\frac{\nu}{\nu-q}} \cdot C_\nu^q \|u\|_X^q - \lambda\|\beta\|_1.$$

Therefore, from the definition of the energy functional $\mathcal{A}_\lambda$ and using the condition $(\mathbf{A_4})$, we get

$$\begin{aligned}
\mathcal{A}_\lambda(u) &\geq a(u) - \lambda\|\alpha\|_{\frac{\nu}{\nu-q}} \cdot C_\nu^q \|u\|_X^q - \lambda\|\beta\|_1 \\
&\geq c\|u\|^p - \lambda\|\alpha\|_{\frac{\nu}{\nu-q}} \cdot C_\nu^q \|u\|_X^q - \lambda\|\beta\|_1.
\end{aligned}$$

Taking into account $(\mathbf{F_2})$ and the fact that $q \in (0,p)$, it follows that $\mathcal{A}_\lambda(u) \to +\infty$, whenever $\|u\|_X \to +\infty$. This completes the proof.

Lemma 3.3.2. *If the conditions hold then for every $\lambda > 0$ the functional $\mathcal{A}_\lambda : X \to \mathbb{R}$ satisfies the Palais-Smale condition.*

Proof. The proof of this lemma is similar to the proofs of the Lemma 2.2.2 and of [59, Theorem 5.1.1].

Lemma 3.3.3. *Assume that $(\mathbf{F_3}), (\mathbf{F_4})$ and $(\mathbf{A_5})$ are satisfied. Then, for all $H \in H_*$, we have*

$$\mathcal{A}_\lambda(u^H) \leq \mathcal{A}_\lambda(u), \forall u \in X.$$

Proof. From $(\mathbf{A_5})$, we have that $a(u^H) \leq a(u)$. Therefore, using $(\mathbf{F_3}), (\mathbf{F_4})$ and taking inspiration from the proof of Lemma 4.6. in Squassina [92], we obtain

$$\int_{\mathbb{R}^N} F(x, u(x))dx \leq \int_{\mathbb{R}^N} F(x, u^H(x))dx.$$

Hence, by the definition of $\mathcal{A}_\lambda$, we have that for all $\lambda > 0$:

$$\begin{aligned} \mathcal{A}_\lambda(u) &= \frac{1}{p}a(u) - \lambda \int_{\mathbb{R}^N} F(x, u(x))dx \\ &\geq \frac{1}{p}a(u^H) - \lambda \int_{\mathbb{R}^N} F(x, u^H(x))dx \\ &= \mathcal{A}_\lambda(u^H). \end{aligned}$$

Proof of Theorem 3.3.1 is similar to the proof of Theorem 2.2.1 so it is left to the reader.

Particular case. Let $V : \mathbb{R}^N \to \mathbb{R}$ a function such that:

$(\mathbf{V1})$ $V_0 := \inf_{x\in\mathbb{R}^N} V(x) > 0$;

$(\mathbf{V2})$ For every $M > 0$, we have $\text{meas}(\{x \in \mathbb{R}^N : b(x) \leq M\}) < \infty$;

$(\mathbf{V3})$ For $x, y \in \mathbb{R}^N$, if $|x| \leq |y|$ then $V(x) \leq V(y)$.

The space $H = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2 dx < \infty\}$, equipped with the inner product

$$\langle u, v\rangle_H = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx$$

is a Hilbert space. It is known that H is compactly embedded into $L^s(\mathbb{R}^n)$ for $s \in [2, 2^*)$ (see Bartsch and Wang [9]).

A particular case of the problem (P_λ^2) can be formulated as follows: Find a positive $u \in H$ such that for each $v \in H$ we have

$$\int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx + \int_{\mathbb{R}^N} F_y^0(x, u(x) - v(x))dx \geq 0. \qquad (\mathbf{P}'_\lambda).$$

Similarly to the proof of Theorem 3.3.1, we can prove the next result:

Lemma 3.3.4. *If $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the conditions $(\mathbf{F1})$, $(\mathbf{F_2})$, $(\mathbf{F_3})$, $(\mathbf{F_4})$ and $(\mathbf{V1}) - (\mathbf{V3})$, then there exists two nontrivial solutions of the problem $(\mathbf{P_\lambda^2})$, which are invariants by the spherical cap symmetrization.*

Proof. Theorem 3.3.1 can be applied since the conditions (**A1**) – (**A5**) are fulfilled. Indeed, the assumptions (**A1**) – (**A5**) follow from the fact that $a(u) = \frac{1}{2}\langle u, u\rangle$. On the other hand, the condition (**V3**), implies (**A5**). Then, by Theorem 3.3.1, it follows that problem $(\mathbf{P}^{\mathbf{2}}_{\lambda})$ has two nontrivial solutions, which are invariants by the spherical cap symmetrization.

Chapter 4

Differential inclusions on Riemannian manifolds

This chapter explores a wide class of elliptic differential inclusions on non-compact, complete Riemannian manifolds, involving the Laplace–Beltrami operator and a Hardy-type singular term. We establish conditions for existence, non-existence, and multiplicity of solutions, depending on the nonlinear term and the manifold's curvature. Our approach relies on nonsmooth variational techniques, isometric actions, and eigenvalue properties on Riemannian manifolds. These results were first presented in [54] by Kristály, Mezei and Szilák.

4.1 The problem

Various geometric/physical phenomena can be reduced to finding solutions for the problem

$$\mathcal{L}u(x) = \alpha(x)f(u(x)),\ \ x \in \Omega, \tag{P}$$

where Ω is an open domain in an ambient metric measure space, $\mathcal{L}$ is an elliptic-type operator, $\alpha : \Omega \to \mathbb{R}$ is a measurable potential, and $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear function having certain regularity and growth properties. Such problems arise from the Yamabe problem on compact/non-compact Riemannian manifolds, the standing Schrödinger equation in $\mathbb{R}^n$ $(n \geq 2)$, Dirichlet and Neumann problems on bounded/unbounded domains, etc. Wide range of strategies and theories have been applied in the last century in order to investigate problem (P), as variational methods, fixed point arguments, sub- and super-solution techniques, etc.

An important class of problems within (P) appears when the nonlinear term $f(x,\cdot)$ is not necessarily continuous; such a relevant example appears in the description of the von Kármán adhesive plates, see Panagiotopoulos [81]. Due to the jumping effect of $f(x,\cdot)$, as a first approach, problem (P) need not has any solution. However, from physical reasons, we expect to obtain certain equilibrium states of the phenomena described by means of problem (P). Accordingly, a natural way to handle the aforementioned discontinuity situation is to 'fill the gaps', defining a differential inclusion associated with problem (P). More precisely, if f is locally essentially bounded on $\mathbb{R}$, we consider instead of the value $f(t)$ the interval $[\underline{f}(t), \overline{f}(t)]$, where

$$\underline{f}(t) = \lim_{\delta\to 0^+} \operatorname{essinf}_{|s-t|<\delta} f(s), \quad \overline{f}(t) = \lim_{\delta\to 0^+} \operatorname{esssup}_{|s-t|<\delta} f(s);$$

here, $\operatorname{essinf}_A f = \sup\{a \in \mathbb{R} : f(x) \geq a \text{ for a.e. } x \in A\}$ and $\operatorname{esssup}_A f = -\operatorname{essinf}_A(-f)$ whenever $A \neq \emptyset$. In this way, we replace (P) by the differential inclusion problem

$$\mathcal{L}u(x) \in \alpha(x)\partial F(u(x)),\ x \in \Omega, \tag{DI}$$

where $F(t) = \displaystyle\int_0^t f(s)ds$ is a locally Lipschitz function, and $\partial F(t) = [\underline{f}(t), \overline{f}(t)]$, $t \in \mathbb{R}$. Hereafter, ∂F stands for the subdifferential of F at $t \in \mathbb{R}$ in the sense of Clarke [28].

Differential inclusion problems, similar to (DI), may appear on not necessarily Euclidean structures; indeed, in certain circumstances the domain Ω can be a subset of a *curved space* (Riemannian or Finsler manifolds, sub-Riemannian structures as Heisenberg or Carnot groups, etc.), while the operator $\mathcal{L}$ may reflect the geometric feature of the ambient space.

In the present paper we consider the variational differential inclusion

$$\mathcal{L}_{x_0}u(x) \equiv -\Delta_g u(x) - \mu\frac{u(x)}{d_g^2(x_0,x)} + u(x) \in \lambda\alpha(x)\partial F(u(x)), \quad x \in M, \tag{4.1}$$

where (M,g) is an n-dimensional complete Riemannian manifold, $n \geq 3$ (endowed with its canonical measure dv_g), Δ_g is the Laplace-Beltrami operator on (M,g), $d_g : M \times M \to \mathbb{R}$ is the distance function associated with the Riemannian metric g, $x_0 \in M$ is a fixed point, $\alpha : M \to \mathbb{R}$ is a measurable potential, $\mu, \lambda \in \mathbb{R}$ are some parameters, $F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function and ∂F stands for the Clarke subdifferential of F. An element $u \in H^1(M)$ is a solution of (4.1) if there exists a measurable selection $x \mapsto \xi_x \in \partial F(u(x))$ such that the map $x \mapsto \alpha(x)\xi_x w(x)$ belongs to $L^1(M)$ for every test-function $w \in H^1(M)$ and one has

$$\begin{aligned} \int_M \nabla_g u(x)\nabla_g w(x)\mathrm{d}v_g \quad &- \quad \mu \int_M \frac{u(x)w(x)}{d_g^2(x_0,x)}\mathrm{d}v_g + \int_M u(x)w(x)\mathrm{d}v_g \\ &= \quad \lambda \int_M \alpha(x)\xi_x w(x)\mathrm{d}v_g. \end{aligned} \tag{4.2}$$

One can readily observe that (4.2) reduces to the fact that u is a weak solution of

$$-\Delta_g u(x) - \mu \frac{u(x)}{d_g^2(x_0,x)} + u(x) = \lambda\alpha(x)f(u(x)), \quad x \in M, \tag{4.3}$$

whenever f is continuous (and consequently, F is of class C^1 and $\partial F(t) = F'(t) = f(t)$).

On one hand, variational elliptic differential inclusions as (4.1) – or slightly different versions of them formulated in terms of variational-hemivariational inequalities – have been deeply studied in the last three decades, mostly in Euclidean spaces (both for bounded and unbounded domains), see e.g. Bonanno, D'Aguì and Winkert [13], Candito and Livrea [20], Carl and Le [21, 22], Carl, Le and Motreanu [23], Costea, Kristály and Varga [29], Gasiński and Papageorgiou [38], Kristály and Varga [58], Liu, Liu and Motreanu [64], Liu, Livrea, Motreanu and Zeng [65], Migórski, Ochal and Sofonea [72], Motreanu and Panagiotopoulos [77], Panagiotopoulos [81], Varga [97], etc.

On the other hand, various forms of (4.3) have been investigated both on compact and non-compact Riemannian manifolds (mostly without the singular term), see e.g. Berchio, Ferrero and Grillo [10], Bonanno, Molica Bisci and Rădulescu [14], Jaber [41], Lisei and Varga [62], Liu and Liu [63], Molica Bisci and Pucci [74], Molica Bisci and Secchi [75], Molica Bisci and Vilasi [76], etc. As expected, on non-compact manifolds additional restrictions and approaches are needed to compensate the lack of compactness.

We shall focus to a broad class of *non-compact Riemannian manifolds* and prove various non-existence, existence and multiplicity results for the differential inclusion problem (4.1), by assuming certain *curvature* hypotheses and *growths* for the function F (at the origin and at infinity). In fact, we consider two classes of Riemannian manifolds having different curvature restrictions; namely, we assume that a complete, non-compact Riemannian manifold (M, g) satisfies one of the conditions:

(i) $\mathbf{K} \leq -\kappa$ for some $\kappa \geq 0$, where $\mathbf{K}$ is the sectional curvature of the Cartan-Hadamard manifold[1] (M, g);

(ii) $\mathsf{Ric}_{(M,g)} \geq 0$, where $\mathsf{Ric}_{(M,g)}$ is the Ricci curvature on (M, g).

The 'clash' of (i) and (ii) is precisely the Euclidean space $\mathbb{R}^n$ endowed with the usual metric. In the case (ii), i.e., when $\mathsf{Ric}_{(M,g)} \geq 0$, a crucial role is played by the *asymptotic volume ratio*

$$\mathsf{AVR}_{(M,g)} = \lim_{r\to\infty} \frac{V_g(B_x(r))}{\omega_n r^n},$$

where V_g stands for the volume in (M, g), $B_x(r) = \{y \in M : d_g(x, y) < r\}$ is the ball of radius $r > 0$ and center $x \in M$, while $\omega_n = \pi^{n/2}/\Gamma(1 + n/2)$ is the volume of the Euclidean unit ball in $\mathbb{R}^n$. By Bishop-Gromov comparison principle it turns out that the asymptotic volume ratio is well-defined (i.e., independent of the choice of $x \in M$) and

$$\mathsf{AVR}_{(M,g)} \in [0, 1]. \tag{4.4}$$

4.2 Auxiliary results

In this section we recall those notions and results that are crucial to carry out our proofs. Before to do this, we fix some notations. If (M, g) is a complete Riemannian manifold, the Sobolev space $H^1(M)$ over M is the completion of $C_0^\infty(M)$ with respect to the norm

$$\|u\|_{H_1} = \left(\int_M |\nabla_g u|^2 \mathrm{d}v_g + \int_M u^2 \mathrm{d}v_g\right)^{1/2},$$

while the L^q-Lebesgue norm ($q \geq 1$) is

$$\|u\|_{L^q} = \left(\int_M |u|^q \mathrm{d}v_g\right)^{1/q},$$

with the supremum-norm for $q = +\infty$.

[1]Complete, simply connected Riemannian manifold with non-positive sectional curvature.

Functional inequalities and spectral estimates on Riemannian manifolds.

I. Cartan-Hadamard manifolds.

Throughout this subsection, let (M,g) be an n-dimensional Cartan-Hadamard manifold, $n \geq 3$. We notice that in this geometric context, there exists $C_n > 0$ such that

$$\|u\|_{L^{2^*}} \leq C_n \left(\int_M |\nabla_g u|^2 \mathrm{d}v_g \right)^{1/2}, \quad \forall u \in C_0^\infty(M),$$

see e.g. Hebey [39, Chapter 8], where $2^* = 2n/(n-2)$ is the critical Sobolev exponent. Moreover, the best Sobolev embedding constant C_n is precisely its Euclidean counterpart AT_n, provided by Aubin [7] and Talenti [95], whenever the Cartan-Hadamard conjecture holds on (M,g) (e.g. in dimensions 3 and 4). In high-dimensions, the sharp constant $C_n > 0$ is not known; however, a non-optimal form can be given by means of the Croke-constant as in Hebey [39, p. 239].

A density argument combined with a simple interpolation shows that the Sobolev space $H^1(M)$ is continuously embedded into $L^q(M)$ for every $q \in [2, 2^*]$; more precisely, there exists $K_q^- > 0$ such that

$$\|u\|_{L^q} \leq K_q^- \|u\|_{H^1}, \quad \forall u \in H^1(M). \tag{4.5}$$

Let $x_0 \in M$ be fixed. Then the *Hardy inequality* holds on (M,g), which reads as

$$\frac{(n-2)^2}{4} \int_M \frac{u^2(x)}{d_g^2(x_0,x)} \mathrm{d}v_g \leq \int_M |\nabla_g u|^2 \mathrm{d}v_g, \quad \forall u \in H^1(M), \tag{4.6}$$

where $\frac{(n-2)^2}{4}$ is sharp and never achieved, see e.g. D'Ambrosio and Dipierro [31], and Kristály [47].

In addition, if the sectional curvature has the property $\mathbf{K} \leq -\kappa$ for some $\kappa > 0$, then *McKean's spectral gap theorem* asserts that

$$\gamma_{(M,g)} := \inf_{u \in H^1(M) \setminus \{0\}} \frac{\displaystyle\int_M |\nabla_g u|^2 \mathrm{d}v_g}{\displaystyle\int_M u^2 \mathrm{d}v_g} \geq \frac{(n-1)^2}{4} \kappa. \tag{4.7}$$

The inequality (4.7) is sharp, see e.g. on the n-dimensional hyperbolic space $\mathbb{H}_\kappa^n$ with constant sectional curvature $\mathbf{K} = -\kappa$; we also notice that the infimum in (4.7) is not achieved by any function $u \in H^1(M)$;

II. Riemannian manifolds with non-negative Ricci curvature.

In this subsection we consider an n-dimensional ($n \geq 3$) complete non-compact Riemannian manifold (M, g) with $\mathsf{Ric}_{(M,g)} \geq 0$. As we already noticed in (4.4), the asymptotic volume ratio $\mathsf{AVR}_{(M,g)} \in [0, 1]$ provides deep geometric information about the manifold; for instance, $\mathsf{AVR}_{(M,g)} = 1$ if and only if (M, g) is isometric to the Euclidean space $\mathbb{R}^n$. Quantitatively speaking, closer value of $\mathsf{AVR}_{(M,g)}$ to 1 implies topologically closer manifold (M, g) to the Euclidean space $\mathbb{R}^n$, expressed in terms of the trivialization of higher homotopy groups of M, see Munn [78].

In the geometric context when (M, g) is a complete non-compact Riemannian manifold with $\mathsf{Ric}_{(M,g)} \geq 0$, a necessarily and sufficient condition to have the Sobolev embedding is the fact that $\mathsf{AVR}_{(M,g)} > 0$, see Coulhon and Saloff-Coste [30] and Hebey [39]. Moreover, a recent result of Balogh and Kristály [8] asserts that if $\mathsf{AVR}_{(M,g)} > 0$ then

$$\|u\|_{L^{2^*}} \leq \mathsf{AVR}_{(M,g)}^{-\frac{1}{n}} \mathsf{AT}_n \left(\int_M |\nabla_g u|^2 \mathrm{d}v_g \right)^{1/2}, \quad \forall u \in H^1(M),$$

where the constant $\mathsf{AVR}_{(M,g)}^{-\frac{1}{n}} \mathsf{AT}_n$ is sharp; here AT_n stands for the best Sobolev embedding constant in the Euclidean Sobolev inequality on $\mathbb{R}^n$, see Aubin [7] and Talenti [95]. In particular, $H^1(M)$ is continuously embedded into $L^q(M)$ for every $q \in [2, 2^*]$; more precisely, there exists $K_q^+ > 0$ such that

$$\|u\|_{L^q} \leq K_q^+ \|u\|_{H^1}, \quad \forall u \in H^1(M). \tag{4.8}$$

Given $x_0 \in M$ fixed, the *Hardy inequality* on (M, g) is verified as

$$\mathsf{AVR}_{(M,g)}^{\frac{2}{n}} \frac{(n-2)^2}{4} \int_M \frac{u^2(x)}{d_g^2(x_0, x)} \mathrm{d}v_g \leq \int_M |\nabla_g u|^2 \mathrm{d}v_g, \quad \forall u \in H^1(M), \tag{4.9}$$

see Kristály, Mester and Mezei [51]. The sharpness of the constant in (4.9) is not known unless we are in the classical Euclidean setting. However, if we assume that there exists a non-zero function realizing the equality in (4.9), from the proof in [51] (based on a Pólya-Szegő inequality involving the asymptotic volume ratio) we would obtain that $\mathsf{AVR}_{(M,g)} = 1$, i.e., (M, g) is isometric to the Euclidean space $\mathbb{R}^n$, which is a contradiction (since no non-zero extremal function exists in the Hardy inequality in $\mathbb{R}^n$).

Compact embeddings via isometric actions.

According to the previous subsections, the Sobolev space $H^1(M)$ is continuously embedded into $L^q(M)$, $q \in [2, 2^*]$, whenever (M, g) is either a Cartan-Hadamard manifold or a complete non-compact Riemannian manifold with $\mathsf{Ric}_{(M,g)} \geq 0$; however, none of them is compact, which represents an impediment to apply variational arguments on $H^1(M)$. To handle the lack of compactness, we use certain symmetrization à la Lions [61] by means of *isometries* of M.

Let (M, g) be an n-dimensional complete non-compact Riemanian manifold, $n \geq 3$, and – as in the Introduction – $\mathsf{Isom}_g(M)$ be the group of isometries of (M, g). Let G be a subgroup of $\mathsf{Isom}_g(M)$ and $\mathsf{Fix}_M(G)$ be the set of fixed points of the isometry group G in M, see (4.13). Let

$$H^1_G(M) = \{u \in H^1(M) : u \circ \sigma = u, \ \forall \sigma \in G\}$$

be the closed G-invariant subspace of $H^1(M)$. The consequences of the main results in the paper by Farkas, Kristály and Mester [35] state that if one of the following assumptions hold, i.e.,

- (M, g) is a Cartan-Hadamard manifold and $\mathsf{Fix}_M(G)$ is a singleton, or
- $\mathsf{Ric}_{(M,g)} \geq 0$, $\mathsf{AVR}_{(M,g)} > 0$ and G is coercive,

then the space $H^1_G(M)$ can be compactly embedded into $L^q(M)$ for every $q \in (2, 2^*)$.

4.3 Non-existence of solutions

We assume on the potential $\alpha : M \to \mathbb{R}$ that

$(\mathbf{H})_\alpha$: $\alpha \geq 0$ and $\alpha \in L^1(M) \cap L^\infty(M) \setminus \{0\}$.

For the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ we require

$(\mathbf{H})_0$: there exists $C_0 > 0$ such that

$$|\xi| \leq C_0|t|, \ \ \forall \xi \in \partial F(t), \ t \in \mathbb{R}.$$

The first result of the section reads as follows.

Theorem 4.3.1. ([54]) *Let (M, g) be an n-dimensional complete non-compact Riemannian manifold, $n \geq 3$, and assume that the potential $\alpha : M \to \mathbb{R}$ and* *the locally Lipschitz function* $F : \mathbb{R} \to \mathbb{R}$ *satisfy assumptions* $(\mathbf{H})_\alpha$ *and* $(\mathbf{H})_0$, *respectively. Assume in addition that one of the following curvature conditions holds*:

(i) $\mathbf{K} \leq -\kappa$ *for some* $\kappa \geq 0$, (M, g) *is simply connected and*

   (i1) either $\kappa = 0$, $\mu \leq \frac{(n-2)^2}{4}$ *and* $|\lambda| C_0 \|\alpha\|_{L^\infty} \leq 1$,

   (i2) or $\kappa > 0$, $\mu \leq \frac{(n-2)^2}{4}$ *and*

$$(n-2)^2(|\lambda| C_0 \|\alpha\|_{L^\infty} - 1) \leq (n-1)^2 \left(\frac{(n-2)^2}{4} - \mu_+ \right) \kappa,$$

   where $\mu_+ = \max(\mu, 0)$;

(ii) $\mathsf{Ric}_{(M,g)} \geq 0$, $\mu \leq \mathsf{AVR}_{(M,g)}^{\frac{2}{n}} \frac{(n-2)^2}{4}$ *and* $|\lambda| C_0 \|\alpha\|_{L^\infty} \leq 1$.

Then the differential inclusion (4.1) *has only the zero solution.*

Remark 4.3.1. The assertions in (i) show that there is a balance in the sense that when a stronger curvature restriction occurs, the analytic assumption can be relaxed.

Proof. Let $u \in H^1(M)$ be a solution of (4.1), i.e., relation (4.2) holds for every $v \in H^1(M)$. Let us choose $v = u$ in (4.2), obtaining

$$\int_M |\nabla_g u(x)|^2 \mathrm{d}v_g - \mu \int_M \frac{u^2(x)}{d_g^2(x_0, x)} \mathrm{d}v_g + \int_M u^2(x) \mathrm{d}v_g = \tag{4.10}$$
$$= \lambda \int_M \alpha(x) \xi_x u(x) \mathrm{d}v_g,$$

where $\xi_x \in \partial F(u(x))$ is a suitable selection, $x \in M$, such that $x \mapsto \alpha(x)\xi_x v(x)$ belongs to $L^1(M)$. By assumptions $(\mathbf{H})_\alpha$, $(\mathbf{H})_0$ and relation (4.10) we obtain that

$$\int_M |\nabla_g u(x)|^2 \mathrm{d}v_g - \mu \int_M \frac{u^2(x)}{d_g^2(x_0, x)} \mathrm{d}v_g + \int_M u^2(x) \mathrm{d}v_g \leq \tag{4.11}$$
$$\leq |\lambda| C_0 \|\alpha\|_{L^\infty} \int_M u^2(x) \mathrm{d}v_g.$$

Assume by contradiction that $u \neq 0$.

Proof of (i): $\mathbf{K} \leq -\kappa$ for some $\kappa \geq 0$.

Let $\kappa = 0$. If $\mu \leq \frac{(n-2)^2}{4}$, by the Hardy inequality (4.6) and relation (4.11), it turns out that

$$\int_M u^2(x)\mathrm{d}v_g < |\lambda|C_0\|\alpha\|_{L^\infty}\int_M u^2(x)\mathrm{d}v_g;$$

here we used the fact that equality cannot occur in the Hardy inequality (4.6) unless $u = 0$. Consequently, if $|\lambda|C_0\|\alpha\|_{L^\infty} \leq 1$, we arrive to a contradiction, i.e., we necessarily have $u = 0$, concluding the proof of (i1).

Let $\kappa > 0$. Assume first that $0 < \mu \leq \frac{(n-2)^2}{4}$. Then by the Hardy inequality (4.6) we have that

$$\mu\int_M \frac{u^2(x)}{d_g^2(x_0,x)}\mathrm{d}v_g < \frac{4\mu}{(n-2)^2}\int_M |\nabla_g u(x)|^2\mathrm{d}v_g,$$

where we again used the fact that no equality occurs in (4.6) for non-zero functions. Thus, by (4.11) it follows that

$$\left(1 - \frac{4\mu}{(n-2)^2}\right)\int_M |\nabla_g u(x)|^2\mathrm{d}v_g < (|\lambda|C_0\|\alpha\|_{L^\infty} - 1)\int_M u^2(x)\mathrm{d}v_g. \tag{4.12}$$

First, if $|\lambda|C_0\|\alpha\|_{L^\infty} \leq 1$, since $\mu \leq \frac{(n-2)^2}{4}$, relation (4.12) gives a contradiction. Second, if $|\lambda|C_0\|\alpha\|_{L^\infty} > 1$, by our assumption $(n-2)^2(|\lambda|C_0\|\alpha\|_{L^\infty} - 1) \leq (n-1)^2\left(\frac{(n-2)^2}{4} - \mu\right)\kappa$ we obtain that $\mu < \frac{(n-2)^2}{4}$; moreover, relation (4.12) and the assumption imply that

$$\int_M |\nabla_g u(x)|^2\mathrm{d}v_g < \frac{(n-1)^2}{4}\kappa\int_M u^2(x)\mathrm{d}v_g.$$

The latter inequality is in contradiction to McKean's spectral gap theorem, see (4.7). Therefore, we necessarily have $u = 0$, concluding the proof of (i2) for $\mu > 0$.

If $\mu \leq 0$, then our assumption reduces to $|\lambda|C_0\|\alpha\|_{L^\infty} - 1 \leq \frac{(n-1)^2}{4}\kappa$ and by (4.11) one has that

$$\int_M |\nabla_g u(x)|^2\mathrm{d}v_g \leq (|\lambda|C_0\|\alpha\|_{L^\infty} - 1)\int_M u^2(x)\mathrm{d}v_g.$$

Therefore, we obtain that

$$\int_M |\nabla_g u(x)|^2\mathrm{d}v_g \leq \frac{(n-1)^2}{4}\kappa\int_M u^2(x)\mathrm{d}v_g.$$

Since no equality occurs in McKean's spectral gap estimate (4.7) for any non-zero function $u \in H^1(M)$, we arrive to a contradiction. In conclusion, we necessarily have that $u = 0$, which ends the proof of (i2) also for $\mu \leq 0$.

Proof of (ii): $\mathsf{Ric}_{(M,g)} \geq 0$. Since $\mu \leq \mathsf{AVR}_{(M,g)}^{\frac{2}{n}} \frac{(n-2)^2}{4}$, the Hardy inequality from (4.9) (together with the fact that no non-zero function realizes the equality) and relation (4.11) imply that

$$\int_M u^2(x)\mathrm{d}v_g < |\lambda| C_0 \|\alpha\|_{L^\infty} \int_M u^2(x)\mathrm{d}v_g.$$

Consequently, if $|\lambda| C_0 \|\alpha\|_{L^\infty} \leq 1$, we arrive to a contradiction; thus $u = 0$. This ends the proof of (ii). □

4.4 Sub-quadratic case

In order to produce existence or even multiplicity of non-zero solutions to (4.1), we require on the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ the following assumptions:

$(\mathbf{H})_1$: $\lim_{t\to 0} \frac{\max\{|\xi| : \xi \in \partial F(t)\}}{t} = 0$;

$(\mathbf{H})_2$: $\lim_{|t|\to\infty} \frac{\max\{|\xi| : \xi \in \partial F(t)\}}{t} = 0$;

$(\mathbf{H})_3$: $F(0) = 0$ and there exist $t_0^- < 0 < t_0^+$ such that $F(t_0^\pm) > 0$.

Note that $(\mathbf{H})_1$ and $(\mathbf{H})_2$ mean that the function $t \mapsto \max\{|\xi| : \xi \in \partial F(t)\}$ is *superlinear at the origin* and *sublinear at infinity*, respectively; in particular, by using Lebourg's mean value theorem, we observe that F is *sub-quadratic at infinity*. In addition, by the upper semicontinuity of the set-valued function $t \mapsto \partial F(t)$ and conditions $(\mathbf{H})_1$ and $(\mathbf{H})_2$, it turns out that the hypothesis $(\mathbf{H})_0$ is also valid for a suitably large value of $C_0 > 0$; in particular, Theorem 4.3.1 can be applied (under the assumptions $(\mathbf{H})_1$ and $(\mathbf{H})_2$), and for sufficiently 'small' values of $|\lambda|$ only the zero solution exists for the differential inclusion (4.1). However, for 'large' values of $\lambda > 0$, we can guarantee the existence of multiple non-zero solutions for (4.1) by requiring further assumptions on the behavior of the *isometric group* of the Riemannian manifold (M, g). In fact, the latter assumptions are destined to balance the lack of compactness of the Riemannian manifolds we are dealing with.

To state the second result of the section, we denote by $\mathsf{Isom}_g(M)$ the *group of isometries* of the complete Riemannian manifold (M, g). Let G be a subgroup of $\mathsf{Isom}_g(M)$ and

$$\mathsf{Fix}_M(G) = \{x \in M : \sigma(x) = x, \forall \sigma \in G\} \tag{4.13}$$

be the set of *fixed points* of the isometry group G in M. The *G-orbit* of a point $x \in M$ is $\mathcal{O}_G^x = \{\sigma(x) : \sigma \in G\}$. The continuous action of the group G on M is *coercive* if for every $t > 0$ the set $\mathcal{O}_t := \{x \in M : \mathrm{diam}(\mathcal{O}_G^x) \leq t\}$ is bounded, see Skrzypczak and Tintarev [93, 94]; here $\mathrm{diam}(S)$ denotes the diameter of $S \subset M$. A function $u : M \to \mathbb{R}$ is *G-invariant* if $u(x) = u(\sigma(x))$ for every $x \in M$ and $\sigma \in G$.

Theorem 4.4.1. ([54]) *Let (M, g) be an n-dimensional complete non-compact Riemannian manifold, $n \geq 3$, and G be a compact connected subgroup of $\mathsf{Isom}_g(M)$ such that $\mathsf{Fix}_M(G) = \{x_0\}$ for the same $x_0 \in M$ as in problem (4.1). Let $\alpha : M \to \mathbb{R}$ be a potential satisfying $(\mathbf{H})_\alpha$ which depends only on $d_g(x_0, \cdot)$ and the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ satisfying assumptions $(\mathbf{H})_i$, $i \in \{1, 2, 3\}$, respectively. In addition, we assume that one of the following curvature assumptions holds*:

(i) (M, g) is of Cartan-Hadamard-type and $0 \leq \mu < \frac{(n-2)^2}{4}$;

(ii) $\mathsf{Ric}_{(M,g)} \geq 0$, $\mathsf{AVR}_{(M,g)} > 0$, $0 \leq \mu < \mathsf{AVR}_{(M,g)}^{\frac{2}{n}} \frac{(n-2)^2}{4}$ and G is coercive.

Then there exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$ the differential inclusion (4.1) has at least four non-zero G-invariant solutions in $H^1(M)$.

Proof. Step 1. (Truncation and nonsmooth energy functional) Since $F(0) = 0$ (by $(\mathbf{H})_3$), we consider the truncated locally Lipschitz function $F^+(t) = F(t_+)$, $t \in \mathbb{R}$. The *energy functional* $\mathcal{E}^+ : H^1(M) \to \mathbb{R}$ to the slightly modified problem (4.1), considering F^+ instead of F, is defined as

$$\mathcal{E}^+(u) = \frac{1}{2}\mathcal{N}_\mu(u) - \lambda\mathcal{F}^+(u),$$

where

$$\mathcal{N}_\mu(u) = \int_M |\nabla_g u(x)|^2 \mathrm{d}v_g - \mu \int_M \frac{u^2(x)}{d_g^2(x_0, x)} \mathrm{d}v_g + \int_M u^2(x) \mathrm{d}v_g$$

and

$$\mathcal{F}^+(u) = \int_M \alpha(x) F^+(u(x)) \mathrm{d}v_g.$$

On one hand, it is clear that $\mathcal{N}_\mu$ is of class C^1 on $H^1(M)$ and due to the Hardy inequalities (i.e., (4.6) and (4.9)), for the corresponding values of μ from the statement of the theorem, $\mathcal{N}_\mu^{1/2}$ turns out to be equivalent to the usual norm $\|\cdot\|_{H^1}$ on $H^1(M)$, i.e.,

$$c_\mu \|u\|_{H^1}^2 \leq \mathcal{N}_\mu(u) \leq \|u\|_{H^1}^2, \quad \forall u \in H^1(M), \tag{4.14}$$

where

$$0 < c_\mu = \begin{cases} 1 - \frac{4\mu}{(n-2)^2}, & \text{in the case (i);} \\ 1 - \mathsf{AVR}_{(M,g)}^{-\frac{2}{n}} \frac{4\mu}{(n-2)^2} & \text{in the case (ii).} \end{cases}$$

On the other hand, one can prove that $\mathcal{F}^+$ is well-defined and locally Lipschitz on $H^1(M)$. To see this, we first observe that by $(\mathbf{H})_1$ and $(\mathbf{H})_2$, for every $\epsilon > 0$ there exists $\delta_\epsilon \in (0,1)$ such that

$$|\xi| \leq \epsilon t, \ \forall \xi \in \partial F^+(t), \ \forall 0 < t < \delta_\epsilon \ \& \ t > \delta_\epsilon^{-1}. \tag{4.15}$$

Fix $\epsilon_0 > 0$. Since ∂F^+ is an upper semicontinuous set-valued map with non-empty compact values, see Proposition 1.1.1/(a), we also have for some $K_{\epsilon_0} > 0$ that $|\xi| \leq K_{\epsilon_0} t$ for every $\xi \in \partial F^+(t)$ and $t \in [\delta_{\epsilon_0}, \delta_{\epsilon_0}^{-1}]$. The latter fact with (4.15) implies that

$$|\xi| \leq C_{\epsilon_0} t, \ \forall \xi \in \partial F^+(t), \ \forall t > 0,$$

where $C_{\epsilon_0} = \max\{\epsilon_0, K_{\epsilon_0}\}$. Now, let $u \in H^1(M)$ and U_u be any open bounded neighborhood of u in $H^1(M)$, i.e., for some $K > 0$ we have $\|w\|_{H^1} \leq K$ for every $w \in U_u$. If $u_1, u_2 \in U_u$, then by Lebourg's mean value theorem, see Proposition 1.1.3, for a.e. $x \in M$ there exist $\gamma \in [0,1]$ and $\xi_x^\gamma \in \partial F^+((1-\gamma)u_1(x) + \gamma u_2(x))$ such that

$$\begin{aligned} |F^+(u_1(x)) - F^+(u_2(x))| &= |\xi_x^\gamma||u_1(x) - u_2(x)| \\ &\leq C_{\epsilon_0}(|u_1(x)| + |u_2(x)|)|u_1(x) - u_2(x)|. \end{aligned}$$

By Hölder's inequality and the trivial embedding $H^1(M) \subset L^2(M)$, we have that

$$\begin{aligned} |\mathcal{F}^+(u_1) - \mathcal{F}^+(u_2)| &\leq \int_M \alpha(x)|F^+(u_1(x)) - F^+(u_2(x))| \mathrm{d}v_g \\ &\leq 2C_{\epsilon_0}\|\alpha\|_{L^\infty} K \|u_1 - u_2\|_{H^1}, \end{aligned}$$

which means that $\mathcal{F}^+$ is Lipschitz on U_u. The fact that $\mathcal{F}^+$ is well-defined follows in a similar way. Having these properties, a similar argument as in Clarke [28, Section 2.7] (see also Costea, Kristály and Varga [29]) shows that for every closed subspace W of $H^1(M)$ we have that

$$\partial(\mathcal{F}^+|_W)(u) \subseteq \int_M \alpha(x)\partial F^+(u(x)) \mathrm{d}v_g, \quad \forall u \in W;$$

here, $\mathcal{F}^+|_W$ is the restriction of the functional $\mathcal{F}^+$ to the subspace W and the latter inclusion has the following interpretation: to every $\xi \in \partial(\mathcal{F}^+|_W)(u)$ there exists a measurable selection $x \mapsto \xi_x \in \partial F^+(u(x))$ such that the map $x \mapsto \alpha(x)\xi_x w(x)$ belongs to $L^1(M)$ for every $w \in W$ and

$$\langle \xi, w \rangle = \int_M \alpha(x)\xi_x w(x) \mathrm{d}v_g.$$

By using Fatou's lemma, Lebourg's mean value theorem, Lebesgue's dominated convergence theorem, and a careful limiting argument, see e.g. Kristály [49] in the Euclidean setting, it turns out that

$$(\mathcal{F}^+|_W)^0(u; w) \leq \int_M \alpha(x)(F^+)^0(u(x); w(x)) \mathrm{d}v_g, \quad \forall u, w \in W. \tag{4.16}$$

Let $u \in H^1(M)$ be a critical point of $\mathcal{E}^+$, i.e., $0 \in \partial\mathcal{E}^+(u)$. We are going to prove that u is a non-negative solution to the differential inclusion (4.1). First, by Proposition 1.1.1/(g)&(i), we have that

$$\frac{1}{2}\mathcal{N}_\mu'(u) \in \lambda\partial\mathcal{F}^+(u),$$

i.e., for every test-function $w \in H^1(M)$ one has

$$\int_M \nabla_g u(x) \nabla_g w(x) \mathrm{d}v_g - \mu \int_M \frac{u(x)w(x)}{d_g^2(x_0, x)} \mathrm{d}v_g + \int_M u(x)w(x) \mathrm{d}v_g =$$

$$= \lambda \int_M \alpha(x)\xi_x w(x) \mathrm{d}v_g,$$

with the above interpretation for the right hand side. Let $u_- = \min(0, u)$ be the non-positive part of u and note that it belongs to the space $H^1(M)$, see Hebey [39, Proposition 2.5]. If we put $v = u_-$ into the latter relation, we obtain that $\xi_x u_-(x) = 0$ for a.e. $x \in M$ since $\xi_x \in \partial F^+(u(x))$ (thus $\xi_x = 0$ whenever $u(x) < 0$). In consequence, $\mathcal{N}_\mu(u_-) = 0$, thus $u_- = 0$, i.e., $u \geq 0$. In particular, $\xi_x \in \partial F^+(u(x)) = \partial F(u(x))$, therefore the latter relation is precisely (4.2), which means that $u \in H^1(M)$ is a non-negative solution of (4.1).

Step 2. (Isometry actions) Let G be a compact connected subgroup of $\mathsf{Isom}_g(M)$ with the property that $\mathsf{Fix}_M(G) = \{x_0\}$ for the same $x_0 \in M$ as in problem (4.1). The action of G on $H^1(M)$, i.e., $G \times H^1(M) \to H^1(M)$, is defined by

$$(\sigma u)(x) = u(\sigma^{-1}(x)), \ \forall \sigma \in G, \ u \in H^1(M), \ x \in M. \tag{4.17}$$

It is standard to prove that G acts continuously and linearly on $H^1(M)$. For instance, if $\sigma_1, \sigma_2 \in G$, it turns out that for every $u \in H^1(M)$ and $\sigma \in G$ we have

$$(\sigma_1 \circ \sigma_2)u(x) = u((\sigma_1 \circ \sigma_2)^{-1}(x)) = u(\sigma_2^{-1}(\sigma_1^{-1}(x))) =$$

$$= (\sigma_2 u)(\sigma_1^{-1}(x)) = (\sigma_1(\sigma_2 u))(x).$$

Moreover, since G contains isometries of (M, g), the functionals

$$u \mapsto \int_M |\nabla_g u(x)|^2 \mathrm{d}v_g$$

and

$$u \mapsto \int_M u^2(x) \mathrm{d}v_g$$

are both G-invariant; in particular, $\|\sigma u\|_{H^1} = \|u\|_{H^1}$ for every $\sigma \in G$ and $u \in H^1(M)$, i.e., G acts isometrically on $H^1(M)$.

Since $\mathsf{Fix}_M(G) = \{x_0\}$, it turns out that for every $\sigma \in G$ and $y \in M$, we have $d_g(x_0, \sigma(y)) = d_g(\sigma(x_0), \sigma(y)) = d_g(x_0, y)$; therefore, a change of variables implies that

$$\begin{aligned}\int_M \frac{(\sigma u)^2(x)}{d_g^2(x_0, x)} \mathrm{d}v_g(x) &= \int_M \frac{u^2(\sigma^{-1}(x))}{d_g^2(x_0, x)} \mathrm{d}v_g(x) = \int_M \frac{u^2(y)}{d_g^2(x_0, \sigma(y))} \mathrm{d}v_g(\sigma(y)) \\ &= \int_M \frac{u^2(y)}{d_g^2(x_0, y)} \mathrm{d}v_g(y).\end{aligned}$$

In particular, the functional $u \mapsto \mathcal{N}_\mu(u)$ is G-invariant on $H^1(M)$.

Furthermore, since $\alpha : M \to \mathbb{R}$ depends only on $d_g(x_0, \cdot)$, it is also G-invariant, and one can prove by a change of variables that for every $\sigma \in G$ and $u \in H^1(M)$,

$$\begin{aligned}\mathcal{F}^+(\sigma u) &= \int_M \alpha(x) F^+((\sigma u)(x)) \mathrm{d}v_g(x) = \int_M \alpha(x) F^+(u(\sigma^{-1}(x))) \mathrm{d}v_g(x) \\ &= \int_M \alpha(\sigma(y)) F^+(u(y)) \mathrm{d}v_g(\sigma(y)) = \int_M \alpha(y) F^+(u(y)) \mathrm{d}v_g(y) \\ &= \mathcal{F}^+(u),\end{aligned}$$

i.e., $\mathcal{F}^+$ is G-invariant on $H^1(M)$. In conclusion, the energy functional $\mathcal{E}^+ = \mathcal{N}_\mu/2 - \lambda \mathcal{F}^+$ is G-invariant on $H^1(M)$.

The set of fixed points of G over $H^1(M)$, i.e., $\mathsf{Fix}_{H^1(M)}(G)$, is nothing but the closed subset of G-invariant functions of $H^1(M)$. Now, according to the principle of symmetric criticality, see Proposition 1.4.2, if $u \in \mathsf{Fix}_{H^1(M)}(G) =: H^1_G(M)$ is a critical point of the restricted energy functional $\mathcal{E}^+_G := \mathcal{E}^+|_{H^1_G(M)}$ then u is also a critical point of the initial energy functional $\mathcal{E}^+$.

Step 3. (Spectral gap estimate for $\mathcal{F}^+/\mathcal{N}_\mu$ on $H^1_G(M)$.) We are going to prove that for every admissible μ from the statement of the theorem, one has

$$0 < \sup_{u \in H^1_G(M)\setminus\{0\}} \frac{\mathcal{F}^+(u)}{\mathcal{N}_\mu(u)} < +\infty. \tag{4.18}$$

Let $q \in (2, 2^*)$ and fix arbitrarily $\epsilon > 0$ together with the number $\delta_\epsilon > 0$ appearing in (4.15). By the boundedness of the function $t \mapsto \frac{\max|\partial F^+(t)|}{t^{q-1}}$ on $[\delta_\epsilon, \delta_\epsilon^{-1}]$ and due to (4.15), there exists $l_\epsilon > 0$ such that

$$0 \le |\xi| \le \epsilon t + l_\epsilon t^{q-1},\ \forall t \ge 0,\ \xi \in \partial F^+(t) = \partial F(t). \tag{4.19}$$

Note that we have

$$0 \le |F^+(t)| \le \epsilon t^2 + l_\epsilon |t|^q,\ \forall t \in \mathbb{R}. \tag{4.20}$$

Indeed, on one hand, by definition $F^+(t) = F(0) = 0$ for every $t \le 0$, thus the latter relation trivially holds. On the other hand, for $t \ge 0$, the estimate (4.19) and Lebourg's mean value theorem immediately imply the required estimate.

Consequently, the estimate (4.20) shows that for every $u \in H^1_G(M)$ we have

$$\begin{aligned} 0 \le |\mathcal{F}^+(u)| &= \left| \int_M \alpha(x) F^+(u(x)) \mathrm{d}v_g \right| \le \int_M \alpha(x) \left|F^+(u(x))\right| \mathrm{d}v_g \\ &\le \|\alpha\|_{L^\infty} \left(\epsilon \|u\|^2_{H^1} + l_\epsilon (K^\pm_q)^q \|u\|^q_{H^1}\right), \end{aligned}$$

where $K^\pm_q > 0$ are the embedding constants from (4.5) and (4.8), respectively. Accordingly, for every $u \in H^1_G(M) \setminus \{0\}$ one has that

$$0 \le \frac{|\mathcal{F}^+(u)|}{\mathcal{N}_\mu(u)} \le c_\mu^{-1} \|\alpha\|_{L^\infty} \left(\epsilon + l_\epsilon (K^\pm_q)^q \|u\|^{q-2}_{H^1}\right),$$

where $c_\mu > 0$ is the constant from (4.14). Due to the fact that $q > 2$ and $\epsilon > 0$ is arbitrarily fixed, it turns out that

$$\frac{\mathcal{F}^+(u)}{\mathcal{N}_\mu(u)} \to 0 \ \text{ as } \ \|u\|_{H^1} \to 0, u \in H^1_G(M). \tag{4.21}$$

The counterpart of (4.21) at 'infinity' reads as

$$\frac{\mathcal{F}^+(u)}{\mathcal{N}_\mu(u)} \to 0 \ \text{ as } \ \|u\|_{H^1} \to +\infty, u \in H^1_G(M). \tag{4.22}$$

Indeed, combining the boundedness of $t \mapsto \frac{\max|\partial F^+(t)|}{t^{1/2}}$ on $[\delta_\epsilon, \delta_\epsilon^{-1}]$ with the estimate (4.15), one can find $L_\epsilon > 0$ such that

$$0 \leq |\xi| \leq \epsilon t + L_\epsilon t^{1/2}, \ \forall t \geq 0, \ \xi \in \partial F^+(t) = \partial F(t).$$

Due to hypothesis $(\mathbf{H})_\alpha$, one has that $\alpha \in L^4(M)$. Then using Lebourg's mean value theorem and Hölder's inequality, we can proceed as before, obtaining

$$0 \leq |\mathcal{F}^+(u)| \leq \int_M \alpha(x)\left|F^+(u(x))\right| \mathrm{d}v_g \leq \epsilon \|\alpha\|_{L^\infty} \|u\|_{H^1}^2 + L_\epsilon \|\alpha\|_{L^4} \|u\|_{H^1}^{\frac{3}{2}}. \quad (4.23)$$

Consequently, for every $u \in H_G^1(M) \setminus \{0\}$ we have

$$0 \leq \frac{|\mathcal{F}^+(u)|}{\mathcal{N}_\mu(u)} \leq c_\mu^{-1}\left(\epsilon\|\alpha\|_{L^\infty} + L_\epsilon \|\alpha\|_{L^4} \|u\|_{H^1}^{-\frac{1}{2}}\right).$$

This estimate together with the arbitrariness of $\epsilon > 0$ immediately imply (4.22).

In particular, (4.21) and (4.22) imply that the second inequality in (4.18) holds. In order to check the first inequality in (4.18), we recall by $(\mathbf{H})_3$ that there exists $t_0^+ > 0$ such that $F(t_0^+) > 0$. Moreover, by $(\mathbf{H})_\alpha$, since $\alpha \neq 0$ and it depends only on $d_g(x_0, \cdot)$, there exists an open x_0-centered annulus on M with radii $0 \leq r < R$, i.e. $A_{x_0}(r, R) = \{x \in M : r < d_g(x_0, x) < R\}$, such that $\operatorname{essinf}_{A_{x_0}(r,R)} \alpha = \alpha_0 > 0$. For sufficiently small $\epsilon > 0$ (e.g. $\epsilon < (R-r)/3$), we consider the function $w_\epsilon : M \to \mathbb{R}$ defined by

$$w_\epsilon(x) = \begin{cases} \frac{t_0^+}{\epsilon}(d_g(x_0, x) - r) & \text{if} \ \ d_g(x_0, x) \in (r, r+\epsilon), \\ t_0^+ & \text{if} \ \ d_g(x_0, x) \in [r+\epsilon, R-\epsilon], \\ \frac{t_0^+}{\epsilon}(R - d_g(x_0, x)) & \text{if} \ \ d_g(x_0, x) \in (R-\epsilon, R), \\ 0 & \text{if} \ \ x \notin A_{x_0}(r, R). \end{cases}$$

Note that $w_\epsilon \in H_G^1(M)$ and $w_\epsilon \geq 0$. Moreover,

$$\begin{aligned} \mathcal{F}^+(w_\epsilon) &= \int_M \alpha(x) F(w_\epsilon(x)) \mathrm{d}v_g = \int_{A_{x_0}(r,R)} \alpha(x) F(w_\epsilon(x)) \mathrm{d}v_g \\ &\geq \alpha_0 F(t_0^+) V_g(A_{x_0}(r+\epsilon, R-\epsilon)) \\ &\quad - \|\alpha\|_{L^\infty} \max_{t \in [0, t_0^+]} |F(t)| (V_g(A_{x_0}(r, r+\epsilon)) + V_g(A_{x_0}(R-\epsilon, R))). \end{aligned}$$

By continuity reason, there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$,

$$\mathcal{F}^+(w_\epsilon) \geq \alpha_0 F(t_0^+) V_g(A_{x_0}(r, R))/2 > 0.$$

On the other hand, by (4.14) and the eikonal equation ($|\nabla_g d_g(x_0,\cdot)| = 1$ a.e. on M) we have the estimate

$$\mathcal{N}_\mu(w_\epsilon) \leq \|w_\epsilon\|^2_{H^1} \leq (t_0^+)^2(1+\epsilon^{-2})V_g(A_{x_0}(r,R)) < +\infty.$$

Consequently, it turns out that

$$0 < \frac{\mathcal{F}^+(w_{\epsilon_0/2})}{\mathcal{N}_\mu(w_{\epsilon_0/2})} \leq \sup_{u\in H^1_G(M)\setminus\{0\}} \frac{\mathcal{F}^+(u)}{\mathcal{N}_\mu(u)},$$

which shows the validity of the first inequality in (4.18).

Step 4. (Analytic properties of $\mathcal{E}^+_G$) We shall prove three basic properties of $\mathcal{E}^+_G$ on $H^1_G(M)$, namely, coercivity and boundedness from below, as well as the validity of the nonsmooth Palais-Smale condition.

Let $\lambda > 0$ be arbitrarily fixed and μ be in the admissible range (cf. the statement of the theorem). First, we observe by (4.14) and (4.23) that for every $u \in H^1_G(M)$ we have

$$\begin{aligned}\mathcal{E}^+_G(u) &= \frac{1}{2}\mathcal{N}_\mu(u) - \lambda\mathcal{F}^+(u)\\ &\geq \left(\frac{c_\mu}{2} - \epsilon\lambda\|\alpha\|_{L^\infty}\right)\|u\|^2_{H^1} - \lambda L_\epsilon\|\alpha\|_{L^4}\|u\|^{\frac{3}{2}}_{H^1}.\end{aligned}$$

In particular, for sufficiently small $\epsilon > 0$, e.g. $0 < \epsilon < \frac{c_\mu}{2}\lambda^{-1}\|\alpha\|^{-1}_{L^\infty}$, it follows that $\mathcal{E}^+_G$ is bounded from below and coercive, i.e., $\mathcal{E}^+_G(u) \to +\infty$ whenever $\|u\|_{H^1} \to +\infty$.

Let $\{u_k\}_k \subset H^1_G(M)$ be a Palais-Smale sequence for $\mathcal{E}^+_G$, i.e., for some $M > 0$, one has $|\mathcal{E}^+_G(u_k)| \leq M$ and $m(u_k) \to 0$ as $k \to \infty$, where $m(u) = \min\{\|\xi\|_* : \xi \in \partial\mathcal{E}^+_G(u)\}$. We want to prove that, up to a subsequence, $\{u_k\}_k$ strongly converges to some element in $H^1_G(M)$. Being $\mathcal{E}^+_G$ coercive, the sequence $\{u_k\}_k \subset H^1_G(M)$ is bounded in $H^1_G(M)$. Therefore, due to the fact that $H^1_G(M)$ can be compactly embedded into $L^q(M)$, $q \in (2,2^*)$, see §4.2, up to a subsequence, one has that

$$u_k \to u \text{ weakly in } H^1_G(M); \tag{4.24}$$

$$u_k \to u \text{ strongly in } L^q(M),\ q \in (2,2^*). \tag{4.25}$$

By Proposition 1.1.1/(g) and the definition of $\mathcal{E}^+_G$ we have that

$$(\mathcal{E}^+_G)^0(u_k; u-u_k) = \frac{1}{2}\langle\mathcal{N}'_\mu(u_k), u-u_k\rangle + \lambda(-\mathcal{F}^+)^0(u_k; u-u_k);$$

$$(\mathcal{E}_G^+)^0(u;u_k-u)=\frac{1}{2}\langle\mathcal{N}_\mu'(u),u_k-u\rangle+\lambda(-\mathcal{F}^+)^0(u;u_k-u).$$

Note that

$$\frac{1}{2}\langle\mathcal{N}_\mu'(u_k),u-u_k\rangle+\frac{1}{2}\langle\mathcal{N}_\mu'(u),u_k-u\rangle=-\mathcal{N}_\mu(u_k-u).$$

By adding the above relations and using Proposition 1.1.1/(e), it turns out that

$$\begin{aligned}\mathcal{N}_\mu(u_k-u) &= \lambda\left((\mathcal{F}^+)^0(u_k;-u+u_k)+(\mathcal{F}^+)^0(u;-u_k+u)\right)\\ &\quad -(\mathcal{E}_G^+)^0(u_k;u-u_k)-(\mathcal{E}_G^+)^0(u;u_k-u). \end{aligned}\tag{4.26}$$

In the sequel, we are going to estimate the terms in the right hand side of (4.26). First, by inequality (4.16), Proposition 1.1.1/(b) and (4.19) together with the fact that $\partial F^+(t)=\{0\}$ for $t\le 0$, we have

$$\begin{aligned} I_k^1 &:= (\mathcal{F}^+)^0(u_k;-u+u_k)+(\mathcal{F}^+)^0(u;-u_k+u)\\ &\le \int_M \alpha(x)\left[(F^+)^0(u_k(x);u_k(x)-u(x))+(F^+)^0(u(x);u(x)-u_k(x))\right]\mathrm{d}v_g\\ &= \int_M \alpha(x)\left[\max\{\xi_k(u_k(x)-u(x)):\xi_k\in\partial F^+(u_k(x))\}\right.\\ &\qquad\left.+\max\{\xi(u(x)-u_k(x)):\xi\in\partial F^+(u(x))\}\right]\mathrm{d}v_g\\ &\le \|\alpha\|_{L^\infty}\int_M[\epsilon(|u_k(x)|+|u(x)|)+l_\epsilon(|u_k(x)|^{q-1}+|u(x)|^{q-1})]|u(x)\\ &\qquad -u_k(x)|\mathrm{d}v_g\\ &\le 2\epsilon\|\alpha\|_{L^\infty}(\|u_k\|_{H^1}^2+\|u\|_{H^1}^2)+l_\epsilon\|\alpha\|_{L^\infty}(\|u_k\|_{L^q}^{q-1}+\|u\|_{L^q}^{q-1})\|u_k-u\|_{L^q}. \end{aligned}$$

By the arbitrariness of $\epsilon>0$ and the convergence property (4.25), the latter estimate shows that

$$\limsup_{k\to\infty} I_k^1\le 0. \tag{4.27}$$

Let $\xi_k\in\partial\mathcal{E}_G^+(u_k)$ be such that $m(u_k)=\|\xi_k\|_*$. Thus, we have that

$$I_k^2:=(\mathcal{E}_G^+)^0(u_k;u-u_k)\ge\langle\xi_k,u-u_k\rangle\ge-\|\xi_k\|_*\|u-u_k\|_{H^1}.$$

Consequently, since $m(u_k)=\|\xi_k\|_*\to 0$ as $k\to\infty$, we have that

$$\liminf_{k\to\infty} I_k^2\ge 0. \tag{4.28}$$

Moreover, for every $\xi \in \partial\mathcal{E}_G^+(u)$, we also have that $I_k^3 := (\mathcal{E}_G^+)^0(u; u_k - u) \geq \langle \xi, u_k - u\rangle$; thus, by the weak convergence property (4.24) we have that

$$\liminf_{k\to\infty} I_k^3 \geq 0. \tag{4.29}$$

By the estimates (4.27)-(4.29) and relation (4.26) we have that

$$0 \leq \limsup_{k\to\infty} \mathcal{N}_\mu(u_k - u) \leq \limsup_{k\to\infty} I_k^1 - \liminf_{k\to\infty} I_k^2 - \liminf_{k\to\infty} I_k^3 \leq 0,$$

i.e., $\mathcal{N}_\mu(u_k - u) \to 0$ as $k \to \infty$. Due to (4.14), it turns out that $u_k \to u$ strongly in the H^1-norm as $k \to \infty$, which is the desired property.

Step 5. (Local minimum point for $\mathcal{E}_G^+$: first solution) Let

$$\lambda_0^+ := \inf_{\substack{u\in H_G^1(M)\\ \mathcal{F}^+(u)>0}} \frac{\mathcal{N}_\mu(u)}{2\mathcal{F}^+(u)}.$$

Due to Step 3, see (4.18), one has that $0 < \lambda_0^+ < \infty$.

If we fix $\lambda > \lambda_0^+$, one can find $\tilde{w}_\lambda \in H_G^1(M)$ with $\mathcal{F}^+(\tilde{w}_\lambda) > 0$ such that

$$\lambda > \frac{\mathcal{N}_\mu(\tilde{w}_\lambda)}{2\mathcal{F}^+(\tilde{w}_\lambda)} \geq \lambda_0^+.$$

Thus, by the latter inequality we have

$$C_\lambda^1 := \inf_{H_G^1(M)} \mathcal{E}_G^+ \leq \mathcal{E}_G^+(\tilde{w}_\lambda) = \frac{1}{2}\mathcal{N}_\mu(\tilde{w}_\lambda) - \lambda\mathcal{F}^+(\tilde{w}_\lambda) < 0.$$

Due to the fact that $\mathcal{E}_G^+$ is bounded from below and verifies the nonsmooth Palais-Smale condition (see Step 4), C_λ^1 is a critical value of $\mathcal{E}_G^+$, see Chang [26, Theorem 3.5], i.e., there exists $u_\lambda^1 \in H_G^1(M)$ such that $\mathcal{E}_G^+(u_\lambda^1) = C_\lambda^1 < 0$ and $0 \in \partial\mathcal{E}_G^+(u_\lambda^1)$. In particular, $u_\lambda^1 \neq 0$ (since $\mathcal{E}_G^+(u_\lambda^1) < 0 = \mathcal{E}_G^+(0)$), and by the principle of symmetric criticality, u_λ^1 is a critical point also for the initial energy functional (see Step 2), i.e., $0 \in \partial\mathcal{E}^+(u_\lambda^1)$. According to (the final part of) Step 1, $u_\lambda^1 \in H_G^1(M)$ is a non-negative solution to the differential inclusion (4.1).

Step 6. (Minimax-type critical point for $\mathcal{E}_G^+$: second solution) Let $\lambda > \lambda_0^+$. Due to (4.20), for sufficiently small $\epsilon > 0$ (e.g., $\frac{c_\mu}{4} > \epsilon\lambda\|\alpha\|_{L^\infty}$) and for every $u \in H_G^1(M)$ one has that

$$\begin{aligned} \mathcal{E}_G^+(u) &= \frac{1}{2}\mathcal{N}_\mu(u) - \lambda\mathcal{F}^+(u) \geq \\ &\geq \left(\frac{c_\mu}{2} - \epsilon\lambda\|\alpha\|_{L^\infty}\right)\|u\|_{H^1}^2 - \lambda\|\alpha\|_{L^\infty} l_\epsilon (K_q^\pm)^q \|u\|_{H^1}^q, \end{aligned}$$

where $q \in (2, 2^*)$ and $K_q^{\pm} > 0$ are the embedding constants from (4.5) and (4.8), respectively. Let

$$\rho_\lambda = \min \left\{ \|\tilde{w}_\lambda\|_{H^1}, \left(\frac{\frac{c_\mu}{2} - \epsilon\lambda\|\alpha\|_{L^\infty}}{2\lambda\|\alpha\|_{L^\infty} l_\epsilon (K_q^{\pm})^q} \right)^{\frac{1}{q-2}} \right\}.$$

The choice of $\rho_\lambda > 0$ and Step 4 show that

$$\inf_{\|u\|_{H^1} = \rho_\lambda; u \in H_G^1(M)} \mathcal{E}_G^+(u) \geq \frac{1}{2} \left(\frac{c_\mu}{2} - \epsilon\lambda\|\alpha\|_{L^\infty} \right) \rho_\lambda^2 > 0 = \mathcal{E}_G^+(0) > \mathcal{E}_G^+(\tilde{w}_\lambda).$$

The latter estimate shows that the functional $\mathcal{E}_G^+$ has the mountain pass geometry. On account of Step 4, since $\mathcal{E}_G^+$ satisfies the nonsmooth Palais-Smale condition, we can apply the mountain pass theorem for locally Lipschitz functions, see e.g. Kourogenis and Papageorgiou [46] or Kristály, Motreanu and Varga [57, Theorem 2], guaranteeing the existence of $u_\lambda^2 \in H_G^1(M)$ with the properties $0 \in \partial\mathcal{E}_G^+(u_\lambda^2)$ and

$$\mathcal{E}_G^+(u_\lambda^2) = C_\lambda^2 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{E}_G^+(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1]; H_G^1(M)) : \gamma(0) = 0,\ \gamma(1) = \tilde{w}_\lambda\}.$$

Since

$$C_\lambda^2 \geq \inf_{\|u\|_{H^1} = \rho_\lambda; u \in H_G^1(M)} \mathcal{E}_G^+(u) > 0,$$

it is clear that $0 \neq u_\lambda^2 \neq u_\lambda^1$. The rest of the proof is similar to the end of Step 5, which shows that $u_\lambda^2 \in H_G^1(M)$ is a non-negative solution to the differential inclusion (4.1), different from u_λ^1.

Step 7. (Repetition of Steps 1-6 for $\mathcal{E}^-$) Let $F^-(t) = F(t_-)$, $t \in \mathbb{R}$, where $t_- = \min(t, 0)$. The locally Lipschitz energy functional $\mathcal{E}^- : H^1(M) \to \mathbb{R}$ is defined as

$$\mathcal{E}^-(u) = \frac{1}{2}\mathcal{N}_\mu(u) - \lambda\mathcal{F}^-(u),$$

where

$$\mathcal{F}^-(u) = \int_M \alpha(x) F^-(u(x)) \mathrm{d}v_g.$$

One can show that if $u \in H^1(M)$ is a critical point of $\mathcal{E}^-$, i.e., $0 \in \partial\mathcal{E}^-(u)$, then it is a non-positive solution of (4.1), cf. Step 1.

By using the isometry action (4.17), one can prove that $\mathcal{F}^-$ is G-invariant on $H^1_G(M)$, and if $u \in \mathsf{Fix}_{H^1(M)}(G) =: H^1_G(M)$ is a critical point of $\mathcal{E}^-_G := \mathcal{E}^-|_{H^1_G(M)}$ then $0 \in \partial\mathcal{E}^-(u)$ as well, cf. Step 2.

Instead of the spectral gap estimate (4.18), one can prove

$$0 < \sup_{u \in H^1_G(M)\setminus\{0\}} \frac{\mathcal{F}^-(u)}{\mathcal{N}_\mu(u)} < +\infty,$$

cf. Step 3, and similar analytic properties are valid for $\mathcal{E}^-_G$ as in Step 4 (i.e, coercivity, boundedness from below, and the validity of the nonsmooth Palais-Smale condition). Here, we use again the compact embedding results from §4.2.

Finally, if

$$\lambda_0^- := \inf_{\substack{u \in H^1_G(M) \\ \mathcal{F}^-(u)>0}} \frac{\mathcal{N}_\mu(u)}{2\mathcal{F}^-(u)},$$

by the previous part we know that $0 < \lambda_0^- < \infty$ and similarly to Steps 5 and 6, we can guarantee for every $\lambda > \lambda_0^-$ a local minimum point $u^3_\lambda \in H^1_G(M)$ of $\mathcal{E}^-_G$ with $\mathcal{E}^-_G(u^3_\lambda) < 0$ and a minimax-type point $u^4_\lambda \in H^1_G(M)$ of $\mathcal{E}^-_G$ with $\mathcal{E}^-_G(u^4_\lambda) > 0$; in particular, $u^3_\lambda \neq u^4_\lambda$ and none of them is trivial. These elements are G-invariant, non-positive solutions to the differential inclusion (4.1).

If we choose $\lambda_0 = \max(\lambda_0^+, \lambda_0^-)$, we can apply the above arguments, providing four different, non-zero G-invariant solutions to the differential inclusion (4.1) for every $\lambda > \lambda_0$, two of them being non-negative and the other two being non-positive. The proof is complete. $\square$

Remark 4.4.1. Examples of Riemannian manifolds with the above curvature restrictions and isometric actions are presented in Kristály [48] and Farkas, Kristály and Mester [35] in the setting (i), and Balogh and Kristály [8] in the framework (ii).

4.5 Super-quadratic case

As we already noticed, assumptions $(\mathbf{H})_1$ and $(\mathbf{H})_2$ imply that F is sub-quadratic at infinity. In the sequel, we establish a counterpart of Theorem 4.4.1 whenever F is *super-quadratic at infinity*. More precisely, we assume that the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:

$(\mathbf{H})_4$: $F(0) = 0$ and there exist $\nu > 2$ and $C > 0$ such that

$$2F(t) + F^0(t; -t) \leq -C|t|^\nu, \quad \forall t \in \mathbb{R};$$

$(\mathbf{H})_5$: there is $q \in \left(2, 2 + \frac{4}{n}\right)$ such that $\max\{|\xi| : \xi \in \partial F(t)\} = O(|t|^{q-1})$ as $|t| \to \infty$.
Here, $F^0(t; s)$ is the generalized directional derivative of F at the point $t \in \mathbb{R}$ and direction $s \in \mathbb{R}$. Note that by $(\mathbf{H})_1$ and $(\mathbf{H})_4$, F is super-quadratic at infinity.

Theorem 4.5.1. ([54])*Let (M, g) be an n-dimensional complete non-compact Riemannian manifold, $n \geq 3$, and G be a compact connected subgroup of $\mathsf{Isom}_g(M)$ such that $\mathsf{Fix}_M(G) = \{x_0\}$ for the same $x_0 \in M$ as in problem (4.1). Let $\alpha \in L^\infty(M)$ be a potential which depends only on $d_g(x_0, \cdot)$ and $\operatorname{essinf}_{x\in M}\alpha(x) = \alpha_0 > 0$, while the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ satisfies the assumptions $(\mathbf{H})_i$, $i \in \{1, 4, 5\}$, respectively. If one of the curvature assumptions* (i) *or* (ii) *holds from Theorem 4.4.1, then for every $\lambda > 0$ the differential inclusion* (4.1) *has at least a non-zero G-invariant solution in $H^1(M)$. In addition, if F is an even function,* (4.1) *has infinitely many distinct G-invariant solutions in $H^1(M)$.*

Proof. We divide again the proof into some steps.

Step 1. (Functional setting) In view of the previous section, this part is standard. Indeed, the energy functional $\mathcal{E} : H^1(M) \to \mathbb{R}$ is defined as

$$\mathcal{E}(u) = \frac{1}{2}\mathcal{N}_\mu(u) - \lambda\mathcal{F}(u),$$

where

$$\mathcal{F}(u) = \int_M \alpha(x)F(u(x))\mathrm{d}v_g.$$

Note that by $(\mathbf{H})_1$ and $(\mathbf{H})_5$, for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|\xi| \leq \epsilon|t| + C_\epsilon|t|^{q-1}, \ \forall t \in \mathbb{R}, \ \xi \in \partial F(t). \tag{4.30}$$

Consequently, one has

$$|F(t)| \leq \epsilon t^2 + C_\epsilon|t|^q, \ \forall t \in \mathbb{R}. \tag{4.31}$$

Since $2 < q < 2 + \frac{4}{n} < 2^*$, by using Lebourg's mean value theorem 1.1.3 and (4.30), one can prove that $\mathcal{F}$ is well-defined and locally Lipschitz on $H^1(M)$. It is now standard to show that any critical point $u \in H^1(M)$ of $\mathcal{E}$ is a solution of (4.1).

Step 2. (Isometry actions) By using the action (4.17), one can prove in a similar way as in §4.4 that $\mathcal{E}$ is G-invariant on $H^1(M)$. Moreover, the principle of symmetric criticality (Proposition 1.4.2) implies that if $u \in \mathsf{Fix}_{H^1(M)}(G) =: H^1_G(M)$ is a critical point of $\mathcal{E}_G := \mathcal{E}|_{H^1_G(M)}$ then u is also a critical point of $\mathcal{E}$.

Step 3. (Super-quadraticity of F at infinity) We are going to prove that

$$F(t) \geq \frac{C}{\nu - 2}|t|^\nu, \ \forall t \in \mathbb{R}, \tag{4.32}$$

where $\nu > 2$ and $C > 0$ come from hypothesis $(\mathbf{H})_4$; this means in particular that F is super-quadratic at infinity (as $\nu > 2$). To do this, let $h : \mathbb{R} \to \mathbb{R}$ be defined by

$$h(t) = t^{-2}F(t) - \frac{C}{\nu - 2}|t|^{\nu-2}, \ t \neq 0,$$

and $h(0) = 0$. Note that h is well-defined and locally Lipschitz (indeed, by $(\mathbf{H})_1$ and $F(0) = 0$ we have that $F(t) = o(t^2)$ as $t \to 0$). By Proposition 1.1.1/(g), one has

$$\partial h(t) = -2t^{-3}F(t) + t^{-2}\partial F(t) - C|t|^{\nu-4}t, \ \forall t \in \mathbb{R} \setminus \{0\}.$$

We shall prove (4.32) for $t \geq 0$, the case $t \leq 0$ being similar. Let $t > 0$; then by Lebourg's mean value theorem, there exist $\theta \in (0, t)$ and $\xi_h \in \partial h(\theta)$ such that $h(t) = h(t) - h(0) = \xi_h t$. In turn, there exists $\xi_F \in \partial F(\theta)$ such that $\xi_h = -2\theta^{-3}F(\theta) + \theta^{-2}\xi_F - C\theta^{\nu-3}$ and by $(\mathbf{H})_4$ we have that

$$\begin{aligned} h(t) &= \xi_h t = (-2\theta^{-3}F(\theta) + \theta^{-2}\xi_F - C\theta^{\nu-3})t \\ &= -\theta^{-3}(2F(\theta) + \xi_F(-\theta) + C\theta^\nu)t \\ &\geq -\theta^{-3}(2F(\theta) + F^0(\theta; -\theta) + C\theta^\nu)t \\ &\geq 0, \end{aligned}$$

which concludes the proof. In particular, combining (4.31) with (4.32), we necessarily have that

$$\nu \leq q. \tag{4.33}$$

Step 4. (Nonsmooth Cerami compactness condition for $\mathcal{E}_G$) Let $\{u_k\}_k \subset H^1_G(M)$ be a Cerami sequence for $\mathcal{E}_G$, i.e., for some $M > 0$, one has $|\mathcal{E}_G(u_k)| \leq M$ and $(1 + \|u_k\|_{H^1})m(u_k) \to 0$ as $k \to \infty$, where $m(u) = \min\{\|\xi\|_* : \xi \in \partial\mathcal{E}_G(u)\}$. Our objective is to prove that, up to a subsequence, $\{u_k\}_k$ strongly converges to some element in $H^1_G(M)$.

We first prove that $\{u_k\}_k$ is bounded in $L^\nu(M)$. For every $k \in \mathbb{N}$, let $\xi_k \in \partial\mathcal{E}_G(u_k)$ be such that $\|\xi_k\|_* = m(u_k)$. We observe that

$$\mathcal{E}_G^0(u_k; u_k) \geq \langle \xi_k, u_k \rangle \geq -\|\xi_k\|_* \|u_k\|_{H^1} \geq -(1 + \|u_k\|_{H^1}) m(u_k).$$

Since $(1 + \|u_k\|_{H^1}) m(u_k) \to 0$ as $k \to \infty$, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$ one has that $\mathcal{E}_G^0(u_k; u_k) \geq -1$. Consequently, Proposition 1.1.1/(g), inequality (4.16) (which is also valid due to (4.30)) and hypothesis $(\mathbf{H})_4$ imply for every $k \in \mathbb{N}$ that

$$\begin{aligned}
2M + 1 &\geq 2\mathcal{E}_G(u_k) - \mathcal{E}_G^0(u_k; u_k) \\
&= \mathcal{N}_\mu(u_k) - 2\lambda\mathcal{F}(u_k) - \frac{1}{2}\langle \mathcal{N}_\mu'(u_k); u_k \rangle - \lambda(-\mathcal{F})^0(u_k; u_k) \\
&= -\lambda\left(2\mathcal{F}(u_k) + \mathcal{F}^0(u_k; -u_k)\right) \\
&\geq -\lambda \int_M \alpha(x)\left(2F(u_k(x)) + F^0(u_k(x); -u_k(x))\right) \mathrm{d}v_g \\
&\geq \lambda C \int_M \alpha(x)|u_k(x)|^\nu \mathrm{d}v_g.
\end{aligned}$$

Since $\alpha \in L^\infty(M)$ and $\mathrm{essinf}_{x\in M}\alpha(x) = \alpha_0 > 0$, the latter estimate implies that

$$2M + 1 \geq \lambda C \alpha_0 \|u_k\|_{L^\nu}^\nu, \ \forall k \in \mathbb{N}.$$

Consequently, $\{u_k\}_k$ is bounded in $L^\nu(M)$.

Now, we prove that $\{u_k\}_k$ is bounded in $H_G^1(M)$. By (4.31), for every small $\epsilon > 0$ there exists $\tilde{C}_\epsilon > 0$ such that for every $k \in \mathbb{N}$,

$$\begin{aligned}
M \geq \mathcal{E}_G(u_k) &= \frac{1}{2}\mathcal{N}_\mu(u_k) - \lambda\mathcal{F}(u_k) \\
&\geq \left(\frac{c_\mu}{2} - \epsilon\lambda\|\alpha\|_{L^\infty}\right)\|u_k\|_{H^1}^2 - \lambda\tilde{C}_\epsilon\|\alpha\|_{L^\infty}\|u_k\|_{L^q}^q.
\end{aligned}$$

In particular, if $\frac{c_\mu}{4} > \epsilon\lambda\|\alpha\|_{L^\infty}$, then there exists $M_\epsilon > 0$ and $\overline{C}_\epsilon > 0$ such that

$$\|u_k\|_{H^1}^2 \leq M_\epsilon + \overline{C}_\epsilon\|u_k\|_{L^q}^q, \ \forall k \in \mathbb{N}. \tag{4.34}$$

On account of (4.33), we distinguish two cases:

a) $\nu = q$. Since $\{u_k\}_k$ is bounded in $L^\nu(M)$ and $\nu = q$, by (4.34) we also have that $\{u_k\}_k$ is bounded in $H_G^1(M)$.

b) $\nu < q$. Let $\eta \in (0,1)$ be such that $\frac{1}{q} = \frac{1-\eta}{\nu} + \frac{\eta}{2^*}$. By (4.34) and a standard interpolation inequality we have that

$$\begin{aligned}\|u_k\|_{H^1}^2 &\le M_\epsilon + \overline{C}_\epsilon \|u_k\|_{L^q}^q \le M_\epsilon + \overline{C}_\epsilon \|u_k\|_{L^\nu}^{(1-\eta)q} \|u_k\|_{L^{2^*}}^{\eta q} \\ &\le M_\epsilon + \overline{C}_\epsilon (K_q^\pm)^{\eta q} \|u_k\|_{L^\nu}^{(1-\eta)q} \|u_k\|_{H^1}^{\eta q},\end{aligned} \tag{4.35}$$

where $K_q^\pm > 0$ are the embedding constants from (4.5) and (4.8), respectively. Since $q < 2 + \frac{4}{n}$, we have that $\nu > 2 > \frac{n(q-2)}{2}$. We observe that $\nu > \frac{n(q-2)}{2}$ together with $\frac{1}{q} = \frac{1-\eta}{\nu} + \frac{\eta}{2^*}$ is equivalent to $\eta q < 2$. The latter inequality and (4.35) imply that $\{u_k\}_k$ is bounded in $H_G^1(M)$.

Now, we can proceed as in Step 4, see §4.4; in this way we conclude that $\{u_k\}_k$ strongly converges (up to a subsequence) to some element in $H_G^1(M)$.

Step 5. (Existence/multiplicity of critical points for $\mathcal{E}_G$) Under the assumptions of the theorem, one can prove as above that $\mathcal{E}_G$ has the mountain pass geometry. By Step 4 and on account of the mountain pass theorem for locally Lipschitz functions, see e.g. Kourogenis and Papageorgiou [46], we conclude the existence of a non-zero critical point for $\mathcal{E}_G$. When F is even, we may apply the nonsmooth fountain theorem involving the Cerami compactness condition, see e.g. Kristály [49], guaranteeing the existence of a sequence of critical points for the functional $\mathcal{E}_G$. All these points are G-invariant solutions to the differential inclusion (4.1), which concludes the proof. □

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ISBN 978-606-37-2784-9