

Ábel Lőrinci

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Introduction to Mathematical Logic and Set Theory

Presa Universitară Clujeană

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Preface

Logic is the study and use of valid reasoning. Logic has two aspects: *informal*, that is, the study of arguments in natural language, and *formal*, that is, of the study of inferences from the point of view of form, or in other words, the study of abstract rules of deduction. The earliest studies of formal logic are due to Aristotle. When we use abstract symbols in the formal study of inferences, we speak of symbolic logic; usually this is divided into propositional logic and predicate logic.

Mathematical logic is part of Mathematics and of Logic. Its role is to rigorously define the idea of the truth value of a statement and to explore the application of formal (symbolic) logic methods in different branches of mathematics. Also, mathematical logic deals with the application of mathematical methods and techniques to the study of formal logic.

The development of mathematical logic was strongly motivated by the study of the foundations of mathematics, a study begun in the 19th century, and has important applications in philosophy or linguistics, but also in more recent fields such as computer science (logic programming, artificial intelligence, etc.).

Nowadays, mathematical logic is divided into four subdomains, each focusing on distinct aspects, but obviously the demarcation lines are not strict:

- Set theory, which studies abstract collections of objects and the correspondences between them, playing an important role in the foundations of mathematics.
- Proof theory, which essentially means the formal analysis of mathematical proofs.
- Model theory, which is the formal study of mathematical structures, closely related to abstract algebra.

- Recursion theory (or computability theory), which studies the effective computability of functions defined on the set of natural numbers, having an important role for the foundations of computer science.

In this introductory course dedicated to first year students from the Faculty of Mathematics and Computer Science we will touch on a small part of the mentioned subjects, often in an informal manner. Also included are some basic topics of Algebra, Arithmetic and Combinatorics closely related to the above, but which usually go beyond the framework of Mathematical Logic.

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