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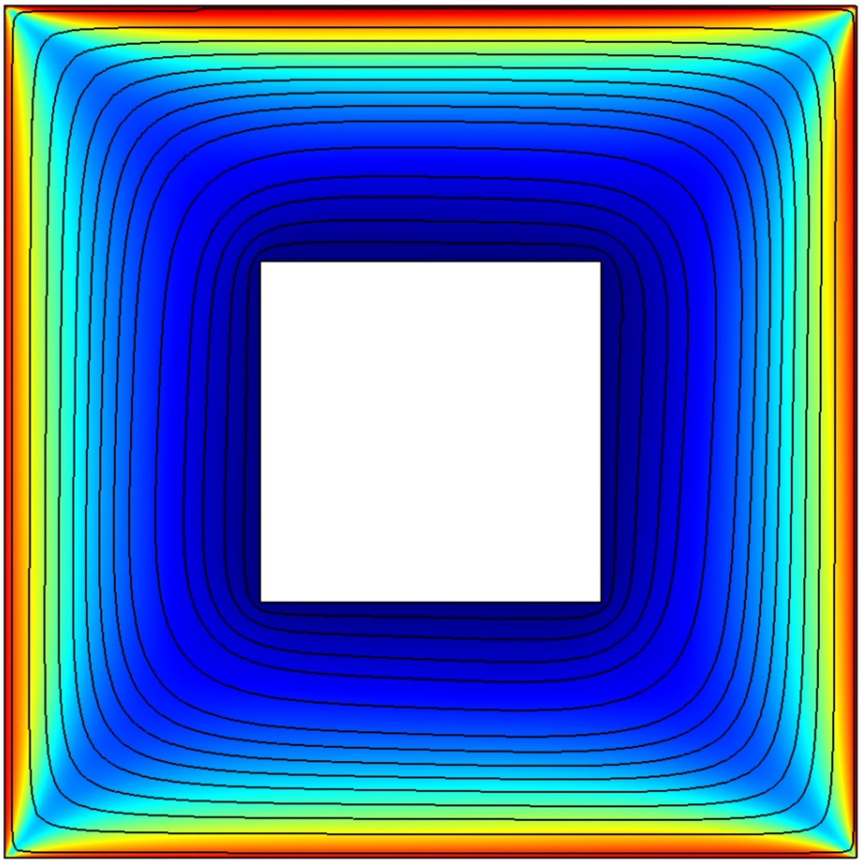

CONTRIBUTIONS TO THE THEORY OF ELLIPTIC BOUNDARY VALUE PROBLEMS AND THEIR APPLICATIONS IN FLUID MECHANICS

Presa Universitară Clujeană

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Contributions to the Theory of Elliptic Boundary Value Problems and Their Applications in Fluid Mechanics

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Contents

Preface

The purpose of this monograph is the treatment of important elliptic boundary value problems for systems of partial differential equations (PDEs) that arise in Fluid Mechanics by using the methods of potential theory and a fixed point theorem. We have treated various boundary value problems as the Dirichlet, Robin-Dirichlet, transmission, Robin-transmission in the linear case as well as the non-linear case. We have provided suggestive numerical examples for a practical problem with multiple applications, while the objective is to complete the theoretical study, which is presented in the first three chapters.

In what follows, let $\mathsf{D} \subset \mathbb{R}^n$, $n \geq 2$ be a bounded Lipschitz domain and we denote its boundary by Γ. Let us consider $\mathcal{P}$, a matrix-valued function, whose entries are essentially bounded functions. We introduce the generalized Brinkman system by

$$\Delta \mathbf{v} - \mathcal{P}\mathbf{v} - \nabla p = \mathbf{f}, \quad \text{div } \mathbf{u} = 0, \text{ in } \mathsf{D}, \tag{0.0.1}$$

where the pair $(\mathbf{v}, p)$ represents the velocity and pressure fields of the considered fluid flow and $\mathbf{f}$ is a given, external force which acts on the fluid flow. In the special case $\mathcal{P} = \alpha\mathbb{I}$, where $\alpha > 0$ is a given constant, the system (0.0.1) becomes the classical Brinkman system,

$$\Delta \mathbf{v} - \alpha\mathbf{v} - \nabla p = \mathbf{f}, \quad \text{div } \mathbf{v} = 0, \text{ in } \mathsf{D}. \tag{0.0.2}$$

If we consider $\mathcal{P} = 0$ in the system (0.0.1), we obtain the well-known Stokes system,

$$\Delta \mathbf{v} - \nabla p = \mathbf{f}, \quad \text{div } \mathbf{v} = 0, \text{ in } \mathsf{D}. \tag{0.0.3}$$

Now, let us also consider the generalized Darcy-Forchheimer-Brinkman system

$$\Delta \mathbf{v} - \mathcal{P}\mathbf{v} - k|\mathbf{v}|\mathbf{v} - \beta(\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla p = \mathbf{f}, \quad \text{div } \mathbf{v} = 0, \text{ in } \mathsf{D}, \tag{0.0.4}$$

where k, β are positive, essentially bounded functions on D. In the special case $\mathcal{P} = \alpha\mathbb{I}$, where $\alpha > 0$ is a given constant and $k, \beta > 0$ are given constants, the system (0.0.4) reduces to the classical Darcy-Forchheimer-Brinkman system

$$\Delta \mathbf{v} - \alpha\mathbf{v} - k|\mathbf{v}|\mathbf{v} - \beta(\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla p = \mathbf{f}, \quad \text{div } \mathbf{v} = 0, \text{ in } \mathsf{D}. \tag{0.0.5}$$

Let us mention the fact that the Darcy-Forchheimer-Brinkman system is used in problems in which the inertia of the fluid is not negligible (see, e.g., [115]).

Finally, for $\mathcal{P} = 0$, $k = 0$ and $\beta > 0$ a given constant, the system (0.0.4) becomes the Navier-Stokes system

$$\Delta \mathbf{v} - \beta(\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla p = \mathbf{f}, \quad \text{div } \mathbf{v} = 0, \text{ in } \mathsf{D}. \tag{0.0.6}$$

For additional details regarding the Navier-Stokes equations we refer the reader to [54], [134], [130], [137].

In this work, we will concern ourselves with the coupling of these aforementioned PDE systems. In these transmission problems, we will deal specifically with two types of configurations. The geometry of these configurations is thoroughly specified in Chapter 1. Moreover, in these problems, we consider the following boundary conditions

$$\mathrm{Tr}_{\mathsf{D}_+}\mathbf{v}_+ - \mathrm{Tr}_{\mathsf{D}_-}\mathbf{v}_- = \mathbf{g},\ \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{v}_+, p_+, \mathbf{f}_+) - \mathbf{t}_{\mathsf{D}_-}(\mathbf{v}_-, p_-, \mathbf{f}_-) + \mathbf{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{v}_+ = \mathbf{h},\ \text{on } \Gamma, \qquad (0.0.7)$$

which will be referred as *transmission conditions*, where the trace operator Tr, the conormal derivative operator $\mathbf{t}$ and the matrix-valued function $\mathbf{L}$ are described in the latter.

We have considered the generalized Brinkman system (0.0.1), which we obtained by substituting the constant $\alpha > 0$ (in the system (0.0.2)) with a matrix-valued function $\mathcal{P}$ whose entries are essentially bounded functions. In this case, by this aforementioned generalization, we move towards the concept of an anisotropic Brinkman system. The purpose that we have in mind is that of investigating fluid flow in porous media, in the case that our porous medium has variable porosity or permeability. For additional details, see, e.g., [77], [78], [79].

Let us provide some insight for the practical motivation for the study of transmission problems. Note that, transmission problems appear as a mathematical model for the study of environmental problems where free air flow is interacting with evaporation from soils and or the transvascular exchange between blood flow in vessels and the surrounding tissue (for additional details [71] and the references therein). Also, transmission problems are used to model the production of electric energy in proton exchange membrane fuel cells (for additional details, see [75] and the references therein). In addition, we mention that employing transmission problems one can study the geophysical flow of water which pass through porous rocks or porous soil (for additional details, see [76] and the references therein). The anisotropic Stokes system is used to describe certain processes (for example, processes in physics, engineering, industry) in which the flow of immiscible fluids or the flow of nonhomogeneous fluids with density dependent viscosity are involved (cf. [28], see also [79]).

In order to study such problems, many techniques can be employed. For linear boundary value problems, we emphasize two approaches, namely, layer potential methods and variational methods, respectively. Also, for the study of nonlinear boundary value problems, one can employ either fixed point theory or topological degree theory.

In the latter, we shall provide a historical overview of the scientific literature that concerns boundary problems.

Let us explore previous works that are concerned with the study of boundary problems in Euclidean setting. We begin with the work of Verchota [142], who established the invertibility property of the classical layer potentials for Laplace's equation, on $L^2(\partial\Omega)$ and subspaces of $L^2(\partial\Omega)$, in the case of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Dahlberg, Kenig and Verchota [34] have obtained well-posedness results for the Dirichlet and traction boundary problems for the Lame system in an arbitary Lipschitz domain in $\mathbb{R}^n$ with L^2-boundary data. They have also investigated the 'slip condition' for the Stokes equations, for boundary data belonging to L^2 boundary spaces accompanied by optimal estimates (see also [35]). Shen [132] has studied constant coefficient elliptic systems in bounded Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, and has obtained L^p resolvent estimates. Amrouche, Girault and Girore [13] have solved the Dirichlet and Neumann boundary value problems for the Laplacian in exterior domains of $\mathbb{R}^n$, $n \geq 2$, while working in weighted Sobolev spaces. Fabes, Mendez and Mitrea [48] have used boundary integral methods for the investigation of inhomogeneous boundary problems for the Laplacian in arbitrary Lipschitz domains with data in Besov spaces. Escauriaza and Mitrea [46] have established existence and uniqueness results for the transmission problem for the Laplacian in the setting of complementary Lipschitz domains in $\mathbb{R}^n$ for $n \geq 2$, while the boundary data was considered in Lebesgue and Hardy spaces.

In what follows, let us name a few papers in which the studies on the Stokes system (0.0.3) were conducted. Nevertheless, the list of publication where this subject is discussed is much more longer. For example, Power and Miranda [121] have investigated the slow viscous flow of an unbounded fluid past a single solid particle. Consequently, they have provided the formulation of integral equations of the second kind for general exterior Stokes flow in dimension $n = 3$. The work of Fabes, Kenig and Verchota [47] is an important contribution of the field of layer potential theory. The authors have used layer potentials in order to obtain existence and uniqueness results for the Dirichlet problem for the Stokes system in an arbitrary Lipschitz domain in $\mathbb{R}^n$, in the case of boundary data in L^2. Dauge [37] has studied the H^s-regularity of solutions of the Stokes system in domains with corners. Girault and Sequeira [56] have investigated the Dirichlet problem for the Stokes system in exterior Lipschitz domains in $\mathbb{R}^n$, $n = 2, 3$. They have used a variational technique in order to obtain well-posedness results in the setting of weighted Sobolev spaces. Power [120] has extended the method used in [121] to that of the Stokes flow problem in multiple cylinders, in the two-dimensional setting of bounded and unbounded domains. Hence, the author has used the double layer potential in order to obtain uniquely-solvable second order integral equations of Fredholm type for the analyzed flow in this particular setting. Shen [131] has considered the L^p Dirichlet problem for the Stokes system in bounded Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, and has provided well-posedness results for such a problem. Alliot and Amrouche [10] have devoted a study to the Stokes problem in $\mathbb{R}^n$, $n \geq 2$, in weighted Sobolev spaces. This approach allows the authors discuss the decay or growth of solutions at infinity. Alliot and Amrouche [12] have investigated the nonhomogeneous Dirichlet problem for the Stokes system in an exterior, connected, Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$ in weighted Sobolev spaces, in order to account the behavior of the solution at infinity.

Russo and Tartaglione [127] have provided existence and uniqueness results for the Robin type problem associated to the Stokes system and also for the Navier-Stokes system, in a bounded Lipschitz domain in Euclidean setting. They have used layer potential methods in order to show the well-posedness result for the Robin problem for the Stokes system and the well-posedness result for the Robin problem for the Navier-Stokes system was established by employing the result from the linear case together with a fixed point theorem. Mitrea and Mitrea [105] have investigated regularity properties for Green functions associated to second order, strongly elliptic, divergence-form differential operators in bounded Lipschitz domains. Their approach yields an existence and uniquenss result for the Stokes system with Dirichlet condition in a bounded Lipschitz domain of $\mathbb{R}^n$, $n = 2, 3$. Mitrea, Wright and Monniaux [110] have studied the Neumann problem for the Stokes and Navier-Stokes systems, respectively, in Lipschitz domains in $\mathbb{R}^n$, for $n \geq 2$ in the linear case and $n = 3$ in the nonlinear case, and have provided existence, regularity and uniqueness results. Medkova [96] has used a layer potential method in order to determine a weak solution for the Neumann problem for the Stokes system in a bounded Lipschitz domain in Euclidean setting. Tartaglione [135] has studied boundary problems for the Stokes equations in bounded and respectively unbounded domains of $\mathbb{R}^n$, which are sufficiently regular (of $C^{k-1,1}$ class, $k \geq 2$) and has well-posedness of such problems with boundary data in Sobolev spaces. Băcuţă, Hassel, Hsiao and Sayas [17] have proposed a fully discrete method, which is based on an integral equation (which, in turn is based on a single-layer potential representation of the velocity), for the exterior Dirichlet boundary problem for the Stokes equations in two or three dimensions. They have also discussed the single layer potential associated to the Brinkman equations in Lipschitz domains.

The linear, elliptic Brinkman system (0.0.2) was also investigated by a great deal of researchers. McCracken [94] has studied the Dirichlet problem for the Stokes resolvent system on the resolvent problem for the Stokes system on half-space of in $\mathbb{R}^3$ and provided the well-posedness of the Dirichlet problem in some L^p spaces. Deuring [38] has constructed solutions in L^p-spaces for the Dirichlet problem for the resolvent Stokes system in the exterior of a bounded domain with $\mathcal{C}^2$ boundary belonging to $\mathbb{R}^3$. Farwig and Sohr [49] have shown that the Dirichlet problem for the Stokes resolvent system admits a unique solution in weighted Sobolev spaces, in the setting of an exterior $\mathcal{C}^{1,1}$ domain of $\mathbb{R}^n$, $n \geq 2$. Shen [133] has obtained L^p resolvent estimates for the Stokes system in the setting of Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, by employing layer potential methods in his study. Kohr, Lanza de Cristoforis and Wendland [72] have investigated Robin type boundary problems for the Brinkman system and the Darcy-Forchheimer-Brinkman system in Lipschitz domains in Euclidean setting. They treat also mixed Dirichlet-Robin and transmission boundary value problems for the Brinkman systems in the setting of bounded creased Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, as well as the Navier problem for the Brinkman system in a bounded Lipschitz domain of $\mathbb{R}^3$. Kohr, Lanza de Cristoforis and Wendland [74] have obtained an existence result for the Poisson problem for a semilinear Brinkman system on a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$ with Dirichlet or Robin conditions on the boundary. These results were obtained by making use of the well-posedness results obtained in the linear case and the Schauder fixed point theorem. Medkova [99] has investigated the Dirichlet problem for the resolvent Stokes system in the setting of bounded and unbounded domains with compact Lyapunov boundary. The author has provided a well-posedness result in the bounded domain case and an existence result in the unbounded domain case. Moreover, an existence result for the Dirichlet problem for the Darcy-Forchheimer-Brinkman has been obtained in [99].

Now, let us focus on previous studies that aim to investigate boundary value problems for nonlinear equations, such as the Navier-Stokes equations (0.0.6) or the Darcy-Forchheimer-Brinkman equations (0.0.5). We mention the contribution of Alliot and Amrouche [11], who have studied regularity properties of the weak solutions of the steady-state Navier-Stokes system in exterior domains of $\mathbb{R}^3$ and, in particular, have obtained a decomposition result for the pressure and certain sufficient conditions for the velocity to vanish at infinity. Russo and Tartaglione [128] have studied the Robin problem for the Oseen and Navier-Stokes systems in an C^1-class, exterior domain of $\mathbb{R}^3$. They have used a layer potential approach in order to show the existence of a solution for the Robin problem for the Oseen system and for existence result for the Robin problem for the Navier-Stokes system, they have employed a fixed point method. Amrouche and Nguyen [14] have investigated the exterior, homogeneous, Dirichlet problem for the Navier-Stokes system in an exterior Lipschitz domain in $\mathbb{R}^3$, in the setting of weighted Sobolev spaces. Russo and Tartaglione [129] have used a variational approach and fixed point theorems to obtain existence results for the Navier problem for the Navier-Stokes system in bounded Lipschitz domains and exterior Lipschitz domains in $\mathbb{R}^3$. Kohr, Lanza de Cristoforis and Wendland [74] obtained an existence and uniqueness result for the Dirichlet problem for the semilinear Darcy-Forchheimer-Brinkman system in the case of small boundary data.

Researchers have also devoted themselves to the investigation of boundary problems in the setting of manifolds. We highlight some works in the later. Let us begin by noting that Mitrea, Mitrea, Mitrea and Taylor [107] have treated boundary problems for the Hodge-Laplacian in the setting of Riemannian manifolds. In their analysis, the authors employ potential theory techniques. Also, Dindos and Mitrea [39] employed the method of boundary integral equations to obtain the well-posedness of the Poisson problem for the Stokes system in Lipschitz domains in the setting of smooth, compact Riemannian manifolds. In [82], Kohr, Pintea and Wendland have used a layer potential approach in order to investigate a certain type of general pseudodifferential matrix operators defined on Lipschitz domains in compact Riemannian manifolds. They apply their findings to Dirichlet-transmission problems for general Brinkman operators. The authors have proposed a useful approach, by which, well-posedness results of certain boundary value problems can be derived by using well-posedness results for transmission-type problems. Kohr, Mikhailov and Wendland [76] have investigated transmission-type boundary value problems for the Navier-Stokes and Darcy-Forchheimer-Brinkman systems in complementary Lipschitz domains in a compact Riemannian manifold of dimension m, $m = 2, 3$. Their approach is based on layer potential techniques combined with fixed point arguments.

Let us point out some papers that deal with transmission-type problems. Mitrea and Taylor [112] have developed layer potential methods for partial differential equations on Lipschitz domains in smooth, connected and compact Riemannian manifolds of dimension $m \geq 3$. The authors use these techniques in order to solve Dirichlet and Neumann boundary value problems for the Laplace-Beltrami operator whose boundary data belong to Besov spaces. Mitrea and Taylor [113] have provided well-posedness results for the Dirichlet problem for the Stokes system and for the initial boundary value problem for the Navier-Stokes system with Dirichlet boundary condition. They have obtained these results in Lipschitz domains of compact Riemannian manifolds. Kohr, Lanza de Cristoforis and Wendland [73] have investigated the existence of a solution for the nonlinear Neumann-transmission problem for the Stokes and Brinkman systems in Lipschitz domains in Euclidean setting. They have used a layer potential approach and the Leray-Schauder degree theory in order to achieve this. Fericean and Wendland [51] have established a well-posedness result for a Dirichlet-transmission problem for the Stokes and Brinkman systems in Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$. This result was obtain by employing the methods of layer potential theory.

Fericean, Groşan, Kohr and Wendland [50] used a layer potential technique to prove an existence result for an interface problem of Robin-transmission type for the Stokes and Brinkman systems in Lipschitz domains of $\mathbb{R}^n$. An application which involves the exterior three-dimensional Stokes flow is also considered. Medkova [97] has employed the method of integral equations in order to provide well-posedness results of transmission problems, Robin-transmission problem and Dirichlet-transmission problem for the Brinkman system in the setting of complementary Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$. The author in [98] has used the method of integral equations to find well-posedness results for transmission problems associated to the Stokes equations in complementary domains of $\mathbb{R}^3$ with Lipschitz boundaries. Kohr, Lanza de Cristoforis, Mikhailov and Wendland [71] have obtained well-posedness results for a transmission problem for the Darcy-Forchheimer-Brinkman and Stokes system in complementary Lipschitz domains in $\mathbb{R}^3$. Their approach proposes a layer potential technique combined with a fixed point theorem. Kohr, Lanza and Wendland [75] have investigated a Robin-transmission problem for the Darcy-Forchheimer-Brinkman and Navier-Stokes systems in two adjacent and bounded Lipschitz domains in $\mathbb{R}^n$, $n = 2, 3$. They have formulated both linear and non-linear transmission and boundary conditions for this problem. Their analysis employs layer potential techniques and fixed point arguments. The authors in [75] have studied a Robin-transmission boundary value problem for the Darcy-Forchheimer-Brinkman and Navier-Stokes systems in two adjacent Lipschitz domains in $\mathbb{R}^n$, $n = 2, 3$, with linear transmission and linear Robin boundary conditions. In addition, they have also investigated this particular transmission problem, in the case of nonlinear Robin and transmission boundary conditions.

Let us also mention important works that concern the investigation of variable-coefficient PDE systems and boundary value problems for such systems. Duffy [43] has provided a model for an anisotropic incompressible viscous fluid. In this case, the equations of state of such a fluid involve an anisotropic physical constant tensor. Mikhailov [100] has reduced mixed boundary value problems for a second-order quasi-linear elliptic PDE with variable coefficients to direct or two-operator direct quasi-linear localized boundary-domain integro-differential equations (BDIEs). Mitrea, Mitrea and Shi [108] have investigated variable coefficient transmission boundary value problems in the setting of bounded Lipschitz domains defined on non-smooth manifolds of dimension $n \geq 2$. Mazzucato and Nistor [93] have established a regularity result for the anisotropic linear elasticity equation with mixed boundary condition on a curved polyhedral domain in $\mathbb{R}^3$, which is allowed to have cracks. This result has been obtained in weighted Sobolev spaces. The paper [25] is concerned with the analysis of direct segregated BDIEs for the Dirichlet, Neumann and mixed boundary value problems of a Laplace-like PDE with variable coefficient, in domains with cracks, in $\mathbb{R}^3$. They have shown that these boundary domain integral equations are equivalent to the original crack type boundary value problems. Mikhailov [102] has studied second order elliptic PDE systems with non-smooth coefficients on interior or exterior Lipschitz domains with compact boundaries in Euclidean setting and showed that the canonical conormal derivatives and the classical conormal derivatives for such systems coincide. Chkadua, Mikhailov and Natroshvili [26] have shown that the Dirichlet, Neumann and Robin boundary value problem for scalar second order divergence-form elliptic PDEs with variable matrix coefficients are equivalent to some systems of localized boundary-domain singular integral equations in $\mathbb{R}^3$.

Barton and Mayboroda [22] have treated boundary problems and layer potentials for a divergence-form elliptic operator. This operator has bounded measurable t-independent coefficients which belong to spaces of fractional smoothness. They also obtain existence and uniqueness results for non-homogeneous boundary value problems for this operator. Barton [21] has used the Babuška-Lax-Milgram theorem in order to introduce layer potential for elliptic differential operators, while generalizing Green's formula, jump relations and other properties. Choi and Lee [29] have constructed the Green function for the (bounded) measurable-coefficient stationary Stokes system in a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$. They have considered this construction under the hypothesis that the weak solutions of this particular Stokes system are Hölder continuous inside the bounded Lipschitz domain. Choi and Yang [31] have studied the fundamental solution of the measurable-coefficient stationary Stokes system in $\mathbb{R}^n$, $n \geq 3$. Choi, Dong and Kim [28] have investigated the conormal derivative problem for the stationary Stokes equations with irregular coefficients in Sobolev spaces defined on Reifenberg flat domains. These irregular coefficients have been considered measurable in one direction and have SMO in the other directions. Choi, Dong and Kim [30] have constructed Green functions for stationary, measurable-coefficient, Stokes systems with conormal derivative boundary condition on bounded domains of $\mathbb{R}^n$, $n \geq 3$. They have provided existence, uniqueness, estimates for the Green function in the case that the weak solutions of the Stokes system are continuous inside the domain. Dong and Kim [41] have studied the stationary Stokes system with variable coefficients, which are measurable in one direction, in a Reifenberg flat domain. In addition, they establish well-posedness results in standard Sobolev spaces and in Muckenhoupt type weighted Sobolev spaces as well. Dong and Kim [40] have investigated solutions of the stationary Stokes system with variable coefficients in bounded Lipschitz domains. They have considered the coefficients of the strongly elliptic operator to be measurable in one direction. In the studies [41] and [40], Dong and Kim note that a variable coefficient Stokes system can be used to model inhomogeneous fluids with density dependent viscosity.

Mikhailov [103] has investigated segregated direct boundary-domain integral equations which are associated to boundary value problems for divergence-type PDE with variable coefficient in the setting of domains in $\mathbb{R}^n$, $n \geq 3$, with Lipschitz boundaries. The author has investigated the equivalence of boundary-domain integral equations to the original boundary value problems, solvability, uniqueness, invertibility of boundary-domain integral equations operators in Sobolev (Bessel potential) spaces. Kohr and Wendland [86] have obtained, in the setting of Lipschitz domains on compact Riemannian manifolds, well-posedness results for the Dirichlet boundary value problems for the L^∞-variable coefficients Stokes and Navier-Stokes PDE systems. In order to obtain these results, the authors have employed a variational approach. Using the tools illustrated in this work, they were able to define layer potential operators for the non-smooth Stokes system on Lipschitz surfaces in compact Riemannian manifolds and provide mapping properties for these operators. Furthermore, they have obtained an existence result for the Dirichlet problem for non-smooth variable coefficient Navier-Stokes system by combining the results in the linear case with a fixed point argument.

Mikhailov and Portillo [104] have analyzed a boundary value problem of mixed type for the stationary, compressible, Stokes equation with variable viscosity, in an exterior domain of $\mathbb{R}^3$. The authors have obtained two BDIEs systems which they show that are equivalent to the considered boundary value problem. Kohr, Mikhailov and Wendland [78] have investigated transmission problems for the anisotropic Stokes and Navier-Stokes systems with L^∞ strongly elliptic coefficient tensor in the setting of complementary Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$. The well-posedness of transmission-type problems that involve the anisotropic Stokes system was extracted by a variational method, and, as a consequence, the author have introduced volume and layer potentials for the anisotropic Stokes system with L^∞ strongly elliptic coefficient tensor and mapping properties for these operators were also established. These aforementioned potentials were used to establish the well-posedness of certain linear transmission problems. The well-posedness results in the linear case, together with a fixed point argument, have led the authors to obtain well-posedness results in the non-linear case as well. Kohr, Mikhailov and Wendland [79] have studied the anisotropic Stokes system with L^∞ viscosity tensor coefficient which fulfills an ellipticity condition for symmetric matrices such that their trace is equal to zero. They have provided a layer potential theory for this PDE system, in L^2-based weighted Sobolev spaces on Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$. Their approach is rooted in the investigation of particular transmission problems for the anisotropic Stokes system. After introducing the layer potentials and the volume potential, they employ these potentials to analyze Dirichlet and Neumann boundary value problems for the anisotropic Stokes system.

Kohr, Mikhailov and Wendland [77] have investigated the anisotropic Stokes system with L^∞ viscosity tensor coefficient which satisfies an ellipticity condition in terms of symmetric matrices with zero matrix trace. For such a system, they have obtained well-posedness results for Dirichlet and transmission problems in Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, with data belonging to standard and weighted Sobolev spaces. Moreover, the authors also treat Dirichlet and transmission problems for the anisotropic Navier-Stokes system in bounded Lipschitz domains in $\mathbb{R}^3$. The main tools, that were employed in the investigation, are mixed variational formulations and the Leray-Schauder theorem. Kohr and Precup [84] have provided a theoretical analysis for coupled systems of Navier-Stokes type with non-homogeneous reaction-type terms. They have used variational, fixed point and matrix theory methods to obtain existence results. Their approach is inspired by the works of Nield and Kuznetsov (see [116], [117]). Kohr and Precup [85] have used a variational approach and fixed point index theory in order to analyze a Dirichlet boundary value problem for a general coupled systems of stationary Navier-Stokes type equations with variable coefficients and non-homogeneous reaction type terms in a bounded domain of $\mathbb{R}^n$, $n \leq 3$.

Similar notions and techniques that are employed in this book can are used in other fields as well. For example, Baias, Popa and Rasa [18] have studied the Ulam stability for a linear difference equation, which contains a bounded and linear operator, in the setting of Banach spaces. Moreover, for a finite dimensional Banach space and the presence of a Fredholm operator in their considered equation, their study yielded stability results. Moreover, the authors in [119] are concerned with establishing a stability result for a nonconstant coefficient linear differential operator. Cîmpean and Popa [32] have investigated the Hyers-Ulam stability of a linear differential equation of higher order with constant coefficients in Aoki-Rassias sense. Precup [124] has studied a class of semilinear elliptic variational systems and has obtained localization and existence results of positive nontrivial solutions by making use of critical point theorems and the technique of inverse-positive matrices. Let us also mention that Precup [123] has provided techniques of nonlinear analysis which are used in the investigation of nonlinear integral equations, for example, fixed point theorems. Marinoschi [92] has studied nonlinear boundary problems which model water flow in porous media (for example, the interaction of rainfall water with the soil). The authors in [23] investigate a optimization problem that models a diffusive flow in a non-homogeneous porous medium. In addition, Marinoschi [91] has studied a time control problem for the linearized Navier-Stokes periodic flow in a channel in two dimensions. We also remark the contributions of Barbu and Marinoschi [19], [20] related to an optimal control method to some optical flow problems.

Note that boundary value problems can be investigated also from a numerical point of view. This has lead to the development of diverse numerical methods (finite differences, finite volumes, finite element) whose purpose is to find numerical solutions for various boundary value problems (see also [126]). In the latter we discuss studies that are concerned with the numerical treatment of these problems. Ghia, Ghia and Shin [55] have used the vorticity-stream function formulation for the incompressible Navier-Stokes equations in dimension $n = 2$. The model problem that they have employed is the driven flow in a square cavity. Vafai [139] has analyzed the effects that occur in the case of variable porosity and inertial forces on convective flow and heat transfer in porous media. AlAmiri [3] has investigated numerically a lid-driven flow in a stable, thermally-stratified, two-dimensional square cavity, which contains a water-saturated porous media. Guo and Zhao [60] have proposed a lattice Boltzmann model for an isothermal incompressible flow in porous media and they have included the porosity into the equilibrium distribution and a force term to the evolution equation (to account for the drag forces of the medium), i.e., the Darcy term and the Forchheimer term. Gupta and Kalita [61] have discussed a new paradigm that can be used to solve the Navier-Stokes equations, which is based on a streamfunction-velocity formulation. Their proposed approach avoids the difficulties that can appear in the computation of vorticity values and the difficulties that appear in the endeavor of solving the pressure equations of the classical velocity-pressure formulation for the Navier-Stokes equations. Erturk, Corke and Gökçŏl [45] have used the stream function and vorticity formulation for the Navier-Stokes in order to numerically investigate the 2-D steady incompressible driven cavity flow. He and Wang [66] have used Navier slip boundary condition in order to study the driven cavity flow. Their results illustrate that the Navier slip boundary condition removes the corner singularity that appears in the case of no-slip boundary condition.

Erturk [44] has provided a brief survey on the studies which concern the driven cavity flow in two dimensions, where physical, mathematical and numerical properties are examined. Yang, Xue and Mahias [144] have concerned themselves with the investigation of the lid-driven rectangular cavity containing a porous Brinkman-Forchheimer medium. AlAmiri [4] has investigated an incompressible, laminar mixed-convection heat transfer in square lid-driven cavity in the presence of a porous block. In this analysis, the Navier-Stokes equations appear in order to represent the transport phenomena. The author has provided comparisons of streamlines, isotherms and other characteristics. Gutt and Groşan [62] have studied the flow of an incompressible viscous fluid through a porous medium in a square cavity of dimension $n = 2$. They analyze this problem theoretically and numerically, as well. Grosan, Sheremet, Pop and Pop [59] have investigated, numerically, the thermophoretic transport of small particles by convection in a differentially heated square cavity which has a wavy wall. They solve numerically the governing partial differential equations and highlight the effect of thermophoresis, while observing the influence of the number of undulations on heat transfer and fluid flow. Bondarenko, Sheremet, Ozotop and Abu-Hamdeh [24] have analyzed mixed convection in alumina or water nanoliquid cavity with two porous blocks which are adherent and have different permeability and porosity. Groşan, Pătrulescu and Pop [58] have proposed a mathematical model which contains the Brinkman PDE system in order to discuss the steady free convection in a square differentially heated cavity which is filled by a bidisperse porous medium.

The monograph consists of four chapters.

- **Chapter 1** contains an overview of the notions that are used. In Section 1.1 we define the concept of a Lipschitz domain, we discuss some useful notations. Moreover, we provide two assumptions (see Assumption 1.1.6 and Assumption 1.1.7, respectively) which describe the geometric setting in which we investigate our boundary problems. These problems are analyzed in the following chapters. In Section 1.1, we introduce the function spaces that are necessary for our analysis. In this section, we introduce Sobolev spaces in Lipschitz domains in the Euclidean setting. We provide their definition, their norms and also the Sobolev Embedding Theorem (see Theorem 1.1.15). Next, we describe Sobolev spaces on Lipschitz boundaries in the Euclidean setting. Next, we discuss weighted Sobolev spaces in $\mathbb{R}^3$ as presented in the work of Hanouzet [65]. This subsection which concerns weighted Sobolev spaces contains also a definition which specifies how a function tends to a constant at infinity in the sense of Leray (see Definition 1.1.16) and a useful corollary (see Corollary 1.1.17). We end this section by discussing the (Gagliardo) trace operator in the case of classical Sobolev spaces and also in the case of weighted Sobolev spaces (see Lemma 1.1.18 and Remark 1.1.19). In Section 1.2 we describe the Stokes operator and the Brinkman operator. For each of these operators, we give their corresponding conormal derivative operators (see Definition 1.2.3, Lemma 1.2.4, Definition 1.2.5, Lemma 1.2.6). We also introduce a generalized version of the Brinkman system (see Subsection 1.2.2). In this subsection we investigate this operator and provide its associated conormal derivate operator (see Definition 1.2.14 and Lemma 1.2.15). Section 1.3 is concerned with the introduction of fundamental solution of the Stokes system (see Relation (1.3.1)) and its associated stress and pressure tensors (see Relation (1.3.4)). Further on, in this section we give the Newtonian potentials and layer potentials for the Stokes system (see Definition 1.3.1, Definition 1.3.3 and Definition 1.3.5). We also provide their mapping properties (see Theorem 1.3.2, Theorem 1.3.15, Theorem 1.3.20), their jump properties (see Lemma 1.3.8) and their growth condition (see Relation (1.3.26)). In a similar manner, in Section 1.4 we give the fundamental solution of the Brinkman system (see Relation (1.4.2)) and its associated stress and pressure tensors (see Relation (1.4.4)). Next, we provide the Newtonian potentials and layer potentials for the Brinkman system (see Definition 1.4.1, Definition 1.4.3, Definition 1.4.5). For each of these operators we provide mapping properties (see Theorem 1.4.2, Theorem 1.4.4, Theorem 1.4.7). We end this section with by giving the jump properties (see Lemma 1.4.8) and growth properties (see Relation (1.4.28)).

- **Chapter 2** is concerned with existence and uniqueness results of transmission type problems for linear PDE systems. We begin this chapter by providing a well-posedness result for the Dirichlet-type problem for the Brinkman system in an exterior Lipschitz domain in $\mathbb{R}^3$ (see Theorem 2.1.2). This result is achieved by employing layer potential methods. Next, an existence and uniqueness result is given for the transmission problem for the generalized Brinkman equations and Stokes equations in $\mathbb{R}^3$ (see Theorem 2.2.2 and Theorem 2.2.3). This result is obtained by using the well-posedness result for the transmission problem for the Stokes system in complementary Lipschitz domains in $\mathbb{R}^3$ (see Theorem 2.2.6 and Theorem 2.2.7) together with certain Fredholm operator theory techniques. We continue by providing a well-posedness result for the transmission problem for the classical and generalized Brinkman equations in $\mathbb{R}^3$ (see Theorem 2.3.1). This result is obtained by using the existence and uniqueness result for the transmission problem for the Stokes and Brinkman systems in complementary Lipschitz domains in $\mathbb{R}^3$ (see Theorem 2.3.3) and Fredholm operator theory methods. In the last section of this chapter, we have the well-posedness result for a Robin-transmission problem for the Brinkman equations in $\mathbb{R}^n$, $n \geq 2$ (see Theorem 2.4.1). This result holds true after undertaking a layer potential analysis. In addition, by using a similar procedure as in the case of Theorem 2.4.1, we provide an existence and uniqueness result for a limiting Robin-transmission problem for the Brinkman equations in $\mathbb{R}^n$, $n \geq 2$ (see Theorem 2.4.2). As a consequence of Theorem 2.4.2, we are able to derive an existence and uniqueness result for the Robin-Dirichlet problem for the Brinkman system (see Corollary 2.4.3). The content of this chapter is based on the papers [7], [8], [9].

- In **Chapter 3** we discuss a generalization of the Darcy-Forchheimer-Brinkman equations (see Relation (3.1.1)). Also, we have provided a useful lemma (see Lemma 3.1.3). Then, we give an existence and uniqueness result for the transmission problem for the generalized Darcy-Forchheimer-Brinkman and Stokes equations in $\mathbb{R}^3$ (see Theorem 3.2.1). This result is obtained by employing the well-posedness result which was obtained in the linear case (see Theorem 2.2.3 of Chapter 2) with a fixed point method. Next, we present an existence and uniqueness result for the transmission problem for the generalized Darcy-Forchheimer-Brinkman and Brinkman equations in $\mathbb{R}^3$ (see Theorem 3.3.1). In order to obtain this result, we use the existence and uniqueness result established in the linear case (see Theorem 2.3.1 of Chapter 2) together with a fixed point technique. We also have an existence and uniqueness result for the Robin-transmission problem for the Darcy-Forchheimer-Brinkman equations in $\mathbb{R}^n$, $n = 2, 3$ (see Theorem 3.4.1). This is possible due to the application of the well-posedness result which was obtained in the linear case (see Theorem 2.4.1 of Chapter 2) together with a fixed point theorem. Similar arguments are employed in order to get a well-posedness result for a limiting Robin-transmission problem Darcy-Forchheimer-Brinkman equations in $\mathbb{R}^n$, $n = 2, 3$ (see Theorem 3.4.2). Finally, due to Theorem 3.4.2, we are able to obtain an existence result for the Robin-Dirichlet problem for the Darcy-Forchheimer-Brinkman equations in $\mathbb{R}^n$, $n = 2, 3$ (see Corollary 3.4.3). The content of this chapter is based on the papers [5], [6], [9].

- Lastly, the goal of **Chapter 4** is to give a numerical analysis in order to determine a numerical solution for the Robin-Dirichlet boundary problem for the Darcy-Forchheimer-Brinkman equations. This numerical study concerns the lid-driven porous cavity problem with Navier slip boundary condition in the presence of a solid body. The geometric setting of this problem can be seen in Figure 4.1. In order to solve this problem, first we write our mathematical model (see Relation (4.1.1)), we conduct a non-dimensional analysis (see Relation (4.1.5)). To get a numerical solution, we use a numerical software, namely COMSOL Multiphysics. Then, we determine the optimal grid for our analysis (see Table 4.1) and we validate our model via comparison with existent results (see Figure 4.2). Finally, we investigate the impact of dimensionless slip length (see Subsection 4.1.4). The content of this chapter is based on the paper [9].

The monograph ends with a section dedicated to further research directions, a section dedicated to conclusions and an appendix consisting of two sections. One is devoted to Agmon-Douglis-Nirenberg-elliptic systems in Euclidean setting and another one to the main properties of Fredholm operators.

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1

Layer Potential Methods for the Stokes and Brinkman systems in Lipschitz domains

This chapter establishes the functional setting in which we will analyze our boundary values problems for the Stokes, Brinkman, Navier-Stokes and Darcy-Forchheimer-Brinkman equations. To this end, we recall definitions, notations and properties that we will use throughout this work.

Hence, we will introduce the concepts of a bounded Lipschitz domain and an unbounded (or exterior) Lipschitz domain in (the Euclidean setting of) $\mathbb{R}^n$, where $n \geq 2$. We will also place an emphasis on the case $n = 3$ in whose setting, we have obtained many of our well-posedness results. Next, we will recall the definitions of the Sobolev spaces in the Euclidean setting and their properties, which are most relevant to our study. In addition, we will also discuss the Gagliardo Trace Lemma which allows us to define the trace operator in the setting of Sobolev spaces. This previous operator is involved in the boundary conditions of the boundary value problems that we study.

Further, we will study the Stokes and Brinkman systems. In the case of these two systems, we will discuss their associated conormal derivative operators. These operators, again, will appear in the boundary conditions of the boundary value problems that we treat in the latter.

One important aspect that we wish to point out is that, in this chapter, we deal with a generalized version of the Brinkman system. Our original results involve these particular systems of PDEs.

Finally, we conclude this chapter with two very important sections. These sections contain the layer potential operators associated to the Stokes and Brinkman equations, respectively. These operators are used in the proof of our well-posedness results, due to the fact that with their help, we are able to construct solutions for our boundary value problems. The sources that were used in the preparation of this chapter are [1], [2], [63], [68], [70], [95], [122], [136], [138], [143].

1.1 Functional Setting

This section is dedicated to the description of the main notions that are used all through this work. First of all, we define the concept of Lipschitz domain and we describe important notations that we use throughout this work. Also, we describe the geometry of the Lipschitz domains that are involved in the boundary problems that we will study in the latter. Next, we provide an overview of Sobolev spaces in $\mathbb{R}^n$, on Lipschitz domains and Lipschitz boundaries. Some properties of these Sobolev spaces are also given. Moreover, we recall the concept of a weighted Sobolev space in the exterior of a bounded Lipschitz domain in $\mathbb{R}^3$. We end this section with the useful Gagliardo trace lemma.

1.1.1 Lipschitz domains

We will review the definition of a bounded Lipschitz domain and introduce the spaces in which we seek our solutions for our boundary value problems. Also, we will discuss the systems that are encountered in our study, and describe the operators that appear in our boundary conditions. Let us provide in the latter the definition of the concept of a Lipschitz domain (cf., e.g., [109, Def. 2.1], see also, [64, Def. 2.1]).

Definition 1.1.1. *Let* $\mathsf{D} \subset \mathbb{R}^n$*,* $n \geq 2$ *be a nonempty, open and bounded set. Denote by* Γ *the boundary of the set* D*. We say that* D *is a bounded Lipschitz domain if for any* $\boldsymbol{x} \in \Gamma$*, there are some constants* $r_1, r_2 > 0$*, a coordinate system* $(y_1, ..., y_n) = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ *that is isometric to the canonical one and has its origin at* $\boldsymbol{x}$*, and a Lipschitz function* $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$*, such that*

$$\mathsf{D} \cap \mathcal{C}(r_1, r_2) = \{y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |y'| < r_1 \text{ and } \psi(y') < y_n < r_2\},$$

where

$$\mathcal{C}(r_1, r_2) := \{y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |y'| < r_1,\ |y_n| < r_2\} \subseteq \mathbb{R}^n.$$

Next, we state some useful remarks.

Remark 1.1.2. *In this work, we will use the repeated index summation convention.*

Remark 1.1.3. *In this work, we use the notation a.e. instead of almost everywhere.*

Remark 1.1.4. *If* X *denotes a Banach space, its topological dual is denoted by* X'*.*

Remark 1.1.5. *If* Y *is an open subset of* $\mathbb{R}^n$*,* $n \geq 2$*, then we denote the duality pairing between two dual spaces defined on* Y *by* $\langle \cdot, \cdot \rangle_Y$*.*

In the latter, we will state some assumptions that allow us to represent the geometry of the Lipschitz domains, the setting where our problems will be formulated.

Assumption 1.1.6. *Let* $\mathsf{D}_+ := \mathsf{D} \subsetneq \mathbb{R}^n$*,* $n \geq 2$*, be a bounded Lipschitz domain with connected boundary* Γ*. Denote by* $\mathsf{D}_- := \mathbb{R}^n \setminus \overline{\mathsf{D}}$ *the complementary (exterior) Lipschitz domain (see Figure 1.1).*

Assumption 1.1.7. *Let* $\mathsf{D} \subseteq \mathbb{R}^n$*,* $n \geq 2$*, be a bounded Lipschitz domain with connected boundary* Γ_-*. Assume that* D_+ *is a bounded Lipschitz domain, with connected boundary denoted by* Γ_+*, such that* $\overline{\mathsf{D}}_+ \subset \mathsf{D}$ *and let* $\mathsf{D}_- := \mathsf{D} \setminus \overline{\mathsf{D}}_+$*. Hence, the boundary of* D_- *has two connected components, namely,* Γ_+ *and* Γ_- *(see Figure 1.2).*

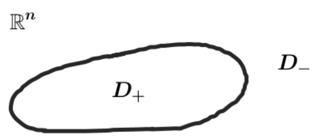


Figure 1.1: The complementary Lipschitz domains D_+ and D_- in $\mathbb{R}^n$.

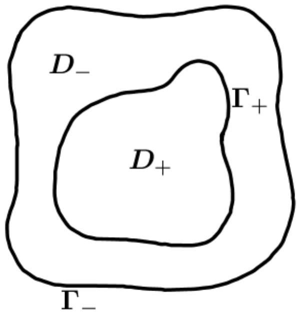


Figure 1.2: A bounded Lipschitz domain $\mathsf{D} = \overline{\mathsf{D}}_+ \cup \mathsf{D}_-$ which satisfies Assumption 1.1.7

1.1.2 On Sobolev spaces in Lipschitz domains

The purpose of this section is to provide an overview of Sobolev spaces in an Euclidean setting in $\mathbb{R}^n$. These spaces are used in the investigation of (weak) solutions of certain PDEs, for which no classical solution can be found. We will use these spaces in the latter.

In the latter, $\mathbb{Z}_+$ denotes the set of non-negative integers and the vector $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_+^n$ is called a multi-index. Let us set $|\alpha| = \sum_{i=1}^n \alpha_j$. We introduce the differential operator

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} ... \partial {x_n}^{\alpha_n}}. \tag{1.1.1}$$

Moreover, we also introduce the differential operator

$$D_k := \frac{1}{i} \frac{\partial}{\partial x^k}, \; i^2 = -1. \tag{1.1.2}$$

Now, we denote by $\mathsf{D} \subseteq \mathbb{R}^n$, $n \geq 2$, either a bounded Lipschitz domain or an exterior Lipschitz domain or $\mathbb{R}^n$. In the case of a bounded Lipschitz domain or an exterior Lipschitz domain D, we denote the boundary of such domains by Γ.

Note that space $\mathcal{C}(\overline{\mathsf{D}})$ is the space of continuous functions on $\overline{\mathsf{D}}$ and it is endowed with the sup-norm.

For a function $g : \mathsf{D} \to \mathbb{R}$, we define the support of g by

$$\text{supp } g := \overline{\{x \in \mathsf{D} \mid g(x) \neq 0\}}. \tag{1.1.3}$$

We denote by $\mathcal{C}^\infty(\mathsf{D})$ the space of infinitely differentiable functions defined on D. We also denote by $\mathcal{C}_0^\infty(\mathsf{D})$ the space of infinitely differentiable functions, that vanish in some neighborhood of Γ. Let us note that if $g \in \mathcal{C}_0^\infty(\mathsf{D})$ then $g|_\Gamma = 0$. Also, if $g \in \mathcal{C}_0^\infty(\mathsf{D})$ then the set (1.1.3) is compact in D. We also introduce the vector function spaces $\mathcal{C}^\infty(\mathsf{D})^n$ and $\mathcal{C}_0^\infty(\mathsf{D})^n$ by

$$\begin{aligned} \mathcal{C}^\infty(\mathsf{D})^n &:= \{\mathbf{u} : \mathsf{D} \to \mathbb{R}^n \mid \mathbf{u} = (u_1, ..., u_n), u_i \in \mathcal{C}^\infty(\mathsf{D}), i = \overline{1, n}\}, \\ \mathcal{C}_0^\infty(\mathsf{D})^n &:= \{\mathbf{u} : \mathsf{D} \to \mathbb{R}^n \mid \mathbf{u} = (u_1, ..., u_n), u_i \in \mathcal{C}_0^\infty(\mathsf{D}), i = \overline{1, n}\}. \end{aligned} \tag{1.1.4}$$

For $p \in [1, \infty)$, the Lebesgue space $L^p(\mathsf{D})$ of (equivalence classes of) measurable functions, p-th power, absolute value Lebesgue integrable on D is given by

$$L^p(\mathsf{D}) := \left\{ u : \mathsf{D} \to \mathbb{R} \;\middle|\; \int_{\mathsf{D}} |u(x)|^p \mathrm{d}x < \infty \right\} \tag{1.1.5}$$

and its norm is given by

$$||u||_{L^p(\mathsf{D})} := \left(\int_{\mathsf{D}} |u(x)|^p \mathrm{d}x \right)^{\frac{1}{p}}, \tag{1.1.6}$$

for $1 \leq p < \infty$. Also, we define the space of vector functions $L^p(\mathsf{D})^n$ by

$$L^p(\mathsf{D})^n := \{\mathbf{u} : \mathsf{D} \to \mathbb{R}^n \mid \mathbf{u} = (u_1, ..., u_n), u_i \in L^p(\mathsf{D}), i = \overline{1, n}\}. \tag{1.1.7}$$

Note that the space $L^\infty(\mathsf{D})$ is the space of (equivalence classes of) essentially bounded functions on D. Its norm is given by

$$||u||_{L^\infty(\mathsf{D})} := \mathrm{esssup}_{x \in \mathsf{D}} |u(x)|. \tag{1.1.8}$$

The quantity in the right hand side of relation (1.1.8) is called the essential supremum of u. It is the smallest number ϵ such that the set $\{x \in \mathsf{D} \mid u(x) > \epsilon\}$ has Lebesgue measure equal to zero. In addition, we define the space $L^\infty(\mathsf{D})^n$ by

$$L^\infty(\mathsf{D})^n := \{\mathbf{u} : \mathsf{D} \to \mathbb{R}^n \mid \mathbf{u} = (u_1, ..., u_n), u_i \in L^\infty(\mathsf{D}), i = \overline{1, n}\}. \tag{1.1.9}$$

In the latter, we will also use the space

$$L^\infty(\mathsf{D})^{n \times n} := \{\mathbf{U} : \mathsf{D} \to \mathbb{R}^n \times \mathbb{R}^n \mid \mathbf{U} = (u_{ij}), u_{ij} \in L^\infty(\mathsf{D}),\ i, j = \overline{1, n}\} \tag{1.1.10}$$

Note that, for $p \in (1, \infty)$, the topological dual of the space $L^p(\mathsf{D})$ is the space $L^q(\mathsf{D})$, where $\frac{1}{p} + \frac{1}{q} = 1$. In addition the dual of the space $L^1(\mathsf{D})$ is the space $L^\infty(\mathsf{D})$. Let us note that, for $1 \leq p \leq \infty$, the space $L^p(\mathsf{D})$ is a Banach space. In addition, $L^2(\mathsf{D})$ is a Hilbert space.

In the latter, let us view the space $\mathcal{C}_0^\infty(\mathsf{D})$ as a topological vector space. Then, let us introduce the spaces $\mathcal{D}(\mathsf{D})$ and $\mathcal{D}'(\mathsf{D})$.

Definition 1.1.8. *The Schwarz space of test functions $\mathcal{D}(\mathsf{D})$ is the space $\mathcal{C}_0^\infty(\mathsf{D})$ endowed with the inductive limit topology.*

Note that, the space $\mathcal{D}(\mathsf{D})^n$ can be defined in a similar way, namely,

$$\mathcal{D}(\mathsf{D})^n := \{\boldsymbol{\psi} : \mathsf{D} \to \mathbb{R}^n \mid \boldsymbol{\psi} = (\psi_1, ..., \psi_n), \psi_i \in \mathcal{D}(\mathsf{D}), i = \overline{1, n}\}. \tag{1.1.11}$$

Definition 1.1.9. *The space of distributions $\mathcal{D}'(\mathsf{D})$ is the space of all linear and continuous functionals on $\mathcal{D}(\mathsf{D})$.*

The space of vector functions $\mathcal{D}'(\mathsf{D})^n$ is given by

$$\mathcal{D}'(\mathsf{D})^n := \{\boldsymbol{\Psi} : \mathsf{D} \to \mathbb{R}^n \mid \boldsymbol{\Psi} = (\Psi_1, ..., \Psi_n), \Psi_i \in \mathcal{D}'(\mathsf{D}), i = \overline{1, n}\}. \tag{1.1.12}$$

Next, we describe the notion of a Sobolev space. Note that, henceforth, we use L^2-based Sobolev spaces that are defined on D. Consequently, we introduce the integer order L^2-based Sobolev spaces as follows.

Definition 1.1.10. *Assume that $k \in \mathbb{Z}_+$. Then, the Sobolev space $H^k(\mathsf{D})$ is defined by*

$$H^k(\mathsf{D}) := \{u \in L^2(\mathsf{D}) \mid D^\alpha u \in L^2(\mathsf{D}), \forall\ \alpha \in \mathbb{Z}_+^n, |\alpha| \leq k\}, \tag{1.1.13}$$

and its norm is given by

$$||u||_{H^k(\mathsf{D})} := \left(\sum_{|\alpha| \leq k} ||D^\alpha u||^2_{L^2(\mathsf{D})} \right)^{\frac{1}{2}}. \tag{1.1.14}$$

We also introduce the Sobolev space $H^k(\mathsf{D})^n$ by

$$H^k(\mathsf{D})^n := \{\mathbf{u} : \mathsf{D} \to \mathbb{R}^n \mid \mathbf{u} = (u_1, ..., u_n), u_i \in H^k(\mathsf{D}), i = \overline{1,n}\}. \tag{1.1.15}$$

Let us also introduce the space $H_0^k(\mathsf{D}) \equiv \mathring{H}^k(\mathsf{D})$ as the closure of $\mathcal{D}(\mathsf{D})$ in $H^k(\mathsf{D})$ with respect to the norm $||\cdot||_{H^k(\mathsf{D})}$. Similarly, we can introduce the space $H_0^k(\mathsf{D})^n \equiv \mathring{H}^k(\mathsf{D})^n$. Moreover, $\mathring{H}^k(\mathbb{R}^n) = H^k(\mathbb{R}^n)$ and $\mathring{H}^k(\mathbb{R}^n)^n = H^k(\mathbb{R}^n)^n$.

The spaces $H^k(\mathsf{D})$ and $\mathring{H}^k(\mathsf{D})$ are Hilbert spaces. Also, let us mention that the Hilbert space $H^k(\mathsf{D})$ is endowed with the inner product

$$(u, v)_{H^k(\mathsf{D})} := \sum_{|\alpha| \geq k} (D^\alpha u, D^\alpha v)^{\frac{1}{2}}_{L^2(\mathsf{D})}. \tag{1.1.16}$$

The following definitions allow us to introduce the fractional order L^2-based Sobolev spaces.

Definition 1.1.11. *Assume that $0 < s < 1$. The fractional order Sobolev space $H^s(\mathsf{D})$ is defined by*

$$H^s(\mathsf{D}) := \left\{ u \in L^2(\mathsf{D}) \;\middle|\; \int_{\mathsf{D}} \int_{\mathsf{D}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \mathrm{d}x \mathrm{d}y < \infty \right\} \tag{1.1.17}$$

and its norm is given by

$$||u||_{H^s(\mathsf{D})} = \left(\int_{\mathsf{D}} |u(x)|^2 \mathrm{d}x + \int_{\mathsf{D}} \int_{\mathsf{D}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \mathrm{d}x \mathrm{d}y \right)^{\frac{1}{2}}. \tag{1.1.18}$$

Definition 1.1.12. *Assume that $0 < s < 1$ and $k \in \mathbb{Z}_+$. Let $\sigma = k + s$. The fractional order Sobolev space $H^\sigma(\mathsf{D})$ is defined by*

$$H^\sigma(\mathsf{D}) := \{u \in H^k(\mathsf{D}) \mid D^\alpha u \in H^s(\mathsf{D}), \forall \alpha \in \mathbb{Z}^n_+,\ 0 \leq |\alpha| \leq k\} \tag{1.1.19}$$

and its norm is given by

$$||u||_{H^\sigma(\mathsf{D})} := \left(\sum_{|\alpha| \leq k} ||D^\alpha u||^2_{H^s(\mathsf{D})} \right)^{\frac{1}{2}}. \tag{1.1.20}$$

By taking into account Definition 1.1.11 and Definition 1.1.12, one can introduce the spaces of vector-valued functions $H^s(\mathsf{D})^n$ and $H^\sigma(\mathsf{D})^n$ component-wise. Moreover, the fractional order Sobolev space $H^\sigma(\mathsf{D})$ is a Hilbert space.

Let us discuss the negative order L^2-based Sobolev spaces. Let $k \in \mathbb{Z}_+$. In order to introduce these Sobolev spaces, let us note that the space $H_0^k(\mathsf{D})$ is the closure of $\mathcal{C}_0^\infty(\mathsf{D})$ in the space $H^k(\mathsf{D})$. In addition,

$$H_0^k(\mathbb{R}^n) = H^k(\mathbb{R}^n). \tag{1.1.21}$$

Similarly, we can define the space of vector functions $H_0^k(\mathsf{D})^n$ component-wise and relation (1.1.21) can be written also for the vector-valued space $H_0^k(\mathbb{R}^n)^n$.

Let us now define the negative order L^2-Sobolev spaces.

Definition 1.1.13. *Assume that $k \in \mathbb{Z}_+$. Then, the negative order Sobolev space $H^{-k}(\mathsf{D})$ is the dual of the space $H_0^k(\mathsf{D})$, i.e.,*

$$H^{-k}(\mathsf{D}) := (H_0^k(\mathsf{D}))', \tag{1.1.22}$$

and its norm is given by

$$||h||_{H^{-k}(\mathsf{D})} := \sup_{u \in H_0^k(\mathsf{D}), u \neq 0} \frac{|\langle h, u \rangle|}{||u||_{H_0^k(\mathsf{D})}}. \tag{1.1.23}$$

Let us note that the vector-valued space $H^{-k}(\mathsf{D})^n$ is defined component-wise. Moreover, we have that $H^{-k}(\mathsf{D})^n = (H_0^k(\mathsf{D})^n)'$. In addition, we have that the density of $\mathcal{C}_0^\infty(\mathsf{D})$ in $H_0^k(\mathsf{D})$ implies the inclusion $H^{-k}(\mathsf{D}) \subset \mathcal{D}'(\mathsf{D})$. Note that the space $H^{-k}(\mathsf{D})$ is a Hilbert space.

Since $C_0^\infty(\mathsf{D})$ is not dense in the space $H^k(\mathsf{D})$, for $k \in \mathbb{Z}_+$, the dual of $H^k(\mathsf{D})$ cannot be embedded as a subspace of the space of distributions $\mathcal{D}'(\mathsf{D})$.

Definition 1.1.14. *Assume that $s \in \mathbb{R}$. Assume that D is a Lipschitz domain in $\mathbb{R}^n$. The space $\tilde{H}^s(\mathsf{D})$ is defined as the closure of $\mathcal{D}(\mathsf{D})$ in $H^s(\mathbb{R}^n)$.*

Moreover, the vector-valued space $\tilde{H}^s(\mathsf{D})^n$ is given by

$$\tilde{H}^s(\mathsf{D})^n := \{\mathbf{u} : \mathsf{D} \to \mathbb{R}^n \mid \mathbf{u} = (u_1, ..., u_n), u_i \in \tilde{H}^s(\mathsf{D}), i = \overline{1,n}\}. \tag{1.1.24}$$

The space $\tilde{H}^s(\mathsf{D})$ can be characterized as

$$\tilde{H}^s(\mathsf{D}) = \{u \in H^s(\mathbb{R}^n) \mid \text{supp } u \subseteq \mathsf{D}\}. \tag{1.1.25}$$

In addition, $\tilde{H}^s(\mathbb{R}^n) = H^s(\mathbb{R}^n)$.

We have the following duality relations

$$(H_0^k(\mathsf{D}))' = H^{-k}(\mathsf{D}),\ (H^k(\mathsf{D}))' = \tilde{H}^{-k}(\mathsf{D}), \tag{1.1.26}$$

for $k \in \mathbb{Z}_+$. Let us mention that the duality relations in (1.1.26) hold also in the case of the vector function Sobolev spaces.

Next, we provide the Sobolev embedding theorem (see, e.g., [1, Theorem 4.12], [2]).

Theorem 1.1.15. *Assume that $k \in \mathbb{Z}_+$. Let $\mathsf{D} \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We have that*

(i) the embedding $H^k(\mathsf{D}) \hookrightarrow \mathcal{C}(\overline{\mathsf{D}})$ is continuous if $k > \frac{n}{2}$.

(ii) the embedding $H^k(\mathsf{D}) \hookrightarrow L^q(\mathsf{D})$ is continuous and compact, for all $q \in [1, \infty)$, if $k = \frac{n}{2}$.

(iii) the embedding $H^k(\mathsf{D}) \hookrightarrow L^q(\mathsf{D})$ is continuous for $\frac{1}{q} = \frac{1}{2} - \frac{k}{n}$, if $k < \frac{n}{2}$.

(iv) the embedding $H^k(\mathsf{D}) \hookrightarrow L^r(\mathsf{D})$ is compact for $1 < r < q$, $\frac{1}{q} = \frac{1}{2} - \frac{k}{n}$, if $k < \frac{n}{2}$.

Sobolev (Bessel potential) spaces on $\mathbb{R}^n$.

In the latter, let us provide a different way of viewing Sobolev spaces. This can be done by employing a powerful tool, namely the Fourier transform. To this end, we recall the Schwartz space of rapidly decreasing functions,

$$\mathcal{S}(\mathbb{R}^n) := \{\psi \in C^\infty(\mathbb{R}^n) \mid \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \psi(x)| < \infty, \forall \text{ multi-indices } \alpha, \beta \in \mathbb{Z}_+^n\}, \tag{1.1.27}$$

and the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n) := (\mathcal{S}(\mathbb{R}^n))'$. In addition, let us introduce the following vector-valued function spaces

$$\begin{aligned} \mathcal{S}(\mathbb{R}^n)^n &:= \{\boldsymbol{\psi} : \mathbb{R}^n \to \mathbb{R}^n \mid \boldsymbol{\psi} = (\psi_1, ..., \psi_n), \psi_i \in \mathcal{S}(\mathbb{R}^n), i = \overline{1,n}\}, \\ \mathcal{S}'(\mathbb{R}^n)^n &:= \{\boldsymbol{\Psi} : \mathbb{R}^n \to \mathbb{R}^n \mid \boldsymbol{\Psi} = (\Psi_1, ..., \Psi_n), \Psi_i \in \mathcal{S}(\mathbb{R}^n), i = \overline{1,n}\}. \end{aligned} \tag{1.1.28}$$

Let $\mathsf{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is the Fourier transform, defined on $\mathcal{S}(\mathbb{R}^n)$ by

$$(\mathsf{F}u)(\zeta) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\zeta x} u(x) \mathrm{d}x, \tag{1.1.29}$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$ and $\zeta \in \mathbb{R}^n$.

The Fourier transform is an isomorphism and its inverse $\mathsf{F}^{-1} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is given by

$$(\mathsf{F}v)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\zeta x} v(\zeta) \mathrm{d}\zeta, \tag{1.1.30}$$

for all $v \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

We extend this isomorphism to the space $\mathcal{S}'(\mathbb{R}^n)$ by the following relation

$$\langle \mathsf{F}h, \psi \rangle := \langle h, \mathsf{F}\psi \rangle, \tag{1.1.31}$$

for all $h \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$, where $\langle \cdot, \cdot \rangle$ represents the duality pairing of two dual spaces.

Now, we use the Fourier transform to give an alternative introduction for the space $H^k(\mathbb{R}^n)$ if $k \in \mathbb{Z}_+$. To this end, let us note that the norms

$$||h||_{H^k(\mathbb{R}^n)} \text{ and } ||\mathsf{F}^{-1}[(1 + |\cdot|^2)^{\frac{k}{2}} \mathsf{F}h]||_{L^2(\mathbb{R}^n)} \tag{1.1.32}$$

are equivalent.

Consequently, for $k \in \mathbb{Z}_+$, the L^2-based (Bessel potential) Sobolev space $H^k(\mathbb{R}^n)$ is given by

$$H^{k,2}(\mathbb{R}^n) := \left\{ h \in \mathcal{S}'(\mathbb{R}^n) \;\middle|\; ||\mathsf{F}^{-1}[(1 + |\zeta|^2)^{\frac{k}{2}} \mathsf{F}h]||_{L^2(\mathbb{R}^n)} < \infty \right\}. \tag{1.1.33}$$

Based on relation (1.1.33), for $s \in \mathbb{R}$ we introduce the L^2-based (Bessel potential) Sobolev spaces with real index by

$$\begin{aligned} &H^{s,2}(\mathbb{R}^n) := \left\{ h \in \mathcal{S}'(\mathbb{R}^n) \;\middle|\; \mathsf{F}^{-1}[(1 + |\zeta|^2)^{\frac{s}{2}} \mathsf{F}h] \in L^2(\mathbb{R}^n) \right\}, \\ &H^{s,2}(\mathsf{D}) := \{ u \in \mathcal{D}'(\mathsf{D}) \mid \exists\, U \in H^s(\mathbb{R}^n) \text{ such that } u = U|_{\mathsf{D}} \}, \end{aligned} \tag{1.1.34}$$

where $|_{\mathsf{D}}$ is the operator of restriction to D. In addition, the space $\tilde{H}^{s,2}(\mathsf{D})$ as the closure of $\mathcal{D}(\mathsf{D})$ in the space $H^{s,2}(\mathbb{R}^n)$.

In addition, for $s \geq 0$, the negative index (Bessel potential) Sobolev spaces are introduced by the following duality relations

$$\tilde{H}^{-s,2}(\mathsf{D}) := (H^s(\mathsf{D}))', \; H^{-s,2}(\mathsf{D}) := (\tilde{H}^{s,2}(\mathsf{D}))'. \tag{1.1.35}$$

Now, let us consider $s := k \in \mathbb{Z}_+$. Then, the space $H^{k,2}(\mathsf{D})$ coincides with the integer order Sobolev space $H^k(\mathsf{D})$ which is introduced in Definition 1.1.10. Hence, throughout this work, both spaces will be denoted by $H^k(\mathsf{D})$.

Let us note that the vector-valued versions of the spaces $H^{s,2}(\mathbb{R}^n)$, $H^{s,2}(\mathsf{D})$ and $\tilde{H}^{s,2}(\mathsf{D})$ will be denoted by $H^{s,2}(\mathbb{R}^n)^n$, $H^{s,2}(\mathsf{D})^n$ and $\tilde{H}^{s,2}(\mathsf{D})^n$. These vector function spaces are defined component-wise. We omit their full description for the sake of brevity. Finally, all L^2-based (Bessel potential) Sobolev spaces considered in the former are Hilbert spaces.

1.1.3 Sobolev spaces on Lipschitz boundaries

In our transmission problems, we are concerned with the behavior of our solutions on the boundary of our domains. Since we are dealing with fields in Sobolev spaces, we have to generalize the concept of restriction to the boundary (from the classical case). To this end, we employ the Sobolev spaces on the boundary.

In order to discuss these spaces, let us proceed by considering a bounded Lipschitz domain $\mathsf{D} \subset \mathbb{R}^n$, $n \geq 2$ and denote its boundary by Γ. Let us consider $(A_i)_{i=\overline{1,m}}$ a finite covering of Γ. Let $(a_i)_{i=\overline{1,m}}$ be a partition of unity, that is, a family of functions $a_i \in \mathcal{D}(\mathbb{R}^n)$, $a_i : \mathbb{R}^n \to [0,1]$, $i = \overline{1,m}$ which have compact support supp $(a_i) \subset A_i$, satisfying

$$\sum_{i=1}^{m} a_i(x) = 1,$$

for all x in an n-dimensional open neighborhood of Γ. Since we deal with Lipschitz domains, the tangent vector and the unit surface measure $\mathrm{d}\sigma_y$ exist for a.a. $y \in \Gamma \cap A_i$. We define the surface integral of g on Γ by

$$\int_{\Gamma} g \mathrm{d}\sigma_y = \sum_{i=1}^{m} \int_{\Gamma \cap A_i} g a_i \mathrm{d}\sigma_y. \tag{1.1.36}$$

Let us take into account the local representation of Γ in order to determine that the integrals in the right hand side of relation (1.1.36) are reduced to integrals over a subset of $\mathbb{R}^n$. Consequently, this definition is independent of the chosen partition of the unity and also independent of the local coordinate representation of Γ.

Let us now define the space $L^2(\Gamma)$ of (equivalence classes of) square-power integrable functions on Γ as the completion of the space $\mathcal{C}^0(\Gamma)$ with respect to the norm

$$||g||_{L^2(\Gamma)} := \left(\int_{\Gamma} |g(y)|^2 \mathrm{d}\sigma \right)^{\frac{1}{2}}.$$

Let $s \in (0,1)$. Define the boundary Sobolev space $H^s(\Gamma)$ as the completion of the space

$$\mathcal{C}_2^0 := \{ f \in \mathcal{C}^0(\Gamma) \mid ||f||_{H^s(\Gamma)} < \infty \},$$

with respect to the norm

$$||f||_{H^s(\Gamma)} := \left\{ ||f||^2_{L^2(\Gamma)} + \int_{\Gamma} \int_{\Gamma} \frac{|f(x) - f(y)|^2}{|x - y|^{n-1+2s}} \mathrm{d}\sigma_x \mathrm{d}\sigma_y \right\}^{\frac{1}{2}}.$$

Let us conclude this part by taking into account that, for $s \in (-1,0)$, we define the Sobolev spaces of negative index by duality, that is, $H^{-s}(\Gamma) = (H^s(\Gamma))'$. As usual, we have $H^0(\Gamma) = L^2(\Gamma)$. The vector-valued versions of the spaces introduced in the former are defined component-wise.

1.1.4 Weighted Sobolev spaces

In this subsection, we will consider the setting provided by Assumption 1.1.6 in the case $n = 3$. We point out that, in this particular case, we work with an exterior (or complementary) Lipschitz domain D_- in $\mathbb{R}^3$. This fact brings an issue to the forefront. Some of our considered transmission problems contain the Stokes system in this complementary Lipschitz domain D_-. Our purpose will be that of taking into account the behavior at infinity of the solutions of our studied boundary value problems. As such, the behavior of these solutions must be included in the spaces that will be used in our analysis and this can be done in terms of weights. Hence, in the setting of $\mathbb{R}^3$, we introduce the weighted Sobolev spaces, as in the work of Hanouzet (see [65]).

Let Assumption 1.1.6 be satisfied for $n = 3$. Let us consider the weight function

$$\rho(\mathbf{x}) := (1 + |\mathbf{x}|)^{\frac{1}{2}}, \text{ for } \mathbf{x} \in \mathbb{R}^3.$$

We introduce the weighted Lebesgue space

$$L^2(\rho^{-1};\mathsf{D}) := \{f : \mathsf{D}_- \to \mathbb{R} \mid \rho^{-1} f \in L^2(\mathsf{D}_-)\}$$

and with its help, we are able to define the weighted Sobolev space

$$\mathcal{H}^1(\mathsf{D}_-) := \{f \in \mathcal{D}(\mathsf{D}_-) \mid \rho^{-1} f \in L^2(\mathsf{D}_-), \nabla f \in L^2(\mathsf{D}_-)^3\},$$

where the vector-function space $L^2(\mathsf{D}_-)^3$ can be described component-wise (as in (1.1.7)). The weighted Sobolev space $\mathcal{H}^1(\mathsf{D}_-)$ is a Hilbert space with respect to the norm

$$||f||_{\mathcal{H}^1(\mathsf{D}_-)} := \left(||\rho^{-1} f||^2_{L^2(\mathsf{D}_-)} + ||\nabla f||^2_{L^2(\mathsf{D})^3}\right)^{\frac{1}{2}}. \tag{1.1.37}$$

Let us introduce also the space

$$\tilde{\mathcal{H}}^1(\mathsf{D}_-) \text{ as the closure of } \mathcal{D}(\mathsf{D}_-) \text{ in } \mathcal{H}^1(\mathbb{R}^3).$$

We introduce the spaces

$$\mathcal{H}^{-1}(\mathsf{D}_-) = (\tilde{\mathcal{H}}^1(\mathsf{D}_-))', \quad \tilde{\mathcal{H}}^{-1}(\mathsf{D}_-) = (\mathcal{H}^1(\mathsf{D}_-))'.$$

Let us remark that $\mathcal{D}(\mathsf{D}_-)$ is dense in the space $\mathcal{H}^1(\mathsf{D}_-)$ and the space $\mathcal{D}(\overline{\mathsf{D}}_-)$ is dense in $\tilde{\mathcal{H}}^1(\mathsf{D}_-)$. In view of the fact that the seminorm

$$|g|_{\mathcal{H}^1(\mathsf{D}_-)} := ||\nabla g||_{L^2(\mathsf{D}_-)^3}$$

is equivalent to the norm (1.1.37) and by the Sobolev inequality (see [1, Theorem 4.31]) we have the embedding

$$\mathcal{H}^1(\mathsf{D}_-) \hookrightarrow L^6(\mathsf{D}_-).$$

Note that the vector-value weighted Sobolev spaces $\mathcal{H}^1(\mathsf{D}_-)^3$ and $\tilde{\mathcal{H}}^{-1}(\mathsf{D}_-)^3$ are given by

$$\begin{aligned} \mathcal{H}^1(\mathsf{D}_-)^3 &:= \{\mathbf{u} : \mathsf{D}_- \to \mathbb{R} \mid \mathbf{u} = (u_1, u_2, u_3), u_i \in \mathcal{H}^1(\mathsf{D}_-), i = \overline{1,3}\}, \\ \tilde{\mathcal{H}}^{-1}(\mathsf{D}_-)^3 &:= \{\mathbf{u} : \mathsf{D}_- \to \mathbb{R} \mid \mathbf{u} = (u_1, u_2, u_3), u_i \in \tilde{\mathcal{H}}^{-1}(\mathsf{D}_-), i = \overline{1,3}\}. \end{aligned} \tag{1.1.38}$$

Finally, let us describe the notion of a function that tends to a constant at infinity in the sense of Leray and a particular result. These concepts will be used in the following chapters (see, e.g., [71, Definition 2.3 and Corollary 2.4] and the references therein).

Definition 1.1.16. *A function $\mathbf{u}$ tends to a constant $\mathbf{u}_\infty$ at ∞, in the sense of Leray if*

$$\lim_{r\to\infty} \int_{\mathcal{S}^2} |\mathbf{u}(ry) - \mathbf{u}_\infty| \mathrm{d}\sigma_y = 0,$$

where $\mathcal{S}^2$ denotes the unit sphere in $\mathbb{R}^3$.

Corollary 1.1.17. *If $u \in \mathcal{H}^1(\mathsf{D}_-)$, then u tends to zero at ∞ in the sense of Leray.*

1.1.5 The trace operator on Sobolev spaces

In this subsection, our aim is to introduce an operator which appears in the boundary conditions of our transmission-type problems that are studied in this work.

The connection between the Sobolev spaces defined on Lipschitz domains and the Sobolev spaces defined on Lipschitz boundaries is given by the following result known as the Gagliardo Trace Lemma (see, e.g., [33], [52], [69, Proposition 3.3], [101, Lemma 2.6]).

Lemma 1.1.18. *(The Gagliardo Trace Lemma) Let Assumption 1.1.6 be satisfied. Then, there exist linear and bounded operators*

$$\mathrm{Tr}_{\mathsf{D}_\pm} : H^1(\mathsf{D}_\pm) \to H^{\frac{1}{2}}(\Gamma), \tag{1.1.39}$$

called the (Gagliardo) trace operators, such that

$$\mathrm{Tr}_{\mathsf{D}_\pm} v = v|_\Gamma, \tag{1.1.40}$$

for all $v \in \mathcal{D}(\overline{\mathsf{D}}_\pm)$. *Moreover, these operators are surjective and have (non-unique) linear and bounded right inverse operators*

$$\mathrm{Tr}^{-1}_{\mathsf{D}_\pm} : H^{\frac{1}{2}}(\Gamma) \to H^1(\mathsf{D}_\pm), \tag{1.1.41}$$

that is $\mathrm{Tr}_{\mathsf{D}_\pm} \circ \mathrm{Tr}^{-1}_{\mathsf{D}_\pm} = \mathbb{I}$.

We end this subsection by pointing out some useful remarks.

Remark 1.1.19. *Similar to Lemma 1.1.18, one can define the exterior trace operator on the weighted Sobolev space* $\mathcal{H}^1(\mathsf{D}_-)$, *that is,* $\mathrm{Tr}_{\mathsf{D}_-} : \mathcal{H}^1(\mathsf{D}_-) \to H^{\frac{1}{2}}(\Gamma)$ *(for additional details, see, e.g., [101, Theorem 2.3, Lemma 2.6], [71, Lemma 2.2]).*

Remark 1.1.20. *Lemma 1.1.18 holds also in the case of vector-valued and matrix-valued functions. For the sake of brevity, we keep the notations* $\mathrm{Tr}_{\mathsf{D}_\pm}$ *and* $\mathrm{Tr}^{-1}_{\mathsf{D}_\pm}$ *in the setting of vector-valued or matrix-valued functions.*

1.2 The Stokes, classical Brinkman and generalized Brinkman operators

In this section we will discuss the operators that appear in this work. These operators are involved in the transmission problems that we study. Recall the $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing functions introduced in relation (1.1.27) and recall that its dual, denoted by $\mathcal{S}'(\mathbb{R}^n)$, is the space of tempered distributions. The vector function spaces $\mathcal{S}(\mathbb{R}^n)^n$ and $\mathcal{S}'(\mathbb{R}^n)^n$ are given by relation (1.1.28).

The *Stokes operator* is given by

$$\mathbb{S} := \begin{bmatrix} \Delta & -\nabla \\ \mathrm{div} & 0 \end{bmatrix} : \mathcal{S}(\mathbb{R}^n)^n \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)^n \times \mathcal{S}(\mathbb{R}^n) \tag{1.2.1}$$

and the operator

$$\mathsf{L}_0 : \mathcal{S}(\mathbb{R}^n)^n \times \mathcal{S}(\mathbb{R}^n)^n \to \mathcal{S}(\mathbb{R}^n)^n, \quad \mathsf{L}_0(\mathbf{v}, p) := \Delta \mathbf{v} - \nabla p. \tag{1.2.2}$$

Let us note that the operator $\mathbb{S}$ introduced in relation (1.2.1) is Agmon-Douglis-Nirenberg elliptic (see Remark A.6 and see also [68], [143]) and this operator $\mathbb{S}$ together with the operator L_0 can be extended to linear and bounded operators, that is,

$$\mathbb{S} : H^1(\mathbb{R}^n)^n \times L^2(\mathbb{R}^n)^n \to H^{-1}(\mathbb{R}^n)^n \times L^2(\mathbb{R}^n), \quad \mathsf{L}_0 : H^1(\mathbb{R}^n)^n \times L^2(\mathbb{R}^n)^n \to H^{-1}(\mathbb{R}^n)^n. \tag{1.2.3}$$

Let $\alpha > 0$ be a given constant. Let us introduce also the *Brinkman operator* as follows

$$\mathcal{B}_\alpha := \begin{bmatrix} (\Delta - \alpha\mathbb{I}) & -\nabla \\ \mathrm{div} & 0 \end{bmatrix} : \mathcal{S}(\mathbb{R}^n)^n \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)^n \times \mathcal{S}(\mathbb{R}^n) \tag{1.2.4}$$

and it associated operator

$$\mathsf{L}_\alpha : \mathcal{S}(\mathbb{R}^n)^n \times \mathcal{S}(\mathbb{R}^n)^n \to \mathcal{S}(\mathbb{R}^n)^n, \quad \mathsf{L}_\alpha(\mathbf{v}, p) := (\Delta - \alpha\mathbb{I})\mathbf{v} - \nabla p. \tag{1.2.5}$$

The operator B_α introduced in relation (1.2.4) is Agmon-Douglis-Nirenberg elliptic (see Remark A.6 and see also [68], [143]) and together with its associated operator L_α are extended to linear and bounded operators, as follows

$$\mathcal{B}_\alpha : H^1(\mathbb{R}^n)^n \times L^2(\mathbb{R}^n)^n \to H^{-1}(\mathbb{R}^n)^n \times L^2(\mathbb{R}^n), \quad \mathsf{L}_\alpha : H^1(\mathbb{R}^n)^n \times L^2(\mathbb{R}^n)^n \to H^{-1}(\mathbb{R}^n)^n. \tag{1.2.6}$$

Finally, we address some notations that we will employ from now on.

Notation 1.2.1. *Consider the spaces of divergence free vector fields*

$$H^1_{\mathrm{div}}(\mathsf{D})^n = \{\mathbf{u} \in H^1(\mathsf{D})^n \mid \mathrm{div}\ \mathbf{u} = 0 \ \mathit{in}\ \mathsf{D}\}, \tag{1.2.7}$$

and

$$\mathcal{H}^1_{\mathrm{div}}(\mathsf{D}_-)^3 := \{\mathbf{u} \in \mathcal{H}^1(\mathsf{D}_-)^3 \mid \mathrm{div}\ \mathbf{u} = 0 \ \mathit{in}\ \mathsf{D}_-\}. \tag{1.2.8}$$

Notation 1.2.2. *Throughout this work, we introduce the operator $\mathring{\mathsf{E}}_\pm$, which represents the extension by zero operator outside $\mathsf{D}_\pm$. More specifically, it allows us to extend functions from $\mathring{H}^1(\mathsf{D}_\pm)$ by zero to $\mathbb{R}^n \setminus \mathsf{D}_\pm$. We keep the same notation $\mathring{\mathsf{E}}_\pm$ in the case of vector-valued spaces.*

1.2.1 The conormal derivative operator associated to the Stokes and Brinkman systems

This subsection is dedicated to the introduction of the conormal derivative operators associated to the Stokes and Brinkman systems. We discuss the classical derivative operator and the generalized conormal derivative operator associated for these systems. In the latter, let $\mathsf{D} \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain with connected boundary Γ.

We will introduce the classical conormal derivative operator as follows. For a pair $(\mathbf{v}, p) \in C^1(\overline{\mathsf{D}}_\pm)^n \times C^0(\overline{\mathsf{D}}_\pm)$ satisfying div $\mathbf{v} = 0$ in $\mathsf{D}_\pm$ we have that classical derivative operator (or traction field) associated to the Stokes or Brinkman operator is provided by the constitutive equation of the Newtonian (viscous) incompressible fluid, i.e.,

$$\mathbf{t}^\pm(\mathbf{v}, p) := \mathrm{Tr}_{\mathsf{D}}\boldsymbol{\sigma}(\mathbf{v}, p)\boldsymbol{\nu}, \tag{1.2.9}$$

where

$$\boldsymbol{\sigma}(\mathbf{v}, p) := -p\mathbb{I} + 2\mathbb{E}(\mathbf{v}) \tag{1.2.10}$$

is the stress tensor and $\mathbb{E}(\mathbf{v})$ is the symmetric part of $\nabla\mathbf{v}$, that is $\mathbb{E}(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^t)$, where the superscript t denotes the transpose. The symbol $\boldsymbol{\nu}$ represents the outward unit normal to D, which is defined a.e. on Γ.

Note that, for $\boldsymbol{\phi} \in \mathcal{D}(\mathbb{R}^n)^n$, we have the following Green identity for the Brinkman system,

$$\pm\langle \mathbf{t}_\alpha^\pm(\mathbf{v},p), \boldsymbol{\phi}\rangle_\Gamma = 2\langle \mathbb{E}(\mathbf{v}), \mathbb{E}(\boldsymbol{\phi})\rangle_{\mathsf{D}_\pm} + \alpha\langle \mathbf{v}, \boldsymbol{\phi}\rangle_{\mathsf{D}_\pm} - \langle p, \operatorname{div}\boldsymbol{\phi}\rangle_{\mathsf{D}_\pm} + \langle \mathsf{L}_\alpha(\mathbf{v},p), \boldsymbol{\phi}\rangle_{\mathsf{D}_\pm}, \tag{1.2.11}$$

where $\alpha > 0$ is a given constant. In particular, for $\alpha = 0$, we obtain the Green identity for the Stokes system,

$$\pm\langle \mathbf{t}^\pm(\mathbf{v},p), \boldsymbol{\phi}\rangle_\Gamma = 2\langle \mathbb{E}(\mathbf{v}), \mathbb{E}(\boldsymbol{\phi})\rangle_{\mathsf{D}_\pm} - \langle p, \operatorname{div}\boldsymbol{\phi}\rangle_{\mathsf{D}_\pm} + \langle \mathsf{L}_0(\mathbf{v},p), \boldsymbol{\phi}\rangle_{\mathsf{D}_\pm}. \tag{1.2.12}$$

Formulas (1.2.11) and (1.2.12) follow after repeated integration by parts.

Formula (1.2.12) suggests the definition of the generalized conormal derivative operator associated to the Stokes system, and the corresponding Green formula in the setting of Sobolev spaces (see, e.g., [114, Theorem 10.4.1], [33, Lemma 3.2], [101, Defintion 3.1, Theorem 3.2]).

Definition 1.2.3. *Let* $\mathsf{D}_+ := \mathsf{D} \subset \mathbb{R}^n$*, be a bounded Lipschitz domain and let* $\mathsf{D}_- := \mathbb{R}^n \setminus \overline{\mathsf{D}}$*. Define the space* $\boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_0)$ *by*

$$\boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_0) := \{(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm) \in H^1(\mathsf{D}_\pm)^n \times L^2(\mathsf{D}_\pm) \times \tilde{H}^{-1}(\mathsf{D}_\pm)^n : \mathsf{L}_0(\mathbf{v}_\pm, p_\pm) = \mathbf{g}_\pm|_{\mathsf{D}_\pm} \text{ and } \operatorname{div}\mathbf{v}_\pm = 0 \text{ in } \mathsf{D}_\pm\}.$$

Then, the generalized conormal derivative operators $\mathbf{t}_{\mathsf{D}_\pm}$ *for the Stokes system in* $\mathsf{D}_\pm$ *are defined on each* $(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm) \in \boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_0)$ *by the following relation:*

$$\begin{aligned}\pm\langle \mathbf{t}_{\mathsf{D}_\pm}(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm), \boldsymbol{\phi}\rangle_\Gamma := {}& 2\langle \mathbb{E}(\mathbf{v}_\pm), \mathbb{E}(\mathrm{Tr}_{\mathsf{D}_\pm}^{-1}\boldsymbol{\phi})\rangle_{\mathsf{D}_\pm} - \langle p_\pm, \operatorname{div}(\mathrm{Tr}_{\mathsf{D}_\pm}^{-1}\boldsymbol{\phi})\rangle_{\mathsf{D}_\pm} \\ &+\langle \mathbf{g}_\pm, \mathrm{Tr}_{\mathsf{D}_\pm}^{-1}\boldsymbol{\phi}\rangle_{\mathsf{D}_\pm}, \forall\, \boldsymbol{\phi} \in H^{\frac{1}{2}}(\Gamma)^n.\end{aligned} \tag{1.2.13}$$

Lemma 1.2.4. *In the setting of Definition* 1.2.3*, the generalized conormal derivative operators*

$$\mathbf{t}_{\mathsf{D}_\pm} : \boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_0) \to H^{-\frac{1}{2}}(\Gamma)^n \tag{1.2.14}$$

are linear and bounded and Definition 1.2.3 *is independent of the choice of a right inverse* $\mathrm{Tr}_{\mathsf{D}_\pm}^{-1} : H^{\frac{1}{2}}(\Gamma)^n \to H^1(\mathsf{D}_+)^n$ *of the trace operator* $\mathrm{Tr}_{\mathsf{D}_\pm} : H^1(\mathsf{D}_+)^n \to H^{\frac{1}{2}}(\Gamma)^n$*. Moreover, the following Green formulas hold*

$$\begin{aligned}\pm\langle \mathbf{t}_{\mathsf{D}_\pm}(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm), \mathrm{Tr}_{\mathsf{D}_\pm}\boldsymbol{\psi}_\pm\rangle_\Gamma := {}& 2\langle \mathbb{E}(\mathbf{v}_\pm), \mathbb{E}(\boldsymbol{\psi}_\pm)\rangle_{\mathsf{D}_\pm} - \langle p_\pm, \operatorname{div}\boldsymbol{\psi}_\pm\rangle_{\mathsf{D}_\pm} \\ &+\langle \mathbf{g}_\pm, \boldsymbol{\psi}_\pm\rangle_{\mathsf{D}_\pm},\end{aligned} \tag{1.2.15}$$

for all $(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm) \in \boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_0)$ *and for any* $\boldsymbol{\psi}_\pm \in H^1(\mathsf{D}_\pm)^n$.

Similarly, formula (1.2.11) suggests the definition of the generalized conormal derivative operator associated to the Brinkman system, (see, e.g., [33, Lemma 3.2], [75, Lemma 2.2], [71, Lemma 2.5]).

Definition 1.2.5. *Let* $\mathsf{D}_+ := \mathsf{D} \subset \mathbb{R}^n$*, be a bounded Lipschitz domain and let* $\mathsf{D}_- := \mathbb{R}^n \setminus \overline{\mathsf{D}}$*. Define the space* $\boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_\alpha)$ *by*

$$\boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_\alpha) := \{(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm) \in H^1(\mathsf{D}_\pm)^n \times L^2(\mathsf{D}_\pm) \times \tilde{H}^{-1}(\mathsf{D}_\pm)^n : \mathsf{L}_\alpha(\mathbf{v}_\pm, p_\pm) = \mathbf{g}_\pm|_{\mathsf{D}_\pm} \text{ and } \operatorname{div}\mathbf{v}_\pm = 0 \text{ in } \mathsf{D}_\pm\}.$$

Then, the generalized conormal derivative operators $\mathbf{t}_{\alpha,\mathsf{D}_\pm}$ *for the Brinkman system in* $\mathsf{D}_\pm$ *are defined on each* $(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm) \in \boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_\alpha)$ *by the following relation:*

$$\begin{aligned}\pm\langle \mathbf{t}_{\alpha,\mathsf{D}_\pm}(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm), \boldsymbol{\phi}\rangle_\Gamma := {}& 2\langle \mathbb{E}(\mathbf{v}_\pm), \mathbb{E}(\mathrm{Tr}_{\mathsf{D}_\pm}^{-1}\boldsymbol{\phi})\rangle_{\mathsf{D}_\pm} + \alpha\langle \mathbf{v}_\pm, \mathrm{Tr}_{\mathsf{D}_\pm}^{-1}\boldsymbol{\phi}\rangle_{\mathsf{D}_\pm} \\ &-\langle p_\pm, \operatorname{div}(\mathrm{Tr}_{\mathsf{D}_\pm}^{-1}\boldsymbol{\phi})\rangle_{\mathsf{D}_\pm} + \langle \mathbf{g}_\pm, \mathrm{Tr}_{\mathsf{D}_\pm}^{-1}\boldsymbol{\phi}\rangle_{\mathsf{D}_\pm}, \forall\, \boldsymbol{\phi} \in H^{\frac{1}{2}}(\Gamma)^n.\end{aligned} \tag{1.2.16}$$

Lemma 1.2.6. *In the setting of Definition* 1.2.5, *the generalized conormal derivative operators*

$$\mathbf{t}_{\alpha,\mathsf{D}_\pm} : \boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_\alpha) \to H^{-\frac{1}{2}}(\Gamma)^n \tag{1.2.17}$$

are linear and bounded and Definition 1.2.5 *is independent of the choice of a right inverse* $\mathrm{Tr}_{\mathsf{D}_\pm}^{-1} : H^{\frac{1}{2}}(\Gamma)^n \to H^1(\mathsf{D}_+)^n$ *of the trace operator* $\mathrm{Tr}_{\mathsf{D}_\pm} : H^1(\mathsf{D}_+)^n \to H^{\frac{1}{2}}(\Gamma)^n$. *Moreover, the following Green formulas hold*

$$\begin{aligned}\pm\langle \mathbf{t}_{\alpha,\mathsf{D}_\pm}(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm), \mathrm{Tr}_{\mathsf{D}_\pm}\boldsymbol{\psi}_\pm\rangle_\Gamma := \ & 2\langle \mathbb{E}(\mathbf{v}_\pm), \mathbb{E}(\boldsymbol{\psi}_\pm)\rangle_{\mathsf{D}_\pm} + \alpha\langle \mathbf{v}_\pm, \boldsymbol{\psi}_\pm\rangle_{\mathsf{D}_\pm} \\ & -\langle p_\pm, \mathrm{div}\ \boldsymbol{\psi}_\pm\rangle_{\mathsf{D}_\pm} + \langle \mathbf{g}_\pm, \boldsymbol{\psi}_\pm\rangle_{\mathsf{D}_\pm},\end{aligned} \tag{1.2.18}$$

for all $(\mathbf{v}_\pm, p_\pm, \mathbf{g}_\pm) \in \boldsymbol{H}^1(\mathsf{D}_\pm, \mathsf{L}_\alpha)$ *and for any* $\boldsymbol{\psi}_\pm \in H^1(\mathsf{D}_\pm)^n$.

Let us end this subsection by pointing out some useful remarks (see also [71, Remark 2.6, Lemma 2.9], [75, Remark 2.4]).

Remark 1.2.7. *For* $\alpha = 0$, *the conormal derivative for the Brinkman system (see Definition 1.2.5) reduces to the conormal derivative for the Stokes system (see Definition 1.2.3).*

Remark 1.2.8. *Let* $\mathsf{D}_+ := \mathsf{D} \subset \mathbb{R}^3$, *be a bounded Lipschitz domain and let* $\mathsf{D}_- := \mathbb{R}^3 \setminus \overline{\mathsf{D}}$. *For* $(\mathbf{v}_-, p_-, \mathbf{g}_-) \in \mathcal{H}^1(\mathsf{D}_-)^3 \times L^2(\mathsf{D}_-) \times \tilde{\mathcal{H}}^{-1}(\mathsf{D}_-)^3$ *satisfying* $\mathsf{L}_0(\mathbf{v}_-, p_-) = \mathbf{g}_-|_{\mathsf{D}_-}$, *the conormal derivative operator* $\mathbf{t}_{\mathsf{D}_-}(\mathbf{v}_-, p_-, \mathbf{g}_-)$ *is well-defined by relation (1.2.13) and a corresponding Green formula similar to relation (1.2.15) holds true in* D_-.

Remark 1.2.9. *Let* $\mathsf{D}_+ := \mathsf{D} \subset \mathbb{R}^3$, *be a bounded Lipschitz domain and let* $\mathsf{D}_- := \mathbb{R}^3 \setminus \overline{\mathsf{D}}$. *Then for* $(\mathbf{v}_-, p_-, \mathbf{g}_-) \in H^1(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-) \times \tilde{H}^{-1}(\mathsf{D}_-)^3$, *such that* $\mathsf{L}_\alpha(\mathbf{v}_-, p_-) = \mathbf{g}_-|_{\mathsf{D}_-}$, *the conormal derivative* $\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{v}_-, p_-, \mathbf{g}_-)$ *is well-defined by relation (1.2.16). In addition, in this case, the Green formula (1.2.18) also holds, in* D_-. *The space* $\mathfrak{M}(\mathsf{D}_-)$ *is provided by Definition 2.1.1.*

Remark 1.2.10. *Let* $\mathsf{D} \subset \mathbb{R}^n$, $n \geq 2$, *be a bounded Lipschitz domain and denote its boundary by* Γ. *In the case* $\Gamma = \Gamma_1 \cup \Gamma_2$, *where* Γ_1 *and* Γ_2 *are connected components of* Γ *such that* $\Gamma_1 \cap \Gamma_2 = \emptyset$, *we define the operator*

$$(\mathbf{t}_{\alpha,\mathsf{D}}(\cdot,\cdot,\cdot))|_{\Gamma_1} : \boldsymbol{H}^1(\mathsf{D}, \mathsf{L}_\alpha) \to H^{-\frac{1}{2}}(\Gamma_1)^n, \tag{1.2.19}$$

by the relation

$$\langle \mathbf{t}_{\alpha,\mathsf{D}}(\mathbf{v}, p, \mathbf{g})|_{\Gamma_1}, \boldsymbol{\Phi}\rangle_{\Gamma_1} := \langle \mathbf{t}_{\alpha,\mathsf{D}}(\mathbf{v}, p, \mathbf{g}), \boldsymbol{\Phi}\rangle_\Gamma, \tag{1.2.20}$$

for all $\boldsymbol{\Phi} \in C^\infty(\mathbb{R}^n)^n$ *which vanish in an open neighborhood of* Γ_2.

Remark 1.2.11. *We will write* $\mathbf{t}_{\alpha,\mathsf{D}}(\mathbf{v}, p)$ *instead of* $\mathbf{t}_{\alpha,\mathsf{D}}(\mathbf{v}, p, \mathbf{0})$.

1.2.2 The generalized Brinkman system and related results

In this monograph, we consider a generalized type Brinkman system. Indeed, the term $\alpha\mathbb{I}$ which appears in the classical Brinkman operator (see relations (1.2.4) and (1.2.5)) has been replaced by another, much more general term. Part of the original results that are included in this book are transmission problem in which this generalized version of the Brinkman system is involved. More recently, this generalized type Brinkman system has also been treated in the much more general setting of variable coefficient PDE systems (see, e.g., [77], [78], [79], [86]).

Hence, for the introduction of this generalized version of the Brinkman system, we consider a bounded Lipschitz domain $\mathsf{D} \subseteq \mathbb{R}^3$. The generalized Brinkman system is given by

$$\mathsf{L}_{\mathcal{P}}(\mathbf{v}, p) := \Delta\mathbf{v} - \mathcal{P}\mathbf{v} - \nabla p = \mathbf{g} \text{ in } \mathsf{D},\ \mathrm{div}\ \mathbf{v} = 0 \text{ in } \mathsf{D}, \tag{1.2.21}$$

where $\mathcal{P} \in L^\infty(\mathsf{D})^{3\times 3}$ such that $\mathcal{P}$ satisfies the following non-negativity condition

$$\langle \mathcal{P}\mathbf{v}, \mathbf{v}\rangle_{\mathsf{D}} \geq c_{\mathcal{P}} \|\mathbf{v}\|^2_{L^2(\mathsf{D})^3}, \ \forall\ \mathbf{v} \in L^2(\mathsf{D})^3, \tag{1.2.22}$$

where $c_{\mathcal{P}} > 0$ is a constant.

The system (1.2.21) is viewed in a distributional sense, that is, for $(\mathbf{v}, p) \in H^1(\mathsf{D})^3 \times L^2(\mathsf{D})$, we have

$$\langle \mathsf{L}_{\mathcal{P}}(\mathbf{v}, p), \boldsymbol{\psi}\rangle_{D_+} = \langle \mathbf{g}, \boldsymbol{\psi}\rangle_{\mathsf{D}},\ \langle \text{div } \mathbf{v}, g_0\rangle_{\mathsf{D}} = 0, \tag{1.2.23}$$

for all $(\boldsymbol{\psi}, g_0) \in \mathcal{D}(\mathsf{D})^3 \times \mathcal{D}(\mathsf{D})$, where

$$\langle \mathsf{L}_{\mathcal{P}}(\mathbf{v}, p), \boldsymbol{\psi}\rangle_{\mathsf{D}} := \langle \Delta\mathbf{v} - \mathcal{P}\mathbf{v} - \nabla p, \boldsymbol{\psi}\rangle_{\mathsf{D}} \ = -\langle \nabla\mathbf{v}, \nabla\boldsymbol{\psi}\rangle_{\mathsf{D}} - \langle \mathcal{P}\mathbf{v}, \boldsymbol{\psi}\rangle_{\mathsf{D}} + \langle p, \text{div } \boldsymbol{\psi}\rangle_{\mathsf{D}}.$$

Also, the continuous embedding $L^2(\mathsf{D}) \hookrightarrow H^{-1}(\mathsf{D})$ implies the linearity and boundedness of the operator

$$\mathsf{L}_{\mathcal{P}} : H^1(\mathsf{D})^3 \times L^2(\mathsf{D}) \to H^{-1}(\mathsf{D})^3 = (\mathring{H}^1(\mathsf{D})^3)'. \tag{1.2.24}$$

Note that, we are able to extract from this generalized version of the Brinkman system the classical Stokes or Brinkman systems, respectively. This fact is emphasized in the following remarks.

Remark 1.2.12. *For $\mathcal{P} \equiv 0$, the system (1.2.21) is the classical Stokes system.*

Remark 1.2.13. *For $\mathcal{P} \equiv \alpha\mathbb{I}$, where $\alpha > 0$ is a constant, the system (1.2.21) is the classical Brinkman system.*

For this generalized version of the Brinkman system, we introduce its associated conormal derivative operator (see, e.g., [7, Lemma 2.4]).

Definition 1.2.14. *Let $\mathsf{D}_+ := \mathsf{D} \subset \mathbb{R}^3$, be a bounded Lipschitz domain and denote its boundary by Γ. Let $\mathcal{P} \in L^\infty(\mathsf{D}_+)^{3\times 3}$ such that condition (1.2.22) is satisfied. Define the space $\boldsymbol{H}^1(\mathsf{D}_+, \mathsf{L}_{\mathcal{P}})$ by*

$$\boldsymbol{H}^1(\mathsf{D}_+, \mathsf{L}_{\mathcal{P}}) := \{(\mathbf{v}_+, p_+, \mathbf{g}_+) \in H^1(\mathsf{D}_+)^3 \times L^2(\mathsf{D}_+) \times \tilde{H}^{-1}(\mathsf{D}_+)^3 : \mathsf{L}_{\mathcal{P}}(\boldsymbol{v}_+, p_+) = \mathbf{g}_+|_{\mathsf{D}_+} \\ \text{and div } \mathbf{v}_+ = 0 \text{ in } \mathsf{D}_+\}.$$

Then, the conormal derivative operator

$$\mathbf{t}_{\mathcal{P},\mathsf{D}_+} : \boldsymbol{H}^1(\mathsf{D}_+, \mathsf{L}_{\mathcal{P}}) \to H^{-\frac{1}{2}}(\Gamma)^3 \tag{1.2.25}$$

for the generalized Brinkman system in D_+ is defined on each $(\mathbf{v}_+, p_+, \mathbf{g}_+) \in \boldsymbol{H}^1(\mathsf{D}_+, \mathsf{L}_{\mathcal{P}})$ by the following relation:

$$\begin{aligned}\langle \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{v}_+, p_+, \mathbf{g}_+), \boldsymbol{\phi}\rangle_\Gamma := {} & 2\langle \mathbb{E}(\mathbf{v}_+), \mathbb{E}(\text{Tr}^{-1}_{\mathsf{D}_+}\boldsymbol{\phi})\rangle_{\mathsf{D}_+} + \langle \mathcal{P}\mathbf{v}_+, \text{Tr}^{-1}_{\mathsf{D}_+}\boldsymbol{\phi}\rangle_{\mathsf{D}_+} \\ & - \langle p_+, \text{div } (\text{Tr}^{-1}_{\mathsf{D}_+}\boldsymbol{\phi})\rangle_{\mathsf{D}_+} + \langle \mathbf{g}_+, \text{Tr}^{-1}_{\mathsf{D}_+}\boldsymbol{\phi}\rangle_{\mathsf{D}_+}, \forall\ \boldsymbol{\phi} \in H^{\frac{1}{2}}(\Gamma)^3.\end{aligned} \tag{1.2.26}$$

Lemma 1.2.15. *In the setting of Definition* 1.2.14, *the conormal derivative operator for the generalized Brinkman system,*

$$\mathbf{t}_{\mathcal{P},\mathsf{D}_+} : \boldsymbol{H}^1(\mathsf{D}_+, \mathsf{L}_{\mathcal{P}}) \to H^{-\frac{1}{2}}(\Gamma)^3 \tag{1.2.27}$$

is linear and bounded, and Definition 1.2.14 *is independent of the choice of a right inverse $\text{Tr}^{-1}_{\mathsf{D}_+} : H^{\frac{1}{2}}(\Gamma)^3 \to H^1(\mathsf{D}_+)^3$ of the trace operator $\text{Tr}_{\mathsf{D}_+} : H^1(\mathsf{D}_+)^3 \to H^{\frac{1}{2}}(\Gamma)^3$. Moreover, the following Green formula holds*

$$\begin{aligned}\langle \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{v}_+, p_+, \mathbf{g}_+), \text{Tr}_{\mathsf{D}_+}\boldsymbol{\psi}_+\rangle_\Gamma := {} & 2\langle \mathbb{E}(\mathbf{v}_+), \mathbb{E}(\boldsymbol{\psi}_+)\rangle_{\mathsf{D}_+} + \langle \mathcal{P}\mathbf{v}_+, \boldsymbol{\psi}_+\rangle_{\mathsf{D}_+} \\ & -\langle p_+, \text{div } \boldsymbol{\psi}_+\rangle_{\mathsf{D}_+} + \langle \mathbf{g}_+, \boldsymbol{\psi}_+\rangle_{\mathsf{D}_+},\end{aligned} \tag{1.2.28}$$

for all $(\mathbf{v}_+, p_+, \mathbf{g}_+) \in \boldsymbol{H}^1(\mathsf{D}_+, \mathsf{L}_{\mathcal{P}})$ and for any $\boldsymbol{\psi}_+ \in H^1_{\text{div}}(\mathsf{D}_+)^n$.

The proof of Lemma 1.2.15 follows similar ideas to those used in the proof of Lemma 1.2.6, i.e., the special case $\mathcal{P} = \alpha\mathbb{I}$, where $\alpha > 0$ is a given constant. For additional details, we refer the reader to [75, Lemma 2.2].

Remark 1.2.16. *Taking into account the definitions of the conormal derivative operators for the Stokes and generalized Brinkman systems, given by (1.2.13) and (1.2.26), we deduce that*

$$\mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{v}, p, \mathbf{g}) = \mathbf{t}_{\mathsf{D}_+}(\mathbf{v}, p, \mathbf{g} + \mathring{\mathsf{E}}_+(\mathcal{P}\mathbf{v})), \tag{1.2.29}$$

where $\mathring{\mathsf{E}}_+$ *denotes the operator of extension by zero outside* D_+.

1.3 Stokes layer potentials and their properties

In this section we give the fundamental solution for the Stokes system in $\mathbb{R}^n$, $n \geq 2$, and with its help we define the layer operators that are involved in the solutions of our transmission problems. The sources that we used for the preparation of this section are [68], [71], [83], [114].

1.3.1 The Stokes system and its fundamental solution

Let $(\mathbf{G}(\cdot,\cdot), \mathbf{P}(\cdot,\cdot)) \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)^{n\times n} \times \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)^n$ denote the fundamental solution of the Stokes system. By $\mathbf{G}(\cdot,\cdot)$ we denote the fundamental velocity tensor and by $\mathbf{P}(\cdot,\cdot)$ we denote the fundamental pressure vector for the Stokes system in $\mathbb{R}^n$.

Note that the fundamental solution of the Stokes system satisfies the equations

$$\Delta_{\mathbf{x}}\mathbf{G}(\mathbf{x},\mathbf{y}) - \nabla_{\mathbf{x}}\mathbf{P}(\mathbf{x},\mathbf{y}) = -\delta_{\mathbf{y}}(\mathbf{x})\mathbb{I},\ \mathrm{div}_{\mathbf{x}}\mathbf{G}(\mathbf{x},\mathbf{y}) = 0, \tag{1.3.1}$$

where the symbol $\delta_{\mathbf{y}}$ denotes the Dirac distribution with mass at $\mathbf{y}$. Also, the differential operators $\Delta_{\mathbf{x}}$, $\nabla_{\mathbf{x}}$ and $\mathrm{div}_{\mathbf{x}}$ act with respect to the variable $\mathbf{x}$.

The components of the fundamental solution $(\mathbf{G}(\mathsf{G}_{jk}), \mathbf{P}(\mathsf{P}_k))$ are given by (see, e.g., [83, p. 38-39], [114, Relation (4.19), Relation (4.20), Relation (4.21)], [141])

$$\mathsf{G}_{jk}(\mathbf{x},\mathbf{y}) := \frac{1}{2\omega_n}\left\{\frac{\delta_{jk}}{(n-2)|\mathbf{y}-\mathbf{x}|^{n-2}} + \frac{x_j x_k}{|\mathbf{y}-\mathbf{x}|^n}\right\}, \quad \mathsf{P}_k(\mathbf{x},\mathbf{y}) = \frac{1}{\omega_n}\frac{x_k}{|\mathbf{y}-\mathbf{x}|^n}, \tag{1.3.2}$$

for $n \geq 3$, and

$$\mathsf{G}_{jk}(\mathbf{x},\mathbf{y}) := \frac{1}{4\pi}\left\{\frac{x_j x_k}{|\mathbf{y}-\mathbf{x}|^2} - \delta_{jk}\log|\mathbf{y}-\mathbf{x}|^{n-2}\right\}, \quad \mathsf{P}_k(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\frac{x_k}{|\mathbf{y}-\mathbf{x}|^2}, \tag{1.3.3}$$

for $n \geq 2$. Note that δ_{jk} denotes the Kronecher symbol and ω_n is the surface measure of the unit sphere $\mathcal{S}^{n-1}$ in $\mathbb{R}^n$.

Let also $\mathbf{S}(\mathsf{S}_{jkl})$ and $\mathbf{R}(\mathsf{R}_{jk})$ denote the associated stress and pressure tensors for the Stokes system. Their components are given by (see, e.g., [83, Chapter 2], [114])

$$\mathsf{S}_{jkl}(\mathbf{x},\mathbf{y}) := \frac{n}{\omega_n}\frac{x_i x_j x_k}{|\mathbf{y}-\mathbf{x}|^{n+2}}, \quad \mathsf{R}_{jk}(\mathbf{x},\mathbf{y}) := -\frac{2}{\omega_n}\left\{-\frac{\delta_{jk}}{|\mathbf{y}-\mathbf{x}|^n} + n\frac{x_j x_k}{|\mathbf{y}-\mathbf{x}|^{n+2}}\right\}, \tag{1.3.4}$$

for $n \geq 2$.

For $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} \neq \mathbf{y}$, the pair $(\mathbf{S}(\mathsf{S}_{jkl}), \mathbf{R}(\mathsf{R}_{jk}))$ satisfies the Stokes system

$$\Delta_{\mathbf{x}}\mathsf{S}_{jkl}(\mathbf{x},\mathbf{y}) - \frac{\partial \mathsf{R}_{jk}(\mathbf{y},\mathbf{x})}{\partial x_k} = 0,\ \frac{\partial \mathsf{S}_{jkl}(\mathbf{x},\mathbf{y})}{\partial x_k} = 0. \tag{1.3.5}$$

1.3.2 The volume potential for the Stokes system and its properties

The purpose of this subsection is to introduce the Newtonian (volume) potential operators associated to the Stokes system and to give their mapping properties. To this end, we consider the Lipschitz domains $\mathsf{D}_{\pm}$ as described in Assumption 1.1.6 and we will take into account the fundamental solution of the Stokes system, that is, the pair $(\mathbf{G}(\cdot,\cdot),\mathbf{P}(\cdot,\cdot)) \in \mathcal{D}'(\mathbb{R}^n\times\mathbb{R}^n)^{n\times n}\times\mathcal{D}'(\mathbb{R}^n\times\mathbb{R}^n)^n$ given by formula (1.3.2) or (1.3.3).

Definition 1.3.1. *For $\mathbf{f}\in H^{-1}(\mathbb{R}^n)^n$, define the Newtonian (volume) velocity and pressure potentials for the Stokes system, by*

$$(\mathcal{N}_{\mathbb{R}^n}\mathbf{f})(\boldsymbol{x}) := -\langle\mathbf{G}(\boldsymbol{x},\cdot),\mathbf{f}\rangle_{\mathbb{R}^n},\ (\mathcal{Q}_{\mathbb{R}^n}\mathbf{f})(\boldsymbol{x}) := -\langle\mathbf{P}(\boldsymbol{x},\cdot),\mathbf{f}\rangle_{\mathbb{R}^n}. \tag{1.3.6}$$

Moreover, the Newtonian (volume) velocity and pressure potentials for the Stokes system corresponding to $\mathsf{D}_{\pm}$, are given by

$$\mathcal{N}_{\mathsf{D}_{\pm}}\mathbf{f} := (\mathcal{N}_{\mathbb{R}^n}\mathbf{f})|_{\mathsf{D}_{\pm}},\ \mathcal{Q}_{\mathsf{D}_{\pm}}\mathbf{f} := (\mathcal{Q}_{\mathbb{R}^n}\mathbf{f})|_{\mathsf{D}_{\pm}}, \tag{1.3.7}$$

where $|_{\mathsf{D}_{\pm}}$ is the restriction operator to $\mathsf{D}_{\pm}$, which acts on vector-valued or scalar-valued functions in $\mathbb{R}^n$.

The following lemma describes the mapping properties of the Newtonian (volume) layer potential operators in the setting of Sobolev spaces (see, e.g., [71, Lemma A.3]).

Theorem 1.3.2. *The Newtonian (volume) velocity and pressure potential operators for the Stokes system, introduced in relation (1.3.6),*

$$\begin{aligned}&\mathcal{N}_{\mathbb{R}^n}: H^{-1}(\mathbb{R}^n)^n\to H^1(\mathbb{R}^n)^n,\ \mathcal{Q}_{\mathbb{R}^n}: H^{-1}(\mathbb{R}^n)^n\to L^2(\mathbb{R}^n),\\ &\mathcal{N}_{\mathbb{R}^3}:\mathcal{H}^{-1}(\mathbb{R}^3)^3\to\mathcal{H}^1(\mathbb{R}^3)^3,\ \mathcal{Q}_{\mathbb{R}^3}:\mathcal{H}^{-1}(\mathbb{R}^3)^3\to L^2(\mathbb{R}^3)\end{aligned} \tag{1.3.8}$$

are linear and continuous operators. Moreover, the Newtonian (volume) velocity and pressure potentials for the Stokes system, introduced in relation (1.3.7),

$$\mathcal{N}_{\mathsf{D}_+}:\tilde{H}^{-1}(\mathsf{D}_+)^n\to H^1(\mathsf{D}_+)^n,\ \mathcal{Q}_{\mathsf{D}_+}:\tilde{H}^{-1}(\mathsf{D}_+)^n\to L^2(\mathsf{D}_+), \tag{1.3.9}$$

and

$$\mathcal{N}_{\mathsf{D}_-}:\tilde{\mathcal{H}}^{-1}(\mathsf{D}_-)^3\to\mathcal{H}^1(\mathsf{D}_-)^3,\ \mathcal{Q}_{\mathsf{D}_-}:\tilde{\mathcal{H}}^{-1}(\mathsf{D}_-)^3\to L^2(\mathsf{D}_-), \tag{1.3.10}$$

in the case $n=3$, are linear and continuous operators as well.

Finally, by taking into account relation (1.3.1), we have that the Newtonian potentials satisfy the following equations (in the sense of distributions):

$$\Delta(\mathcal{N}_{\mathbb{R}^n}\mathbf{f})-\nabla(\mathcal{Q}_{\mathbb{R}^n}\mathbf{f})=\mathbf{f},\ \mathrm{div}\,(\mathcal{N}_{\mathbb{R}^n}\mathrm{f})=0,\ \text{in }\mathbb{R}^n, \tag{1.3.11}$$

and

$$\Delta(\mathcal{N}_{\mathsf{D}_{\pm}}\mathbf{f})-\nabla(\mathcal{Q}_{\mathsf{D}_{\pm}}\mathbf{f})=\mathbf{f},\ \mathrm{div}\,(\mathcal{N}_{\mathsf{D}_{\pm}}\mathrm{f})=0,\ \text{in }\mathsf{D}_{\pm}, \tag{1.3.12}$$

respectively.

1.3.3 Stokes layer potentials and related results

In this subsection, we concern ourselves with the single layer potential and the double layer potential operators associated to the Stokes system. Our purpose is to give their definitions, their mapping properties their jump relations and specify their behavior at infinity. From now on, let Assumption 1.1.6 be satisfied and in addition, we assume that the bounded Lipschitz domain D_+ has a connected boundary Γ.

Firstly, let us focus on the single-layer velocity and pressure potentials, associated to the Stokes system (see, e.g., [114, Relation (4.24), Relation (4.27)]).

Definition 1.3.3. *Let Assumption* 1.1.6 *be satisfied. Let* $\boldsymbol{\varphi} \in H^{-\frac{1}{2}}(\Gamma)^n$*. Define the single-layer velocity potential* $\mathbf{V}_\Gamma\boldsymbol{\varphi}$ *and its associated pressure potential* $\mathcal{Q}^s_\Gamma\boldsymbol{\varphi}$ *for the Stokes system, by*

$$(\mathbf{V}_\Gamma\boldsymbol{\varphi}) := \langle \mathbf{G}(\boldsymbol{x},\cdot), \boldsymbol{\varphi}\rangle_\Gamma,\ (\mathcal{Q}^s_\Gamma\boldsymbol{\varphi}) := \langle \mathbf{P}(\boldsymbol{x},\cdot), \boldsymbol{\varphi}\rangle_\Gamma,\ \forall\ \boldsymbol{x}\in\mathbb{R}^n\setminus\Gamma. \tag{1.3.13}$$

By taking into account relation (1.3.1), we have that the pair $(\mathbf{V}_\Gamma\boldsymbol{\varphi}, \mathcal{Q}^s_\Gamma\boldsymbol{\varphi})$ satisfies the homogeneous Stokes system

$$\Delta(\mathbf{V}_\Gamma\boldsymbol{\varphi}) - \nabla(\mathcal{Q}^s_\Gamma\boldsymbol{\varphi}) = 0, \quad \mathrm{div}\ (\mathbf{V}_\Gamma\boldsymbol{\varphi}) = 0 \tag{1.3.14}$$

in $\mathbb{R}^n \setminus \Gamma$.

The following theorem gives some useful mapping properties for the single layer potential operators associated to the Stokes system (see, e.g., [71, Lemma A.4]).

Theorem 1.3.4. *Let Assumption 1.1.6 be satisfied. Then the following operators*

$$(\mathbf{V}_\Gamma)|_{\mathsf{D}_+} : H^{-\frac{1}{2}}(\Gamma)^n \to H^1(\mathsf{D}_+)^n,\ (\mathcal{Q}^s_\Gamma)|_{\mathsf{D}_+} : H^{-\frac{1}{2}}(\Gamma)^n \to L^2(\mathsf{D}_+) \tag{1.3.15}$$

are linear and bounded. Moreover, for $n = 3$, we have that the operators

$$(\mathbf{V}_\Gamma)|_{\mathsf{D}_-} : H^{-\frac{1}{2}}(\Gamma)^3 \to \mathcal{H}^1(\mathsf{D}_-)^3,\ (\mathcal{Q}^d_\Gamma)|_{\mathsf{D}_-} : H^{-\frac{1}{2}}(\Gamma)^3 \to L^2(\mathsf{D}_-) \tag{1.3.16}$$

are linear and bounded as well, where the weighted Sobolev space $\mathcal{H}^1(\mathsf{D}_-)^3$ is given in relation (1.1.38).

Secondly, we focus on the double-layer velocity and pressure potentials, associated to the Stokes system (see, e.g., [114, Relation (4.25), Relation (4.28)]).

Definition 1.3.5. *Let Assumption* 1.1.6 *be satisfied. Let* $\boldsymbol{\phi} \in H^{\frac{1}{2}}(\Gamma)^n$*. Then, the double-layer velocity potential* $\mathbf{W}_\Gamma\boldsymbol{\phi}$ *and its associated pressure potential* $\mathcal{Q}^d_\Gamma\boldsymbol{\phi}$ *for the Stokes system are defined by*

$$\begin{aligned}(\mathbf{W}_\Gamma\boldsymbol{\phi})_k(\boldsymbol{x}) &:= \int_\Gamma S_{jkl}(\boldsymbol{y},\boldsymbol{x})\nu_l(\boldsymbol{y})\phi_j(\boldsymbol{y})d\sigma_{\boldsymbol{y}},\ \forall\ \boldsymbol{x}\in\mathbb{R}^n\setminus\Gamma,\\ (\mathcal{Q}^d_\Gamma\boldsymbol{\phi})(\boldsymbol{x}) &:= \int_\Gamma R_{jl}(\boldsymbol{x},\boldsymbol{y})\nu_l(\boldsymbol{y})\phi_j(\boldsymbol{y})d\sigma_{\boldsymbol{y}},\ \forall\ \boldsymbol{x}\in\mathbb{R}^n\setminus\Gamma,\end{aligned} \tag{1.3.17}$$

where $\boldsymbol{\nu}(\nu_l)_{l=\overline{1,n}}$ is the outward unit normal to D_+, defined a.e. on Γ.

Note that, in view of relation (1.3.5), we have that the pair $(\mathbf{W}_\Gamma\boldsymbol{\phi}, \mathcal{Q}^d_\Gamma\boldsymbol{\phi})$ satisfies the homogeneous Stokes system

$$\Delta(\mathbf{W}_\Gamma\boldsymbol{\phi}) - \nabla(\mathcal{Q}^d_\Gamma\boldsymbol{\phi}) = 0, \quad \mathrm{div}\ (\mathbf{W}_\Gamma\boldsymbol{\phi}) = 0 \tag{1.3.18}$$

in $\mathbb{R}^n \setminus \Gamma$.

In addition, let us introduce the boundary version of the Stokes double layer velocity potential in the sense of the principal value, as follows (see, e.g., [114, Relation (4.44)]).

Definition 1.3.6. *Define the principal value of* $\mathbf{W}_\Gamma\boldsymbol{\phi}$, *denoted by* $\mathbb{K}_\Gamma\boldsymbol{\phi}$ *and given by:*

$$\begin{aligned}(\mathbb{K}_\Gamma\boldsymbol{\phi})_k(\boldsymbol{x}) :&= \text{p.v.} \int_\Gamma S_{jkl}(\boldsymbol{y},\boldsymbol{x})\nu_l(\boldsymbol{y})\phi_j(\boldsymbol{y})d\sigma_{\boldsymbol{y}} \\ &= \lim_{\varepsilon\to 0}\int_{\Gamma\setminus(\Gamma\cap\overline{B}(x,\varepsilon))} S_{jkl}(\boldsymbol{y},\boldsymbol{x})\nu_l(\boldsymbol{y})\phi_j(\boldsymbol{y})d\sigma_{\boldsymbol{y}},\end{aligned} \tag{1.3.19}$$

for $\boldsymbol{x}\in\Gamma$, *where this limit makes sense.*

Also, the following result provides us with useful mapping properties of the double layer potential operators for the Stokes system (see, e.g., [71, Lemma A.4]).

Theorem 1.3.7. *Let Assumption* 1.1.6 *be satisfied. Then, the following operators*

$$(\mathbf{W}_\Gamma)|_{\mathsf{D}_+} : H^{\frac{1}{2}}(\Gamma)^n \to H^1(\mathsf{D}_+)^n,\ (\mathcal{Q}^d_\Gamma)|_{\mathsf{D}_+} : H^{\frac{1}{2}}(\Gamma)^n \to L^2(\mathsf{D}_+), \tag{1.3.20}$$

are linear and bounded. Moreover, for $n=3$, *we have that the operators*

$$(\mathbf{W}_\Gamma)|_{\mathsf{D}_-} : H^{\frac{1}{2}}(\Gamma)^3 \to \mathcal{H}^1(\mathsf{D}_-)^3,\ (\mathcal{Q}^d_\Gamma)|_{\mathsf{D}_-} : H^{\frac{1}{2}}(\Gamma)^3 \to L^2(\mathsf{D}_-) \tag{1.3.21}$$

are linear and bounded as well.

Let us also provide the lemma which describes the jump relations of the single and double layer potentials for the Stokes system, in the setting of Sobolev spaces (see, e.g., [71, Lemma A.4], [114, Proposition 4.2.2, Proposition 4.2.5, Proposition 4.2.9, Corollary 4.3.2, Theorem 5.3.6, Theorem 5.4.1]).

Lemma 1.3.8. *Let Assumption* 1.1.6 *be satisfied.*

(i) For $\boldsymbol{\varphi}\in H^{-\frac{1}{2}}(\Gamma)^n$ *and* $\boldsymbol{\phi}\in H^{\frac{1}{2}}(\Gamma)^n$, *the following jump relations*

$$\begin{aligned}&\mathrm{Tr}_{\mathsf{D}_+}(\mathbf{V}_\Gamma\boldsymbol{\varphi}) = \mathrm{Tr}_{\mathsf{D}_-}(\mathbf{V}_\Gamma\boldsymbol{\varphi}) =: \mathcal{V}_\Gamma\boldsymbol{\varphi},\\ &\mathrm{Tr}_{\mathsf{D}_\pm}(\mathbf{W}_\Gamma\boldsymbol{\phi}) = \left(\mp\frac{1}{2}\mathbb{I}+\mathbb{K}_\Gamma\right)\boldsymbol{\phi},\\ &\mathbf{t}_{\mathsf{D}_\pm}(\mathbf{V}_\Gamma\boldsymbol{\varphi},\mathcal{Q}^s_\Gamma\boldsymbol{\varphi}) = \left(\pm\frac{1}{2}\mathbb{I}+\mathbb{K}^*_\Gamma\right)\boldsymbol{\varphi},\\ &\mathbf{t}_{\mathsf{D}_+}(\mathbf{W}_\Gamma\boldsymbol{\phi},\mathcal{Q}^d_\Gamma\boldsymbol{\phi}) = \mathbf{t}_{\mathsf{D}_-}(\mathbf{W}_\Gamma\boldsymbol{\phi},\mathcal{Q}^d_\Gamma\boldsymbol{\phi}) =: \mathbb{D}_\Gamma\boldsymbol{\phi}\end{aligned} \tag{1.3.22}$$

hold a.e. on Γ, *where* $\mathbb{K}^*_\Gamma : H^{-\frac{1}{2}}(\Gamma)^n \to H^{-\frac{1}{2}}(\Gamma)^n$ *is the adjoint of the double layer potential operator* $\mathbb{K}_\Gamma : H^{\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n$.

(ii) The following Stokes layer potential operators

$$\begin{aligned}&\mathcal{V}_\Gamma : H^{-\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n,\ \mathbb{K}_\Gamma : H^{\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n,\\ &\mathbb{K}^*_\Gamma : H^{-\frac{1}{2}}(\Gamma)^n \to H^{-\frac{1}{2}}(\Gamma)^n,\ \mathbb{D}_\Gamma : H^{\frac{1}{2}}(\Gamma)^n \to H^{-\frac{1}{2}}(\Gamma)^n,\end{aligned} \tag{1.3.23}$$

are linear and bounded. Moreover, the operator $\mathcal{V}_\Gamma : H^{-\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n$ *is a Fredholm operator of index zero and its kernel, denoted by* Ker $\mathcal{V}_\Gamma$ *(see Definition B.1), is given by*

$$\mathrm{Ker}\ \{\mathcal{V}_\Gamma : H^{-\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n\} = \mathbb{R}\boldsymbol{\nu}. \tag{1.3.24}$$

We have the following useful remark.

Remark 1.3.9. *If f and g are two functions defined in a neighborhood of a point $\boldsymbol{x}$ (which could also be ∞), then*

$$f(\boldsymbol{x}) = O(g(\boldsymbol{x})) \Leftrightarrow \frac{|f(\boldsymbol{x})|}{|g(\boldsymbol{x})|} \text{ is bounded.} \tag{1.3.25}$$

Let us end this subsection by stating the following asymptotic formulas which are satisfied by the Stokes layer potential at infinity (see, e.g., [73, Relation (3.14)])

$$\begin{aligned}
&(\mathbf{V}_\Gamma\boldsymbol{\varphi})(\mathbf{x}) = O(\ln|\mathbf{x}|), \quad (\mathcal{Q}^s_\Gamma\boldsymbol{\varphi})(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \text{ as } |\mathbf{x}| \to \infty,\ n = 2\\
&(\mathbf{V}_\Gamma\boldsymbol{\varphi})(\mathbf{x}) = O(|\mathbf{x}|^{2-n}), \quad (\mathcal{Q}^s_\Gamma\boldsymbol{\varphi})(\mathbf{x}) = O(|\mathbf{x}|^{1-n}) \text{ as } |\mathbf{x}| \to \infty,\ n \geq 3\\
&(\mathbf{W}_\Gamma\boldsymbol{\phi})(\mathbf{x}) = O(|\mathbf{x}|^{1-n}), \quad (\mathcal{Q}^d_\Gamma\boldsymbol{\phi})(\mathbf{x}) = O(|\mathbf{x}|^{-n}) \text{ as } |\mathbf{x}| \to \infty,\ n \geq 2.
\end{aligned} \tag{1.3.26}$$

1.4 Brinkman layer potentials and their properties

In this section we consider the fundamental solution for the Brinkman system in $\mathbb{R}^n$, $n \geq 2$, and then we define the layer potential operators that are useful in the analysis of the transmission problems in the next chapters. The sources used in the preparation of this section are [68], [73], [75], [71].

1.4.1 The Brinkman system and its fundamental solution

Let $\alpha > 0$ be a given constant. Let $(\mathbf{G}^\alpha(\cdot,\cdot), \mathbf{P}^\alpha(\cdot,\cdot)) \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)^{n\times n} \times \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)^n$ denote the fundamental solution of the Brinkman system, where $\mathbf{G}^\alpha(\cdot,\cdot)$ is the fundamental velocity tensor and by $\mathbf{P}^\alpha(\cdot,\cdot)$ is the fundamental pressure vector for the Brinkman system in $\mathbb{R}^n$. Therefore, the pair $(\mathbf{G}^\alpha(\cdot,\cdot), \mathbf{P}^\alpha(\cdot,\cdot))$ satisfies the following equations

$$\Delta_{\mathbf{x}}\mathbf{G}^\alpha(\mathbf{x},\mathbf{y}) - \alpha\mathbf{G}^\alpha(\mathbf{x},\mathbf{y}) - \nabla_{\mathbf{x}}\mathbf{P}^\alpha(\mathbf{x},\mathbf{y}) = -\delta_{\mathbf{y}}(\mathbf{x})\mathbb{I},\ \mathrm{div}_{\mathbf{x}}\mathbf{G}^\alpha(\mathbf{x},\mathbf{y}) = 0. \tag{1.4.1}$$

Recall that $\delta_{\mathbf{y}}$ denotes the Dirac distribution with mass at $\mathbf{y}$ and the differential operators $\Delta_{\mathbf{x}}$, $\nabla_{\mathbf{x}}$ and $\mathrm{div}_{\mathbf{x}}$ act with respect to the variable $\mathbf{x}$.

The components of the fundamental solution $(\mathbf{G}^\alpha(\mathsf{G}^\alpha_{jk}), \mathbf{P}^\alpha(\mathsf{P}^\alpha_k))$ are given by (see, e.g., [73, Relation (2.29)], [141])

$$\begin{aligned}
\mathsf{G}^\alpha_{jk}(\mathbf{x},\mathbf{y}) &:= \frac{1}{2\omega_n}\left\{\frac{\delta_{jk}}{|\mathbf{y}-\mathbf{x}|^{n-2}}E_1(\alpha|\mathbf{y}-\mathbf{x}|) + \frac{x_j x_k}{|\mathbf{y}-\mathbf{x}|^n}E_2(\alpha|\mathbf{y}-\mathbf{x}|)\right\},\\
\mathsf{P}^\alpha_k(\mathbf{x},\mathbf{y}) &= \frac{1}{\omega_n}\frac{x_k}{|\mathbf{y}-\mathbf{x}|^n},
\end{aligned} \tag{1.4.2}$$

where

$$\begin{aligned}
E_1(s) &:= \frac{\left(\frac{s}{2}\right)^{\frac{n}{2}-1}K_{\frac{n}{2}-1}(s)}{\Gamma\left(\frac{n}{2}\right)} + 2\frac{\left(\frac{s}{2}\right)^{\frac{n}{2}}K_{\frac{n}{2}}(s)}{s^2\cdot\mathbf{\Gamma}\left(\frac{n}{2}\right)} - \frac{1}{s^2},\\
E_2(s) &:= \frac{n}{s^2} - 4\frac{\left(\frac{s}{2}\right)^{\frac{n}{2}+1}K_{\frac{n}{2}+1}(s)}{s^2\cdot\Gamma\left(\frac{n}{2}\right)},
\end{aligned} \tag{1.4.3}$$

and K_β is the second kind Bessel function of order $\beta \geq 0$, $\Gamma(\cdot)$ is the Euler Gamma function. Recall that δ_{jk} is the Kronecher symbol and ω_n is the surface measure of the unit sphere S^{n-1} in $\mathbb{R}^n$, $n \geq 2$.

In addition, let $\mathbf{S}^\alpha(\mathsf{S}^\alpha_{jkl})$ and $\mathbf{R}^\alpha(\mathsf{R}^\alpha_{jk})$ be the associated stress and pressure tensors for the Brinkman system. Their components are given by (see, e.g., [73, Relation (2.31) and Relation (2.32)])

$$\begin{aligned}
\mathsf{S}^\alpha_{jkl}(\mathbf{x},\mathbf{y}) &:= \frac{1}{\omega_n}\left\{\delta_{jl}\frac{x_j}{|\mathbf{y}-\mathbf{x}|^n}E_1(\alpha|\mathbf{y}-\mathbf{x}|) + \frac{\delta_{jl}x_i+\delta_{ij}x_l}{|\mathbf{y}-\mathbf{x}|^n}E_2(\alpha|\mathbf{y}-\mathbf{x}|) + \frac{x_ix_jx_l}{|\mathbf{x}|^{n+2}}E_3(\alpha|\mathbf{y}-\mathbf{x}|)\right\},\\
\mathsf{R}^\alpha_{jk}(\mathbf{x},\mathbf{y}) &:= \frac{1}{2\pi}\left\{-(y_i-x_i)\frac{4(y_k-x_k)}{|\mathbf{y}-\mathbf{x}|^4} - (\alpha|\mathbf{y}-\mathbf{x}|^2\log|\mathbf{y}-\mathbf{x}|+2)\frac{\delta_{ik}}{|\mathbf{y}-\mathbf{x}|^2}\right\}, n=2,\\
\mathsf{R}^\alpha_{jk}(\mathbf{x},\mathbf{y}) &:= \frac{1}{\omega_n}\left\{-(y_i-x_i)\frac{2n(y_k-x_k)}{|\mathbf{y}-\mathbf{x}|^{n+2}} + \frac{2\delta_{ik}}{|\mathbf{y}-\mathbf{x}|^n} - \alpha\frac{1}{n-2}\frac{1}{|\mathbf{y}-\mathbf{x}|^{n-2}}\delta_{ik}\right\}, n\geq 3,
\end{aligned} \tag{1.4.4}$$

where

$$\begin{aligned}
E_1(s) &:= 8\frac{\left(\frac{s}{2}\right)^{\frac{n}{2}+1}K_{\frac{n}{2}+1}(s)}{s^2\cdot\Gamma\left(\frac{n}{2}\right)} - \frac{2n}{z^2}+1\\
E_2(s) &:= 8\frac{\left(\frac{s}{2}\right)^{\frac{n}{2}+1}K_{\frac{n}{2}+1}(s)}{s^2\cdot\Gamma\left(\frac{n}{2}\right)} + 2\frac{\left(\frac{s}{2}\right)^{\frac{n}{2}}K_{\frac{n}{2}}(s)}{\Gamma\left(\frac{n}{2}\right)} - \frac{2n}{s^2}\\
E_3(s) &:= -16\frac{\left(\frac{s}{2}\right)^{\frac{n}{2}+2}K_{\frac{n}{2}+2}(s)}{s^2\cdot\Gamma\left(\frac{n}{2}\right)} + \frac{2n(n+2)}{s^2}.
\end{aligned} \tag{1.4.5}$$

For $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$, $\mathbf{x}\neq\mathbf{y}$, the pair $(\mathbf{S}^\alpha(\mathsf{S}^\alpha_{jkl}),\mathbf{R}^\alpha(\mathsf{R}^\alpha_{jk}))$ satisfies the Brinkman system

$$\Delta_{\mathbf{x}}\mathsf{S}^\alpha_{jkl}(\mathbf{x},\mathbf{y}) - \alpha\mathsf{S}^\alpha_{jkl}(\mathbf{x},\mathbf{y}) - \frac{\partial\mathsf{R}^\alpha_{jk}(\mathbf{y},\mathbf{x})}{\partial x_k} = 0,\ \frac{\partial\mathsf{S}^\alpha_{jkl}(\mathbf{x},\mathbf{y})}{\partial x_k} = 0. \tag{1.4.6}$$

1.4.2 The volume potential for the Brinkman system and its properties

The purpose of this subsection is to introduce the Newtonian (volume) potential operators associated to the Brinkman system and to give their mapping properties. To this end, we consider the Lipschitz domains $\mathsf{D}_\pm$ as described in Assumption 1.1.6 and we will take into account the fundamental solution of the Brinkman system, that is, the pair $(\mathbf{G}^\alpha(\cdot,\cdot),\mathbf{P}^\alpha(\cdot,\cdot))\in\mathcal{D}'(\mathbb{R}^n\times\mathbb{R}^n)^{n\times n}\times\mathcal{D}'(\mathbb{R}^n\times\mathbb{R}^n)^n$ given by relation (1.4.2).

Definition 1.4.1. *Let $\alpha>0$ be a given constant. For $\mathbf{f}\in H^{-1}(\mathbb{R}^n)^n$, define the Newtonian (volume) velocity and pressure potentials for the Brinkman system, by*

$$(\mathcal{N}_{\alpha,\mathbb{R}^n}\mathbf{f})(\boldsymbol{x}) := -\langle\mathbf{G}^\alpha(\boldsymbol{x},\cdot),\mathbf{f}\rangle_{\mathbb{R}^n},\ (\mathcal{Q}_{\alpha,\mathbb{R}^n}\mathbf{f})(\boldsymbol{x}) := -\langle\mathbf{P}^\alpha(\boldsymbol{x},\cdot),\mathbf{f}\rangle_{\mathbb{R}^n}. \tag{1.4.7}$$

Moreover, the Newtonian (volume) velocity and pressure potentials for the Brinkman system corresponding to $\mathsf{D}_\pm$, are given by

$$\mathcal{N}_{\alpha,\mathsf{D}_\pm}\mathbf{f} := (\mathcal{N}_{\alpha,\mathbb{R}^n}\mathbf{f})|_{\mathsf{D}_\pm},\ \mathcal{Q}_{\alpha,\mathsf{D}_\pm}\mathbf{f} := (\mathcal{Q}_{\alpha,\mathbb{R}^n}\mathbf{f})|_{\mathsf{D}_\pm}. \tag{1.4.8}$$

Recall that $|_{\mathsf{D}_\pm}$ is the restriction to $\mathsf{D}_\pm$ operator, which acts on vector-valued or scalar-valued distributions in $\mathbb{R}^n$.

The following result describes the mapping properties of the Newtonian (volume) layer potential operators in the setting of Sobolev spaces (see, e.g., [71, Lemma A.3]).

Theorem 1.4.2. *Let $\alpha > 0$ be a given constant. The Newtonian (volume) velocity and pressure potential operators for the Brinkman system, given by relation (1.4.7),*

$$\begin{aligned} \mathcal{N}_{\alpha,\mathbb{R}^n} : H^{-1}(\mathbb{R}^n)^n \to H^1(\mathbb{R}^n)^n,\ \mathcal{Q}_{\alpha,\mathbb{R}^n} &: H^{-1}(\mathbb{R}^n)^n \to L^2(\mathbb{R}^n), \\ \mathcal{Q}_{\alpha,\mathbb{R}^3} &: H^{-1}(\mathbb{R}^3)^3 \to \mathfrak{M}(\mathbb{R}^3) \end{aligned} \tag{1.4.9}$$

are linear and continuous operators. Moreover, the Newtonian (volume) velocity and pressure potentials operators for the Brinkman system, introduced in relation (1.4.8),

$$\mathcal{N}_{\alpha,\mathsf{D}_+} : \tilde{H}^{-1}(\mathsf{D}_+)^n \to H^1(\mathsf{D}_+)^n,\ \mathcal{Q}_{\alpha,\mathsf{D}_+} : \tilde{H}^{-1}(\mathsf{D}_+)^n \to L^2(\mathsf{D}_+) \tag{1.4.10}$$

and

$$\mathcal{N}_{\alpha,\mathsf{D}_-} : \tilde{H}^{-1}(\mathsf{D}_-)^3 \to H^1(\mathsf{D}_-)^3,\ \mathcal{Q}_{\alpha,\mathsf{D}_-} : \tilde{H}^{-1}(\mathsf{D}_-)^3 \to \mathfrak{M}(\mathsf{D}_-), \tag{1.4.11}$$

in the case $n = 3$, are linear and continuous operators as well, while the spaces $\mathfrak{M}(\mathbb{R}^3)$ and $\mathfrak{M}(\mathsf{D}_-)$ are given by Definition 2.1.1.

Finally, by taking into account relation (1.4.1), we have that the Newtonian (volume) potentials for the Brinkman system, introduced in Definition 1.4.1 satisfy the equations (in the sense of distributions)

$$\Delta(\mathcal{N}_{\alpha,\mathbb{R}^n}\mathbf{f}) - \alpha(\mathcal{N}_{\alpha,\mathbb{R}^n}\mathbf{f}) - \nabla(\mathcal{Q}_{\alpha,\mathbb{R}^n}\mathbf{f}) = \mathbf{f},\ \text{div}\ (\mathcal{N}_{\alpha,\mathbb{R}^n}\mathrm{f}) = 0,\ \text{in}\ \mathbb{R}^n, \tag{1.4.12}$$

and

$$\Delta(\mathcal{N}_{\alpha,\mathsf{D}_\pm}\mathbf{f}) - \alpha(\mathcal{N}_{\alpha,\mathsf{D}_\pm}\mathbf{f}) - \nabla(\mathcal{Q}_{\alpha,\mathsf{D}_\pm}\mathbf{f}) = \mathbf{f},\ \text{div}\ (\mathcal{N}_{\alpha,\mathsf{D}_\pm}\mathbf{f}) = 0,\ \text{in}\ \mathsf{D}_\pm, \tag{1.4.13}$$

respectively.

1.4.3 Brinkman layer potentials and related results

In this subsection, we consider the single layer potential and the double layer potential operators associated to the Brinkman system. We give their definitions, their mapping properties, their jump relations and we describe their behavior at infinity. As in the previous section, let Assumption 1.1.6 be satisfied and let D_+ be a bounded Lipschitz domain with connected boundary Γ.

Firstly, let us focus on the single-layer velocity and pressure potentials, associated to the Brinkman system (see, e.g., [75, Relation (3.6)], [73, Relation (3.1)]).

Definition 1.4.3. *Let Assumption* 1.1.6 *be satisfied. Let $\alpha > 0$ be a given constant. Let $\boldsymbol{\varphi} \in H^{-\frac{1}{2}}(\Gamma)^n$. Define the single-layer velocity potential $\mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi}$ and its associated pressure potential $\mathcal{Q}^s_{\alpha,\Gamma}\boldsymbol{\varphi}$ for the Brinkman system, by*

$$(\mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi}) := \langle \mathbf{G}^\alpha(\boldsymbol{x},\cdot), \boldsymbol{\varphi}\rangle_\Gamma,\ (\mathcal{Q}^s_{\alpha,\Gamma}\boldsymbol{\varphi}) := \langle \mathbf{P}^\alpha(\boldsymbol{x},\cdot), \boldsymbol{\varphi}\rangle_\Gamma,\ \forall\ \boldsymbol{x} \in \mathbb{R}^n \setminus \Gamma. \tag{1.4.14}$$

By taking into account relation (1.4.1), we have that the pair $(\mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi}, \mathcal{Q}^s_{\alpha,\Gamma}\boldsymbol{\varphi})$ satisfies the homogeneous Brinkman system

$$\Delta(\mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi}) - \alpha(\mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi}) - \nabla(\mathcal{Q}^s_{\alpha,\Gamma}\boldsymbol{\varphi}) = 0, \quad \text{div}\ \mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi} = 0 \tag{1.4.15}$$

in $\mathbb{R}^n \setminus \Gamma$.

The following theorem provides some useful mapping properties of the single-layer potentials associated to the Brinkman system (see, e.g., [73, Lemma 3.1], [71, Lemma A.8]).

Theorem 1.4.4. *Let Assumption* 1.1.6 *be satisfied. Let* $\alpha > 0$ *be a given constant. Then the following operators*

$$(\mathbf{V}_{\alpha,\Gamma})|_{\mathsf{D}_+} : H^{-\frac{1}{2}}(\Gamma)^n \to H^1(\mathsf{D}_+)^n,\ (\mathcal{Q}^s_{\alpha,\Gamma})|_{\mathsf{D}_+} : H^{-\frac{1}{2}}(\Gamma)^n \to L^2(\mathsf{D}_+), \tag{1.4.16}$$

are linear and bounded. Moreover, for $n = 3$*, we have that the operators*

$$(\mathbf{V}_{\alpha,\Gamma})|_{\mathsf{D}_-} : H^{-\frac{1}{2}}(\Gamma)^3 \to H^1(\mathsf{D}_-)^3,\ (\mathcal{Q}^s_{\alpha,\Gamma})|_{\mathsf{D}_-} : H^{-\frac{1}{2}}(\Gamma)^3 \to L^2(\mathsf{D}_-) \tag{1.4.17}$$

are linear and bounded as well.

Secondly, we focus on the double-layer velocity and pressure potentials, associated to the Brinkman system (see, e.g., [75, Relation (3.7)]).

Definition 1.4.5. *Let Assumption* 1.1.6 *be satisfied. Let* $\alpha > 0$ *be a given constant. Let* $\boldsymbol{\phi} \in H^{\frac{1}{2}}(\Gamma)^n$*. We define the double-layer potential* $\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi}$ *and its associated pressure potential* $\mathcal{Q}^d_{\alpha,\Gamma}\boldsymbol{\phi}$ *for the Brinkman system, by*

$$\begin{aligned}(\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi})_k(\boldsymbol{x}) &:= \int_\Gamma \mathcal{S}^\alpha_{jkl}(\boldsymbol{y},\boldsymbol{x})\nu_l(\boldsymbol{y})\phi_j(\boldsymbol{y})d\sigma_{\boldsymbol{y}},\ \forall\ \boldsymbol{x}\in\mathbb{R}^n\setminus\Gamma,\\ (\mathcal{Q}^d_{\alpha,\Gamma}\boldsymbol{\phi})(\boldsymbol{x}) &:= \int_\Gamma \mathcal{R}^\alpha_{jl}(\boldsymbol{x},\boldsymbol{y})\nu_l(\boldsymbol{y})\phi_j(\boldsymbol{y})d\sigma_{\boldsymbol{y}},\ \forall\ \boldsymbol{x}\in\mathbb{R}^n\setminus\Gamma.\end{aligned} \tag{1.4.18}$$

Recall that $\boldsymbol{\nu}(\nu_l)_{l=\overline{1,n}}$ *is the outward unit normal to* D_+*, defined a.e. on* Γ.

Note that, in view of relation (1.4.6), the pair $(\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi}, \mathcal{Q}^d_{\alpha,\Gamma}\boldsymbol{\phi})$ satisfies the homogeneous Brinkman system

$$\Delta(\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi}) - \alpha(\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi}) - \nabla(\mathcal{Q}^d_{\alpha,\Gamma}\boldsymbol{\phi}) = 0, \quad \text{div}\ (\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi}) = 0 \tag{1.4.19}$$

in $\mathbb{R}^n \setminus \Gamma$.

Moreover, we introduce the boundary version of the Brinkman double layer velocity potential in the sense of principal value, as follows (see, e.g., [75, Relation (3.8)]).

Definition 1.4.6. *Let* $\alpha > 0$ *be a given constant. Define the principal value of* $\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi}$*, denoted by* $\mathbb{K}_{\alpha,\Gamma}\boldsymbol{\phi}$ *and given by:*

$$\begin{aligned}(\mathbb{K}_{\alpha,\Gamma}\boldsymbol{\phi})_k(\boldsymbol{x}) :&= \text{p.v.}\int_\Gamma \mathcal{S}^\alpha_{jkl}(\boldsymbol{y},\boldsymbol{x})\nu_l(\boldsymbol{y})\phi_j(\boldsymbol{y})d\sigma_{\boldsymbol{y}}\\ &= \lim_{\varepsilon\to 0}\int_{\Gamma\setminus(\Gamma\cap\overline{B}(x,\varepsilon))} \mathcal{S}^\alpha_{jkl}(\boldsymbol{y},\boldsymbol{x})\nu_l(\boldsymbol{y})\phi_j(\boldsymbol{y})d\sigma_{\boldsymbol{y}},\end{aligned} \tag{1.4.20}$$

for $\boldsymbol{x} \in \Gamma$*, where this limit makes sense.*

Also, we state some useful mapping properties of the double-layer potentials in the following statement (see, e.g., [73, Lemma 3.1], [71, Lemma A.8]).

Theorem 1.4.7. *Let Assumption* 1.1.6 *be satisfied. Let* $\alpha > 0$ *be a given constant. Then the following operators*

$$(\mathbf{W}_{\alpha,\Gamma})|_{\mathsf{D}_+} : H^{\frac{1}{2}}(\Gamma)^n \to H^1(\mathsf{D}_+)^n,\ (\mathcal{Q}^d_{\alpha,\Gamma})|_{\mathsf{D}_+} : H^{\frac{1}{2}}(\Gamma)^n \to L^2(\mathsf{D}_+) \tag{1.4.21}$$

are linear and bounded. Moreover, for $n = 3$*, we have that the operators*

$$(\mathbf{W}_{\alpha,\Gamma})|_{\mathsf{D}_-} : H^{\frac{1}{2}}(\Gamma)^3 \to H^1(\mathsf{D}_-)^3,\ (\mathcal{Q}^d_{\alpha,\Gamma})|_{\mathsf{D}_-} : H^{\frac{1}{2}}(\Gamma)^3 \to \mathfrak{M}(\mathsf{D}_-) \tag{1.4.22}$$

are linear and bounded as well and the space $\mathfrak{M}(\mathsf{D}_-)$ *is given by Definition 2.1.1.*

Next, we concern ourselves with the jump relations of the single and double layer potentials for the Brinkman system, in the setting of Sobolev spaces (see, e.g., [73, Lemma 3.1], [71, Lemma A.4]).

Lemma 1.4.8. *Let Assumption* 1.1.6 *be satisfied. Let* $\alpha > 0$ *be a given constant.*

(i) Let $\boldsymbol{\varphi} \in H^{-\frac{1}{2}}(\Gamma)^n$ *and* $\boldsymbol{\phi} \in H^{\frac{1}{2}}(\Gamma)^n$*. Then, the following jump formulas*

$$\begin{aligned} &\mathrm{Tr}_{\mathsf{D}_+}(\mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi}) = \mathrm{Tr}_{\mathsf{D}_-}(\mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi}) := \mathcal{V}_{\alpha,\Gamma}\boldsymbol{\varphi}, \\ &\mathrm{Tr}_{\mathsf{D}_\pm}(\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi}) = \left(\mp\frac{1}{2}\mathbb{I} + \mathbb{K}_{\alpha,\Gamma}\right)\boldsymbol{\phi}, \\ &\mathbf{t}_{\alpha,\mathsf{D}_\pm}(\mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi}, \mathcal{Q}^s_{\alpha,\Gamma}\boldsymbol{\varphi}) = \left(\pm\frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\alpha,\Gamma}\right)\boldsymbol{\varphi}, \\ &\mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi}, \mathcal{Q}^d_{\alpha,\Gamma}\boldsymbol{\phi}) = \mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi}, \mathcal{Q}^d_{\alpha,\Gamma}\boldsymbol{\phi}) = \mathbb{D}_{\alpha,\Gamma}\boldsymbol{\phi} \end{aligned} \tag{1.4.23}$$

hold a.e. on Γ*, where* $\mathbb{K}^*_{\alpha,\Gamma} : H^{-\frac{1}{2}}(\Gamma)^n \to H^{-\frac{1}{2}}(\Gamma)^n$ *is the adjoint of the double layer potential operator* $\mathbb{K}_{\alpha,\Gamma} : H^{\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n$*.*

(ii) The following operators

$$\begin{aligned} &\mathcal{V}_{\alpha,\Gamma} : H^{-\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n, \quad \mathbb{K}_{\alpha,\Gamma} : H^{\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n, \\ &\mathbb{K}^*_{\alpha,\Gamma} : H^{-\frac{1}{2}}(\Gamma)^n \to H^{-\frac{1}{2}}(\Gamma)^n, \quad \mathbb{D}_{\alpha,\Gamma} : H^{\frac{1}{2}}(\Gamma)^n \to H^{-\frac{1}{2}}(\Gamma)^n \end{aligned} \tag{1.4.24}$$

are well-defined, linear and continuous. Moreover, the operator $\mathcal{V}_{\alpha,\Gamma} : H^{-\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n$ *is a Fredholm operator of index zero and its kernel, denoted by* Ker $\mathcal{V}_{\alpha,\Gamma}$ *(see Definition B.1), is given by*

$$\mathrm{Ker}\ \{\mathcal{V}_{\alpha,\Gamma} : H^{-\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n\} = \mathbb{R}\boldsymbol{\nu}. \tag{1.4.25}$$

Now, we introduce the operators

$$\begin{aligned} &\mathcal{V}_{\alpha,0,\Gamma} := \mathcal{V}_{\alpha,\Gamma} - \mathcal{V}_\Gamma, \quad \mathbb{K}_{\alpha,0,\Gamma} := \mathbb{K}_{\alpha,\Gamma} - \mathbb{K}_\Gamma, \\ &\mathbb{K}^*_{\alpha,0,\Gamma} := \mathbb{K}^*_{\alpha,\Gamma} - \mathbb{K}^*_\Gamma, \quad \mathbb{D}_{\alpha,0,\Gamma} := \mathbb{D}_{\alpha,\Gamma} - \mathbb{D}_\Gamma, \end{aligned} \tag{1.4.26}$$

which will be called complementary layer potential operators. Note that the operators $\mathcal{V}_\Gamma$, $\mathbb{K}_\Gamma$, $\mathbb{K}^*_\Gamma$ and $\mathbb{D}_\Gamma$ are introduced in Lemma 1.3.8 which concerns the jump formulas for the single layer and double layer potentials associated to the Stokes system. We have the following lemma (see, e.g., [73, Theorem 3.1]).

Lemma 1.4.9. *The complementary layer potential operators*

$$\begin{aligned} &\mathcal{V}_{\alpha,0,\Gamma} : H^{-\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n, \quad \mathbb{K}_{\alpha,0,\Gamma} : H^{\frac{1}{2}}(\Gamma)^n \to H^{\frac{1}{2}}(\Gamma)^n, \\ &\mathbb{K}^*_{\alpha,0,\Gamma} : H^{-\frac{1}{2}}(\Gamma)^n \to H^{-\frac{1}{2}}(\Gamma)^n, \quad \mathbb{D}_{\alpha,0,\Gamma} : H^{\frac{1}{2}}(\Gamma)^n \to H^{-\frac{1}{2}}(\Gamma)^n, \end{aligned} \tag{1.4.27}$$

are compact a.e. on Γ*.*

Now, we provide the asymptotic formulas (see Remark 1.3.9) which specify the behavior at infinity for the Brinkman layer potentials (see, e.g., [73, Relation (3.12)])

$$\begin{aligned} &(\mathbf{V}_{\alpha,\Gamma}\boldsymbol{\varphi})(\mathbf{x}) = O(|\mathbf{x}|^{-n}), \quad (\mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi})(\mathbf{x}) = O(|\mathbf{x}|^{1-n}) \text{ as } |\mathbf{x}| \to \infty,\ n \geq 2 \\ &(\mathcal{Q}^s_{\alpha,\Gamma}\boldsymbol{\varphi})(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad (\mathcal{Q}^d_{\alpha,\Gamma}\boldsymbol{\phi})(\mathbf{x}) = O(\ln|\mathbf{x}|) \text{ as } |\mathbf{x}| \to \infty,\ n = 2 \\ &(\mathcal{Q}^s_{\alpha,\Gamma}\boldsymbol{\varphi})(\mathbf{x}) = O(|\mathbf{x}|^{1-n}), \quad (\mathcal{Q}^d_{\alpha,\Gamma}\boldsymbol{\phi})(\mathbf{x}) = O(|\mathbf{x}|^{2-n}) \text{ as } |\mathbf{x}| \to \infty,\ n \geq 3. \end{aligned} \tag{1.4.28}$$

Remark 1.4.10. *The results presented in Section* 1.3 *and in Section* 1.4*, including the definitions of the layer potentials, can be extended to the case of the Stokes (or Brinkman) system with variable coefficients (which belong to* L^{∞}*) by using a variational approach (see, e.g., [77], [78], [79], [86]).*

2

Linear Boundary Value Problems of Transmission-type related to the Stokes and Brinkman systems

This chapter deals with certain linear transmission-type problems which involve the Stokes system, the Brinkman system and a generalized version of the Brinkman system (see relation (1.2.21)) in the Lipschitz domains in Euclidean setting (see Assumption 1.1.6 and Assumption 1.1.7). The content of this chapter follows the results that were obtained in the papers [7], [8], [9].

We present and prove well-posedness results for the following boundary value problems. Firstly, we treat the Poisson problem of transmission-type for the generalized Brinkman and Stokes systems in complementary Lipschitz domains in $\mathbb{R}^3$. Secondly, we analyze the Poisson problem of transmission-type for the generalized Brinkman and classical Brinkman system in complementary Lipschitz domains in $\mathbb{R}^3$. Next, we look at the Poisson problem of Robin-transmission-type for the Brinkman system in Euclidean setting provided by Assumption 1.1.7.

Let us note that, the Stokes system can be seen as a particular case of the Brinkman system. Even so, we have separated the study of transmission problems involving the Brinkman and Stokes system from the transmission problems involving only the Brinkman system. Such a distinction can be justified in view of different practical applications (see, e.g., [16], [71]). Moreover, we use different solution spaces for each of the studied transmission problems, namely, if we work with the Stokes system in an unbounded, exterior domain, we use weighted Sobolev spaces, while if we work with the Brinkman system in an unbounded, exterior domain, we use the usual Sobolev spaces.

In order to obtain the results that are presented in this chapter, the main tools of investigation that we have employed are layer potential theory and Fredholm operator theory. Indeed, using the Stokes layer potentials, the Brinkman layer potentials, results regarding Fredholm operators and Green formulas we construct unique solutions to our considered boundary problems.

In the latter, let us mention some past works that have contributed to investigation of elliptic boundary value problems. Firstly, let us note the paper of Fabes, Kenig and Verchota [47] which concerns the investigation of the Dirichlet problem for the Stokes system in a Lipschitz domain in $\mathbb{R}^n$ and they provided representation formulas in terms of layer potential for the solution. Costabel [33] has studied simple and double layer potentials for second order linear elliptic differential operators on Lipschitz domains in Euclidean setting and has provided continuity and regularity results. Dalla Riva, Lanza de Cristoforis and Musolino [36] have analyzed basic boundary value problems for the Laplace equation in singularly perturbed domains, with an emphasis on domains with small holes.

Varnhorn [140] has used potential theory to construct an explicit solution of the Stokes rezolven system in a bounded domain in $\mathbb{R}^3$ with C^2-boundary. Also, Varnhorn [141] has provided a theory of solvability for the Stokes system in exterior domains and has analyzed the existence of strong solutions in Sobolev spaces and further properties. Medkova [97] has used the integral equation method in order to obtain L^2-solutions for the transmission problem, Robin-transmission problem and the Dirichlet-transmission problem for the Brinkman system, while in [98], she used the same method in order to study the transmission problem for the Stokes system, in homogeneous Sobolev spaces in Lipschitz domains in $\mathbb{R}^3$. Medkova [99] has also studied the Dirichlet problem for the resolvent Stokes system with bounded boundary data in the setting of bounder and unbounded domains with compact Lyapunov boundary. Chkadua, Mikhailov and Natroshvili [26] have used localized integral potentials associated with the Laplace operator in order to reduce boundary value problems for variable-coefficient divergence-from second-order elliptic PDEs to systems of localized boundary-domain singular integral equations. Escauriaza and Mitrea [46] have used layer potential methods to obtain the well-posedness of the transmission problem for the Laplacian in the presence of a Lipschitz interface with boundary data belonging to Lebesgue and Hardy spaces. The work of Mitrea and Wright [114] concerns also transmission boundary value problems for the Stokes system in Lipschitz domains in the Euclidean setting, for $n \geq 2$.

The authors in [75] have obtained a well-posedness result for a linear Robin-transmission problem for the Stokes and Brinkman systems in adjacent Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$ with linear transmission conditions on the Lipschitz interface and Robin condition on the remaining boundary. Kohr, Wendland and Lanza de Cristoforis [73] have analyzed a nonlinear Neumann-transmission problems for the Stokes and Brinkman systems in Euclidean Lipschitz domains with boundary data in L^p, Sobolev and Besov spaces. Fericean, Groşan, Kohr and Wendland [50] have treated interface boundary value problems of Robin-transmission type for the Stokes and Brinkman systems in Lipschitz domains in $\mathbb{R}^n$ for $n \geq 3$ and with boundary data in L^p or Sobolev spaces. Fericean and Wendland [51] have used layer potential theory in order to obtain well-posedness results for a Dirichlet-transmission problem for the Stokes and Brinkman systems in Lipschitz domains in $\mathbb{R}^n$ for $n \geq 3$. The authors in [71] have obtained a well-posedness result for a transmission problem for the Stokes and Brinkman systems in complementary Lipschitz domains in $\mathbb{R}^3$ in weighted Sobolev spaces by making use of layer potential techniques. In [104], Mikhailov and Portillo have studied a mixed boundary value problem for the stationary compressible Stokes system with variable viscosity in an exterior domain of $\mathbb{R}^3$ by the means of boundary-domain integral equations (BDIEs). Mitrea, Mitrea and Mitrea [106] have proved well-posedness and Fredholm solvability results for boundary value problems for elliptic second-order homogeneous constant coefficient systems in domains of general geometric nature.

Regarding the setting of manifolds, let us mention that Kohr, Pintea and Wendland [82] have developed a potential analysis for certain pseudodifferential matrix operators on Lipschitz domains in compact Riemmanian manifolds and they have studied Dirichlet-transmission problems for Brinkman operators in Lipschitz domains in compact Riemmanian manifolds. Also, Kohr, Mikhailov and Wendland [76] have investigated a linear transmission problem for the Stokes and generalized Brinkman system in two complementary Lipschitz domains in a compact Riemannian manifold of dimension $m \geq 2$.

More recently, a great deal of attention has been directed to the variable coefficient PDEs. Note that the works of Kohr, Mikhailov and Wendland [77], [78], [79] concern the analysis of the anisotropic Stokes system with L^∞ coefficient tensor. They investigate diverse boundary problems, Dirichlet type, transmission type and they have also discussed potentials for this anisotropic system.

2.1 Dirichlet type problem for the Brinkman system in an exterior domain

The goal of this section is twofold. First, we introduce a special function space which is involved in the mapping properties of the Newtonial layer potentials for the Brinkman system (see relation (1.4.10) of Theorem 1.4.2). Secondly, we study an exterior Dirichlet boundary value problem for the Brinkman system, which is involved in the proof of our well-posedness results (see Theorem 2.3.1, Theorem 2.3.3, Theorem 3.3.1).

In the latter, let D be either of the domains $\mathbb{R}^3$, D_+ or D_-, which are described in Assumption 1.1.6, for $n = 3$. Let us introduce the space $H^{-1}_{\text{curl}}(\mathsf{D})^3$ by

$$H^{-1}_{\text{curl}}(\mathsf{D})^3 := \{\mathbf{h} \in H^{-1}(\mathsf{D})^3 : \text{curl } \mathbf{h} = 0\}.$$

Definition 2.1.1. *Let* D *be either of the domains described in Assumption* 1.1.6*, for* $n = 3$*. Define the space* $\mathfrak{M}(\mathsf{D})$ *by*

$$\mathfrak{M}(\mathsf{D}) := \{g \in L^2(\rho^{-1}, \mathsf{D}) : \nabla g \in H^{-1}_{\text{curl}}(\mathsf{D})^3\}. \tag{2.1.1}$$

Now, denote by $\mathfrak{M}'(\mathsf{D})$ the dual of the space $\mathfrak{M}(\mathsf{D})$, we have the following continuous chain of embeddings (see, e.g., [71, (A.24)])

$$L^2(\rho, \mathsf{D}) \subset \mathfrak{M}'(\mathsf{D}) \subset L^2(\mathsf{D}) \subset \mathfrak{M}(\mathsf{D}) \subset L^2(\rho^{-1}, \mathsf{D}) \subset L^2_{\text{loc}}(\mathsf{D}). \tag{2.1.2}$$

In the latter, we concern ourselves with two important results. First, we analyze exterior Dirichlet problem for the classical Brinkman system in the space $H^1(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-)$, where D_- is the exterior Lipschitz domain introduced in Assumption 1.1.6, in the case $n = 3$. The well-posedness result is as follows (see, [7, Theorem A.1], [71, Lemma A.2], and [140, Prop. 4.5] in the case of an exterior domain with a $\mathcal{C}^2$-boundary).

Theorem 2.1.2. *Let Assumption 1.1.6 be satisfied for* $n = 3$*. Let* $\alpha > 0$ *be a given constant. Then, the exterior Dirichlet problem for the Brinkman system*

$$\begin{cases} \Delta \mathbf{u} - \alpha \mathbf{u} - \nabla \pi = 0 \text{ in } \mathsf{D}_-, \\ \text{div } \mathbf{u} = 0 \text{ in } \mathsf{D}_-, \\ \text{Tr}_{\mathsf{D}_-} \mathbf{u} = \mathbf{h} \in H^{\frac{1}{2}}(\Gamma)^3 \text{ on } \Gamma, \\ \mathbf{u}(\mathbf{x}) = O(|\mathbf{x}|^{-2}), \nabla \mathbf{u}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \pi(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \text{ as } |\mathbf{x}| \to \infty, \end{cases} \tag{2.1.3}$$

has a unique solution in the space $H^1(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-)$.

Proof. First of all, we focus on showing the problem (2.1.3) admits at most one solution. In order to achieve this, we consider the difference between two solutions of the problem (2.1.3), which will be denoted by $(\mathbf{u}_0, \pi_0) \in H^1(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-)$.

We proceed as follows. We introduce $\mathsf{D}_R := \mathsf{D}_- \cap B_R(0)$, where $B_R(0)$ is a large ball in $\mathbb{R}^3$, centered at the origin such that $\overline{\mathsf{D}}_+ \subset B_R(0)$. Now, we apply the Green formula (1.2.18) in the case of the bounded domain D_R, we get

$$\begin{aligned} \langle \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_0, \pi_0), \text{Tr}_{\mathsf{D}_+} \mathbf{u}_0 \rangle_{\Gamma_R} - \langle \mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_0, \pi_0), \text{Tr}_{\mathsf{D}_-} \mathbf{u}_0 \rangle_{\Gamma} \\ = 2\langle \mathbb{E}(\mathbf{u}_0), \mathbb{E}(\mathbf{u}_0) \rangle_{\mathsf{D}_R} + \alpha \langle \mathbf{u}_0, \mathbf{u}_0 \rangle_{\mathsf{D}_R}, \end{aligned} \tag{2.1.4}$$

where $\Gamma_R := \partial B_R(0)$ and the $\pm$ signs in the left hand side of (2.1.4) are provided by the orientation of the unit normal to Γ and Γ_R, respectively.

Due to the fact that $\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_0 = 0$ on Γ, we get

$$\langle \mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_0,\pi_0), \mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_0\rangle_\Gamma = 0. \tag{2.1.5}$$

Moreover, in view of the far field conditions from (2.1.3), we have

$$\lim_{R\to\infty} \langle \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_0,\pi_0), \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_0\rangle_{\Gamma_R} = \lim_{R\to\infty} \int_{\Gamma_R} \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_0,\pi_0) \cdot \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_0 \mathrm{d}\sigma_R = 0. \tag{2.1.6}$$

Now, let $R \to \infty$ in (2.1.4) and by employing relations (2.1.5) and (2.1.6) we get

$$2\langle \mathbb{E}(\mathbf{u}_0), \mathbb{E}(\mathbf{u}_0)\rangle_{\mathsf{D}_-} + \alpha\langle \mathbf{u}_0, \mathbf{u}_0\rangle_{\mathsf{D}_-} = 0, \tag{2.1.7}$$

which shows that $\mathbf{u}_0 = 0$ in D_-. Furthermore, the Brinkman equation and the conditions at infinity satisfied by π_0 imply that $\pi_0 = 0$ in D_-.

The previous arguments imply

$$\mathbf{u}_0 = 0,\ \pi_0 = 0,\ \text{in } \mathsf{D}_-,$$

and consequently, our problem (2.1.3) admits at most one solution in the space $H^1(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-)$.

Secondly, we focus on showing that a solution the problem (2.1.3) exists. To fulfill this claim, we seek a solution of our problem (2.1.3), in the form

$$\mathbf{u} = \mathbf{W}_{\alpha,\Gamma}\varphi,\ \pi = \mathcal{Q}^d_{\alpha,\Gamma}\varphi, \tag{2.1.8}$$

where $\boldsymbol{\varphi} \in H^{\frac{1}{2}}(\Gamma)^3$ is unknown.

Now, the pair $(\mathbf{u}, \pi)$ defined in relation (2.1.8) satisfy the Brinkman system as well as the far field conditions in (2.1.3) (see, e.g., [140, Prop. 4.5]). By imposing the exterior Dirichlet boundary condition, we can determine the unknown density $\boldsymbol{\varphi} \in H^{\frac{1}{2}}(\Gamma)^3$. To achieve this, we use the Dirichlet boundary condition of problem (2.1.3) together with relation (1.4.23) of Lemma 1.4.8 and we get

$$\left(\frac{1}{2}\mathbb{I} + \mathbb{K}_{\alpha,\Gamma}\right)\boldsymbol{\varphi} = \mathbf{h}. \tag{2.1.9}$$

Then, our next goal would be to show that the operator

$$\frac{1}{2}\mathbb{I} + \mathbb{K}_{\alpha,\Gamma} : H^{\frac{1}{2}}(\Gamma)^3 \to H^{\frac{1}{2}}(\Gamma)^3, \tag{2.1.10}$$

is an isomorphism.

To show that the operator (2.1.10) is an isomorphism, we will prove that (2.1.10) is Fredholm of index zero and one-to-one as well.

In order to show that the operator (2.1.10) is Fredholm of index zero, we note that we have the following decomposition

$$\frac{1}{2}\mathbb{I} + \mathbb{K}_{\alpha,\Gamma} = \frac{1}{2}\mathbb{I} + \mathbb{K}_\Gamma + \mathbb{K}_{\alpha,0,\Gamma} : H^{\frac{1}{2}}(\Gamma)^3 \to H^{\frac{1}{2}}(\Gamma)^3. \tag{2.1.11}$$

We also know that the operator

$$\frac{1}{2}\mathbb{I} + \mathbb{K}_\Gamma : H^{\frac{1}{2}}(\Gamma)^3 \to H^{\frac{1}{2}}(\Gamma)^3,$$

is a Fredholm operator of index zero (see, e.g., [114, Theorem 10.5.3], see also [39, Proposition 3.5]). Note that, the operator

$$\mathbb{K}_{\alpha,0,\Gamma} : H^{\frac{1}{2}}(\Gamma)^3 \to H^{\frac{1}{2}}(\Gamma)^3,$$

is compact (see Lemma 1.4.9). Thus, the operator (2.1.10) is, indeed, Fredholm of index zero. Moreover, its adjoint

$$\frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\alpha,\Gamma} : H^{-\frac{1}{2}}(\Gamma)^3 \to H^{-\frac{1}{2}}(\Gamma)^3, \tag{2.1.12}$$

is also a Fredholm operator of index zero. Moreover, if we would show that (2.1.12) is an isomorphism, then (2.1.10) is an isomorphism as well.

Hence, we will focus on the operator (2.1.12) and we show that (2.1.12) is one-to-one, or equivalently that

$$\mathrm{Ker}\left\{\frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\alpha,\Gamma} : H^{-\frac{1}{2}}(\Gamma)^3 \to H^{-\frac{1}{2}}(\Gamma)^3\right\} = \{\mathbf{0}\}. \tag{2.1.13}$$

We consider

$$\boldsymbol{\psi} \in \mathrm{Ker}\left\{\frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\alpha,\Gamma} : H^{-\frac{1}{2}}(\Gamma)^3 \to H^{-\frac{1}{2}}(\Gamma)^3\right\} \tag{2.1.14}$$

and we construct the fields

$$\mathbf{u}_0 = \mathbf{V}_{\alpha,\Gamma}\boldsymbol{\psi} \in H^1(\mathsf{D}_+)^3,\ \pi_0 = \mathcal{Q}^s_{\alpha,\Gamma}\boldsymbol{\psi} \in L^2(\mathsf{D}_+). \tag{2.1.15}$$

In addition, by using relation (1.4.23) of Lemma 1.4.8, we find

$$\mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_0, \pi_0) = \left(\frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\alpha,\Gamma}\right)\boldsymbol{\psi} = 0. \tag{2.1.16}$$

and if we take into account Green's formula (1.2.18) from Lemma 1.2.6, we obtain

$$0 = 2\langle \mathbb{E}(\mathbf{u}_0), \mathbb{E}(\mathbf{u}_0)\rangle_{\mathsf{D}_+} + \alpha\langle \mathbf{u}_0, \mathbf{u}_0\rangle_{\mathsf{D}_+}. \tag{2.1.17}$$

Due to the fact that $\alpha > 0$, it follows that that $\mathbf{u}_0 = 0$ in D_+. Consequently, we get $\pi_0 = c \in \mathbb{R}$ in D_+. Using the relation (2.1.16) and the fact that

$$\mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_0, \pi_0) = -c\boldsymbol{\nu}, \tag{2.1.18}$$

we obtain $c = 0$ and consequently, we have

$$\mathbf{u}_0 = 0,\ \pi_0 = 0 \text{ in } \mathsf{D}_+. \tag{2.1.19}$$

The continuity of the single-layer potential operator for the Brinkman system across Γ (see Lemma 1.4.8), implies that

$$\mathcal{V}_{\alpha,\Gamma}\boldsymbol{\psi} = 0 \text{ on } \Gamma \tag{2.1.20}$$

and since

$$\mathrm{Ker}\left\{\mathcal{V}_{\alpha,\Gamma} : H^{-\frac{1}{2}}(\Gamma)^3 \to H^{\frac{1}{2}}(\Gamma)^3\right\} = \mathbb{R}\boldsymbol{\nu},$$

where $\mathbb{R}\boldsymbol{\nu} := \{c\boldsymbol{\nu} : c \in \mathbb{R}\}$ (see, e.g., [72, Lemma 3.1]) implies the existence of a constant $p \in \mathbb{R}$ with the property that $\boldsymbol{\psi} = p\boldsymbol{\nu}$. Now, if we take into account that $\mathcal{Q}^s_{\alpha,\Gamma}\boldsymbol{\nu} = -1$, we obtain

$$\pi_0 = p(\mathcal{Q}^s_{\alpha,\Gamma}\boldsymbol{\nu}) = -p. \tag{2.1.21}$$

It remains to apply relation (2.1.19) to deduce that $\boldsymbol{\psi} = 0$, which shows the operator (2.1.12) is one-to-one.

The previous arguments show that the operator (2.1.12) is an isomorphism and hence the operator (2.1.10) is an isomorphism as well. This implies that the problem (2.1.3) has a unique solution in the form:

$$\mathbf{u} = \mathbf{W}_{\alpha,\Gamma}\left(\left(\frac{1}{2}\mathbb{I} + \mathbb{K}_{\alpha,\Gamma}\right)^{-1}\mathbf{h}\right),\ \pi = \mathcal{Q}^s_{\alpha,\Gamma}\left(\left(\frac{1}{2}\mathbb{I} + \mathbb{K}_{\alpha,\Gamma}\right)^{-1}\mathbf{h}\right).$$

This concludes our proof of our result. □

Lastly, we conclude this section with a result, which shows that if we have a pair $(\mathbf{v}, p) \in H^1(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-)$ satisfies the Brinkman system in an exterior Lipschitz domain, then the far field conditions described in problem (2.1.3) are also satisfied (see [7, Lemma A.2], [71, Lemma A.2]).

Lemma 2.1.3. *Let Assumption 1.1.6 be satisfied for $n = 3$. Let $\alpha > 0$ be a given constant. If the pair $(\mathbf{v}, \mathrm{p}) \in H^1(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-)$ satisfies*

$$\Delta \mathbf{v} - \alpha \mathbf{v} - \nabla \mathrm{p} = 0,\ \text{div}\ \mathbf{v} = 0 \text{ in } \mathsf{D}_-, \tag{2.1.22}$$

then

$$\mathbf{v}(\mathbf{x}) = O(|\mathbf{x}|^{-2}), \nabla \mathbf{v}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \mathrm{p}(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \text{ as } |\mathbf{x}| \to \infty. \tag{2.1.23}$$

Proof. To show this result, we employ steps similar to those in the proof of [71, Lemma A.2]. First, by using the fields $(\mathbf{v}, p) \in H^1(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-)$, we introduce the following exterior Dirichlet problem for the Brinkman system

$$\begin{cases} \Delta \mathbf{u} - \alpha \mathbf{u} - \nabla \pi = 0 \text{ in } \mathsf{D}_-, \\ \text{div}\ \mathbf{u} = 0,\ \text{ in } \mathsf{D}_-, \\ \text{Tr}_{\mathsf{D}_-} \mathbf{u} = \text{Tr}_{\mathsf{D}_-} \mathbf{v} \in H^{\frac{1}{2}}(\Gamma)^3,\ \text{ on } \Gamma, \\ \mathbf{u}(\mathbf{x}) = O(|\mathbf{x}|^{-2}), \nabla \mathbf{u}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \pi(\mathbf{x}) = O(|\mathbf{x}|^{-1}),\ \text{ as } |\mathbf{x}| \to \infty. \end{cases} \tag{2.1.24}$$

We apply Theorem 2.1.2 in order to deduce that problem (2.1.24) has a unique solution $(\mathbf{u}, \pi) \in H^1(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-)$.

Secondly, we have that the exterior Dirichlet problem for the Brinkman system

$$\begin{cases} \Delta \mathbf{w} - \alpha \mathbf{w} - \nabla q = 0 \text{ in } \mathsf{D}_-, \\ \text{div}\ \mathbf{w} = 0 \text{ in } \mathsf{D}_-, \\ \text{Tr}_{\mathsf{D}_-} \mathbf{w} = \text{Tr}_{\mathsf{D}_-} \mathbf{v} \in H^{\frac{1}{2}}(\Gamma)^3, \end{cases} \tag{2.1.25}$$

has a unique solution in the space $H^1(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-)$. This fact can be proved if we apply the Green formula (1.2.18) from Lemma 1.2.6, in the case of the exterior domain.

Now, both pairs $(\mathbf{v}, p)$ and $(\mathbf{u}, \pi)$ in the space $H^1(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-)$ satisfy the exterior Dirichlet problem (2.1.25). It follows that

$$\mathbf{v} = \mathbf{u}, \quad p = \pi,\ \text{in } \mathsf{D}_-. \tag{2.1.26}$$

Finally, the asymptotic relations in (2.1.24) satisfied by $\mathbf{u}$ and π imply the fact that $\mathbf{v}$ and p satisfy the asymptotic relations (2.1.23), as stated. This concludes the proof of our result. □

2.2 Transmission problem for a Brinkman type system and the Stokes system in complementary Lipschitz domains in $\mathbb{R}^3$

In this section we aim to state and prove a well-posedness result, for a transmission-type problem, which was obtained in the setting of Assumption 1.1.6 for $n = 3$, i.e., complementary Lipschitz domains in $\mathbb{R}^3$. We consider a generalized version of the Brinkman system in the bounded Lipschitz domain D_+ and the Stokes system in the complementary Lipschitz set D_-. Also, we have the following assumption that we will use in the latter.

Assumption 2.2.1. *Let $n \geq 2$. Assume that $\mathfrak{L} \in L^\infty(\Gamma)^{n\times n}$ is a symmetric matrix valued function, which satisfies the following non-negativity condition*

$$\langle \mathfrak{L}\boldsymbol{u}, \boldsymbol{u}\rangle_\Gamma \geq 0, \tag{2.2.1}$$

for all $\boldsymbol{u} \in L^2(\Gamma)^n$.

To ensure the clarity of our exposition, we consider the following spaces, namely the space of solutions,

$$\mathsf{X}_w := H^1_{\mathrm{div}}(\mathsf{D}_+)^3 \times L^2(\mathsf{D}_+) \times \mathcal{H}^1_{\mathrm{div}}(\mathsf{D}_-)^3 \times L^2(\mathsf{D}_-) \tag{2.2.2}$$

and the space of given data,

$$\mathsf{Y}_w := \tilde{H}^{-1}(\mathsf{D}_+)^3 \times \tilde{\mathcal{H}}^{-1}(\mathsf{D}_-)^3 \times H^{\frac{1}{2}}(\Gamma)^3 \times H^{-\frac{1}{2}}(\Gamma)^3, \tag{2.2.3}$$

respectively. Note that the space $\mathcal{H}^1_{\mathrm{div}}(\mathsf{D}_-)^3$ is given by relation (1.2.8) and the space $\tilde{\mathcal{H}}^{-1}(\mathsf{D}_-)^3$ is given by relation (1.1.38).

The considered transmission problem of Poisson type for the Stokes and generalized Brinkman systems is

$$\begin{cases} \Delta\mathbf{u}_+ - \mathcal{P}\mathbf{u}_+ - \nabla\pi_+ = \mathbf{f}_+|_{\mathsf{D}_+} \text{ in } \mathsf{D}_+, \\ \Delta\mathbf{u}_- - \nabla\pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \operatorname{div} \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ - \mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_- = \mathbf{g} \text{ on } \Gamma, \\ \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+) - \mathbf{t}_{\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-) + \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ = \mathbf{h} \text{ on } \Gamma, \end{cases} \tag{2.2.4}$$

with the unknown fields $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_w$. Note that, the presence of the Stokes system in the unbounded Lipschitz domain D_- justifies the inclusion of the space $\mathcal{H}^1_{\mathrm{div}}(\mathsf{D}_-)^3$ in the solution space X_w provided in relation (2.2.2).

We will treat the transmission problem (2.2.4) in two separate cases. First, we state and prove the well-posedness result for the transmission problem (2.2.4) in the case that the velocity field $\mathbf{u}_-$ tends to zero at infinity in the sense of Leray. Secondly, we state and prove the well-posedness result in the case that the velocity field $\mathbf{u}_-$ tends to a constant at infinity in the sense of Leray. In the latter, let us denote that particular constant by $\mathbf{u}_\infty \in \mathbb{R}^3$.

Hence, we begin with the well-posedness result that was obtained for the transmission problem (2.2.4), in the case $\mathbf{u}_\infty = 0$ (see [8, Theorem 4.5], [71, Theorem 4.2], [76, Theorem 4.4]).

Theorem 2.2.2. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied, for $n = 3$. Let $\mathcal{P} \in L^\infty(\mathsf{D}_+)^{3\times 3}$ such that condition (1.2.22) holds. Then, for $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) \in \mathsf{Y}_w$ given, the Poisson* *problem of transmission type for Stokes and generalized Brinkman systems (2.2.4) has a unique solution* $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_w$. *Moreover, there is a constant* $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \mathcal{P}, \mathfrak{L}) > 0$ *such that*

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_w} \leq C||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h})||_{\mathsf{Y}_w}, \tag{2.2.5}$$

and $\mathbf{u}_-$ *vanishes at infinity in the sense of Leray.*

Proof. The proof of this result is inspired by the proof of [76, Theorem 4.4]. We will treat two distinct cases. The first case is $\mathcal{P} \equiv 0$ and the second case is $\mathcal{P} \neq 0$.

Case 1. $\mathcal{P} \equiv 0$. We stress the fact that in this case, the transmission problem (2.2.4) becomes the transmission problem for the Stokes system, in complementary Lipschitz domains in $\mathbb{R}^3$. This transmission-type problem is well-posed (see problem (2.2.31), Theorem 2.2.6 and also [98, Proposition 5.1, Theorem 5.1]). Due to the well-posedness of the transmission problem for the Stokes system, in complementary Lipschitz domains in $\mathbb{R}^3$ (see, e.g., Theorem 2.2.6) we get a solution operator

$$\mathsf{S} : \mathsf{Y}_w \to \mathsf{X}_w, \tag{2.2.6}$$

(see also relation (2.2.60)), which maps the given data to the solution of this transmission-type problem (see problem (2.2.31)).

Case 2. $\mathcal{P} \neq 0$. We underline the fact that our objective is to reduce this case, $\mathcal{P} \neq 0$, to a Fredholm equation of the second kind which admits a unique solution, as in the previous case $\mathcal{P} \equiv 0$.

Consequently, our objective is to ensure that our transmission problem (2.2.4) has a unique solution, which depends continuously on the given data. To this end, we begin by showing that *we have at most one solution of the problem* (2.2.4). Hence, we consider the difference between two possible solutions of the problem (2.2.4), and we denote their difference by $(\mathbf{u}^0_+, \pi^0_+, \mathbf{u}^0_-, \pi^0_-) \in \mathsf{X}_w$, which satisfies the the homogeneous version of (2.2.4).

Next, we use Green formulas for the Stokes system (1.2.15) and generalized Brinkman system (1.2.28) in order to obtain the following relations

$$\begin{aligned} &\langle \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{u}^0_+, \pi^0_+), \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}^0_+\rangle_\Gamma = 2\langle \mathbb{E}(\mathbf{u}^0_+), \mathbb{E}(\mathbf{u}^0_+)\rangle_{\mathsf{D}_+} + \langle \mathcal{P}\mathbf{u}^0_+, \mathbf{u}^0_+\rangle_{\mathsf{D}_+}, \\ &\langle \mathbf{t}_{\mathsf{D}_-}(\mathbf{u}^0_-, \pi^0_-), \mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}^0_-\rangle_\Gamma = -2\langle \mathbb{E}(\mathbf{u}^0_-), \mathbb{E}(\mathbf{u}^0_-)\rangle_{\mathsf{D}_-}. \end{aligned} \tag{2.2.7}$$

Now, we subtract $(2.2.7)_2$ from $(2.2.7)_1$ and we have that

$$\begin{aligned} &\langle \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{u}^0_+, \pi^0_+), \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}^0_+\rangle_\Gamma - \langle \mathbf{t}_{\mathsf{D}_-}(\mathbf{u}^0_-, \pi^0_-), \mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}^0_-\rangle_\Gamma = \\ &2\langle \mathbb{E}(\mathbf{u}^0_+), \mathbb{E}(\mathbf{u}^0_+)\rangle_{\mathsf{D}_+} + 2\langle \mathbb{E}(\mathbf{u}^0_-), \mathbb{E}(\mathbf{u}^0_-)\rangle_{\mathsf{D}_-} + \langle \mathcal{P}\mathbf{u}^0_+, \mathbf{u}^0_+\rangle_{\mathsf{D}_+}. \end{aligned} \tag{2.2.8}$$

If we take into account the transmission conditions of the homogeneous version of (2.2.4), we get

$$\langle \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{u}^0_+, \pi^0_+), \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}^0_+\rangle_\Gamma - \langle \mathbf{t}_{\mathsf{D}_-}(\mathbf{u}^0_-, \pi^0_-), \mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}^0_-\rangle_\Gamma = -\langle \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}^0_+, \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}^0_+\rangle_\Gamma,$$

which, in turn, yields the fact that

$$-\langle \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}^0_+, \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}^0_+\rangle_\Gamma = 2\langle \mathbb{E}(\mathbf{u}^0_+), \mathbb{E}(\mathbf{u}^0_+)\rangle_{\mathsf{D}_+} + 2\langle \mathbb{E}(\mathbf{u}^0_-), \mathbb{E}(\mathbf{u}^0_-)\rangle_{\mathsf{D}_-} + \langle \mathcal{P}\mathbf{u}^0_+, \mathbf{u}^0_+\rangle_{\mathsf{D}_+}.$$

Recall that, the operator $\mathfrak{L}$ satisfies the non-negativity condition (2.2.1), hence, we get

$$\begin{aligned} 2\langle \mathbb{E}(\mathbf{u}^0_+), \mathbb{E}(\mathbf{u}^0_+)\rangle_{\mathsf{D}_+} + 2\langle \mathbb{E}(\mathbf{u}^0_-), \mathbb{E}(\mathbf{u}^0_-)\rangle_{\mathsf{D}_-} + \langle \mathcal{P}\mathbf{u}^0_+, \mathbf{u}^0_+\rangle_{\mathsf{D}_+} &= 0, \\ \langle \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}^0_+, \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}^0_+\rangle_\Gamma &= 0. \end{aligned} \tag{2.2.9}$$

In view of relation (2.2.9), while taking into account that $\mathcal{P}$ satisfies the non-negativity condition (1.2.22), we have that $\mathbf{u}_+^0 = 0$ in D_+ and $\mathbb{E}(\mathbf{u}_\pm^0) = 0$ in $\mathsf{D}_\pm$. Also, due to the fact that $\nabla\pi_+^0 = \Delta\mathbf{u}_+^0 - \mathcal{P}\mathbf{u}_+^0 = 0$, it follows that the function π_+^0 is constant, i.e., $\pi_+^0 = c \in \mathbb{R}$ in D_+.

By employing the first transmission condition of the homogeneous version of (2.2.4), we get

$$\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+^0 = \mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-^0 = 0. \tag{2.2.10}$$

Then, we have that the pair $(\mathbf{u}_-^0, \pi_-^0) \in \mathcal{H}^1_{\mathrm{div}}(\mathsf{D}_-)^3 \times L^2(\mathsf{D}_-)$ solves the exterior Dirichlet problem associated to the Stokes system with homogeneous Dirichlet boundary condition. This boundary value problem is well-posed (see, e.g., [56, Theorem 3.4]) and in view of this, we get that $\mathbf{u}_-^0 = 0$ and $\pi_-^0 = 0$ in the set D_-.

The second transmission condition of the homogenous version of (2.2.4), implies the fact that

$$\mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0) = 0. \tag{2.2.11}$$

On the other hand, we have that

$$\mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0) = -c\boldsymbol{\nu}. \tag{2.2.12}$$

By relations (2.2.11) and (2.2.12) we deduce that $c = 0$.

We conclude that

$$\mathbf{u}_\pm^0 = 0,\ \pi_\pm^0 = 0 \text{ in } \mathsf{D}_+, \tag{2.2.13}$$

which proves that the transmission problem (2.2.4) has at most one solution.

Now, our goal is to show *the existence of a solution in the case* $\mathcal{P} \neq 0$. First, we rewrite the transmission problem (2.2.4) in the equivalent form

$$\begin{cases} \Delta\mathbf{u}_+ - \nabla\pi_+ = \mathbf{f}_+|_{\mathsf{D}_+} + \mathcal{P}\mathbf{u}_+, \text{ in } \mathsf{D}_+,\\ \Delta\mathbf{u}_- - \nabla\pi_- = \mathbf{f}_-|_{\mathsf{D}_-}, \text{ in } \mathsf{D}_-,\\ \mathrm{div}\ \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm,\\ \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ - \mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_- = \mathbf{g} \text{ on } \Gamma,\\ \mathbf{t}_{\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+ + \mathring{\mathsf{E}}_+(\mathcal{P}\mathbf{u}_+)) - \mathbf{t}_{\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-) + \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ = \mathbf{h} \text{ on } \Gamma. \end{cases} \tag{2.2.14}$$

Note that, we denote by $\mathring{\mathsf{E}}_+$ the extension by zero operator outside D_+, and due to Remark 1.2.16, the operator $\mathbf{t}_{\mathcal{P},\mathsf{D}_+}$ can be expressed in terms of $\mathbf{t}_{\mathsf{D}_+}$. Moreover, the fact that $\mathcal{P} \in L^\infty(\mathsf{D}_+)^{3\times 3}$ implies that the embedding

$$\mathcal{P}\mathbf{u}_+ \in L^2(\mathsf{D}_+)^3 \hookrightarrow \tilde{H}^{-1}(\mathsf{D}_+)^3, \tag{2.2.15}$$

is compact, and

$$\mathring{\mathsf{E}}_+(\mathcal{P}\mathbf{u}_+) \in \tilde{H}^{-1}(\mathsf{D}_+)^3. \tag{2.2.16}$$

Now, by taking into account the well-posedness result for the transmission problem for the Stokes system in complementary Lipschitz domains in $\mathbb{R}^3$ (see also Theorem 2.2.6), we rewrite the transmission problem (2.2.14) in terms of the solution operator

$$\mathsf{S} : \mathsf{Y}_w \to \mathsf{X}_w, \tag{2.2.17}$$

which maps the given data to the unique solution of transmission problem for the Stokes system (see also relations (2.2.31), (2.2.60) and Theorem 2.2.6). Hence, we obtain

$$(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) = \mathsf{S}(\mathbf{f}_+ + \mathring{\mathsf{E}}_+(\mathcal{P}\mathbf{u}_+), \mathbf{f}_-, \mathbf{g}, \mathbf{h}). \tag{2.2.18}$$

Since S is linear, we have that

$$\mathsf{S}(\mathbf{f}_+ + \mathring{\mathsf{E}}_+(\mathcal{P}\mathbf{u}_+), \mathbf{f}_-, \mathbf{g}, \mathbf{h}) = \mathsf{S}(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) + \mathsf{S}(\mathring{\mathsf{E}}_+(\mathcal{P}\mathbf{u}_+), 0, 0, 0). \tag{2.2.19}$$

Consequently, (2.2.18) is equivalent to

$$(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) - \mathsf{S}(\mathring{\mathsf{E}}_+(\mathcal{P}\mathbf{u}_+), 0, 0, 0) = \mathsf{S}(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}). \tag{2.2.20}$$

In view of the embedding (2.2.15), we deduce that the operator $\mathsf{S}_{\mathcal{P}} : \mathsf{X}_w \to \mathsf{X}_w$, given by

$$\mathsf{S}_{\mathcal{P}}(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) := \mathsf{S}(\mathring{\mathsf{E}}_+(\mathcal{P}\mathbf{u}_+), 0, 0, 0), \tag{2.2.21}$$

is a compact operator.

We consider the operator $\mathsf{A}_{\mathcal{P}} := \mathbb{I} - \mathsf{S}_{\mathcal{P}} : \mathsf{X}_w \to \mathsf{X}_w$. Then, in view of the compactness of the operator $\mathsf{S}_{\mathcal{P}} : \mathsf{X}_w \to \mathsf{X}_w$ (due to the compactness of the embedding (2.2.15)), we have that the operator $\mathsf{A}_{\mathcal{P}} : \mathsf{X}_w \to \mathsf{X}_w$ is a Fredholm operator of index zero. Also, relation (2.2.20) is equivalent to

$$\mathsf{A}_{\mathcal{P}}(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) = \mathsf{S}(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}). \tag{2.2.22}$$

Next, we show that $\mathsf{A}_{\mathcal{P}} : \mathsf{X}_w \to \mathsf{X}_w$ is injective. To this end, we investigate the equation

$$\mathsf{A}_{\mathcal{P}}(\mathbf{u}_+^0, \pi_+^0, \mathbf{u}_-^0, \pi_-^0) = 0, \tag{2.2.23}$$

which is equivalent to the homogeneous version associated to the problem (2.2.4). In the second step of this proof, we have shown that the homogeneous version of (2.2.4) admits only the trivial solution. Hence, the injectivity of $\mathsf{A}_{\mathcal{P}} : \mathsf{X}_w \to \mathsf{X}_w$ is guaranteed.

Consequently, the operator $\mathsf{A}_{\mathcal{P}} : \mathsf{X}_w \to \mathsf{X}_w$ is an isomorphism, which means that, the equation (2.2.22) provides a solution the transmission problem (2.2.4) in the space X_w for the given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) \in \mathsf{Y}_w$. Indeed, the solution is provided by

$$(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) = (\mathsf{A}_{\mathcal{P}}^{-1} \circ \mathsf{S})(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}). \tag{2.2.24}$$

The existence of a solution of the problem (2.2.4) is thus proved.

Lastly, relation (2.2.24) together with the continuity of the operators

$$\mathsf{A}_{\mathcal{P}}^{-1} : \mathsf{X}_w \to \mathsf{X}_w, \quad \mathsf{S} : \mathsf{Y}_w \to \mathsf{X}_w, \tag{2.2.25}$$

(see also relation (2.2.60)), imply the existence of a constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \mathcal{P}, \mathfrak{L}) > 0$ such that relation (2.2.5) is satisfied. Moreover, Corollary 1.1.17 together with the fact that $\mathbf{u}_- \in \mathcal{H}^1(\mathsf{D}_-)^3$ implies that $\mathbf{u}_-$ vanishes at infinity in the sense of Leray. This concludes our proof. □

Now, we provide the well-posedness result for the transmission problem (2.2.4), in the case $\mathbf{u}_\infty \neq 0$ (see, [8, Remark 4.6], [71, Theorem 4.4]).

Theorem 2.2.3. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied, for $n = 3$. Let $\mathcal{P} \in L^\infty(\mathsf{D}_+)^{3\times 3}$ such that condition (1.2.22) holds. Then, for $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty) \in \mathsf{Y}_w \times \mathbb{R}^3$, the Poisson problem of transmission-type for the generalized Brinkman and Stokes systems (2.2.4) has a unique solution*

$$(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \tag{2.2.26}$$

satisfying the condition

$$(\mathbf{u}_+, \pi_+, \mathbf{u}_- - \mathbf{u}_\infty, \pi_-) \in \mathsf{X}_w. \tag{2.2.27}$$

In addition, the corresponding solution operator,

$$\mathsf{T} : \mathsf{Y}_w \times \mathbb{R}^3 \to \mathsf{X}_w, \tag{2.2.28}$$

is linear and bounded, and hence, there exists a constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \mathcal{P}, \mathfrak{L}) > 0$ *such that the unique solution of (2.2.4) satisfies the estimate*

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_- - \mathbf{u}_\infty, \pi_-)||_{\mathsf{X}_w} \leq C||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty)||_{\mathsf{Y}_w \times \mathbb{R}^3}, \tag{2.2.29}$$

and $\mathbf{u}_- - \mathbf{u}_\infty$ *vanishes at infinity in the sense of Leray.*

Proof. To show this result, we employ steps similar to those in the proof of [71, Theorem 4.4]. Let us consider the following change of variables

$$\mathbf{v}_+ := \mathbf{u}_+, \quad \mathbf{v}_- := \mathbf{u}_- - \mathbf{u}_\infty.$$

Consequently, we obtain the following transmission problem

$$\begin{cases} \Delta \mathbf{v}_+ - \mathcal{P}\mathbf{v}_+ - \nabla \pi_+ = \mathbf{f}_+|_{\mathsf{D}_+} \text{ in } \mathsf{D}_+, \\ \Delta \mathbf{v}_- - \nabla \pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \text{div } \mathbf{v}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \text{Tr}_{\mathsf{D}_+}\mathbf{v}_+ - \text{Tr}_{\mathsf{D}_-}\mathbf{v}_- = \mathbf{g} + \mathbf{u}_\infty \text{ on } \Gamma, \\ \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{v}_+, \pi_+, \mathbf{f}_+) - \mathbf{t}_{\mathsf{D}_-}(\mathbf{v}_-, \pi_-, \mathbf{f}_-) + \mathfrak{L}\text{Tr}_{\mathsf{D}_+}\mathbf{v}_+ = \mathbf{h} \text{ on } \Gamma, \end{cases} \tag{2.2.30}$$

which is equivalent to the transmission problem (2.2.4) in the unknowns $(\mathbf{v}_+, \pi_+, \mathbf{v}_-, \pi_-) \in \mathsf{X}_w$. Although, such a problem was shown to be well-posed via Theorem 2.2.2. We note the fact that the vector $\mathbf{u}_\infty$ is present in the right hand side of the transmission condition $(2.2.30)_3$, it will also be present in the estimate (2.2.29).

Now, it remains only to apply Theorem 2.2.2 in obtain the desired conclusion. This completes our proof. ☐

2.2.1 Transmission problem for the Stokes system in complementary Lipschitz domains in $\mathbb{R}^3$

This subsection is dedicated to the study of the transmission problem for the Stokes system in complementary Lipschitz domains in $\mathbb{R}^3$, i.e., the setting of Assumption 1.1.6 for $n = 3$. Although it can be easily seen that the well-posedness result associated to this particular transmission problem is a consequence of Theorem 2.2.2 and Theorem 2.2.3 (see also [71, Theorem 4.2, Theorem 4.4], [76, Lemma 4.1, Lemma 4.2, Theorem 4.3]), respectively, our goal is to show that, indeed, such a problem is well-posed. We want to prove this result by using *a different layer potential approach*, in order to construct the solution to this type of boundary value problem.

Again, let us consider the setting provided by Assumption 1.1.6, for $n = 3$. In addition, let Assumption 2.2.1 be satisfied, for $n = 3$. The considered transmission-type problem for the Stokes system reads as follows

$$\begin{cases} \Delta \mathbf{u}_+ - \nabla \pi_+ = \mathbf{f}_+|_{\mathsf{D}_+} \text{ in } \mathsf{D}_+, \\ \Delta \mathbf{u}_- - \nabla \pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \text{div } \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \text{Tr}_{\mathsf{D}_+}\mathbf{u}_+ - \text{Tr}_{\mathsf{D}_-}\mathbf{u}_- = \mathbf{g} \text{ on } \Gamma, \\ \mathbf{t}_{\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+) - \mathbf{t}_{\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-) + \mathfrak{L}\text{Tr}_{\mathsf{D}_+}\mathbf{u}_+ = \mathbf{h} \text{ on } \Gamma. \end{cases} \tag{2.2.31}$$

Let us describe the steps that we follow in order to show that the transmission problem (2.2.31), is well-posed. Firstly, we will state and prove the following lemma (see, [8, Lemma 4.1, Corollary 4.2]).

Lemma 2.2.4. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied, for $n = 3$. Then, the operators*

$$\begin{aligned} \mathbb{I} + \mathcal{V}_\Gamma \mathfrak{L} : H^{\frac{1}{2}}(\Gamma)^3 \to H^{\frac{1}{2}}(\Gamma)^3, \\ \mathbb{I} + \mathfrak{L}\mathcal{V}_\Gamma : H^{-\frac{1}{2}}(\Gamma)^3 \to H^{-\frac{1}{2}}(\Gamma)^3, \end{aligned} \tag{2.2.32}$$

are isomorphisms.

Proof. We proceed by stating the fact that the mappings

$$\mathfrak{L} : L^2(\Gamma)^3 \to L^2(\Gamma)^3, \ \mathcal{V}_\Gamma : H^{-\frac{1}{2}}(\Gamma)^3 \to H^{\frac{1}{2}}(\Gamma)^3, \tag{2.2.33}$$

are linear and continuous (see also Lemma 1.3.8) and the embedding $L^2(\Gamma)^3 \hookrightarrow H^{-\frac{1}{2}}(\Gamma)^3$ is continuous and compact. Hence, the operator

$$\mathcal{V}_\Gamma \mathfrak{L} : H^{\frac{1}{2}}(\Gamma)^3 \to H^{\frac{1}{2}}(\Gamma)^3, \tag{2.2.34}$$

is linear, continuous, compact and consequently, the operator $(2.2.32)_1$ is Fredholm of index zero.

Since the operators in relation (2.2.33) are self-adjoint, we deduce that the operator $(2.2.32)_2$ is the adjoint of operator $(2.2.32)_1$. Also, $(2.2.32)_2$ is a Fredholm operator of index zero (see, e.g., [68, Theorem 5.3.7]).

Now, in order to show that the Fredholm operators (2.2.32) of index zero are isomorpshims, we need to show that, for example, they are also one-to-one. To this end, we will concern ourselves with proving that $(2.2.32)_2$ is one-to-one.

Let us consider

$$\boldsymbol{\varphi}_0 \in \text{Ker } \{\mathbb{I} + \mathfrak{L}\mathcal{V}_\Gamma : H^{-\frac{1}{2}}(\Gamma)^3 \to H^{-\frac{1}{2}}(\Gamma)^3\}, \tag{2.2.35}$$

or equivalently, that

$$-\mathfrak{L}\mathcal{V}_\Gamma \boldsymbol{\varphi}_0 = \boldsymbol{\varphi}_0. \tag{2.2.36}$$

Next, we introduce the fields

$$\mathbf{u}_0 := \mathbf{V}_\Gamma \boldsymbol{\varphi}_0, \quad \pi_0 = \mathcal{Q}^s_\Gamma \boldsymbol{\varphi}_0. \tag{2.2.37}$$

These fields satisfy the Stokes system in $\mathsf{D}_\pm$ in view of the definition of the single layer potentials for the Stokes system (see Definition 1.3.3 and relation (1.3.14)). Now, relations $(1.3.22)_1$ and $(1.3.22)_3$ of Lemma 1.3.8 imply the following relations

$$\text{Tr}_{\mathsf{D}_+}\mathbf{u}_0 = \text{Tr}_{\mathsf{D}_-}\mathbf{u}_0, \quad \mathbf{t}_{\mathsf{D}_+}(\mathbf{u}_0, \pi_0) - \mathbf{t}_{\mathsf{D}_-}(\mathbf{u}_0, \pi_0) = \boldsymbol{\varphi}_0. \tag{2.2.38}$$

Furthermore, by applying Green formulas for the Stokes system (see relation (1.2.15)) in $\mathsf{D}_\pm$, we get

$$\begin{aligned} \langle \mathbf{t}_{\mathsf{D}_+}(\mathbf{u}_0, \pi_0), \text{Tr}_{\mathsf{D}_+}\mathbf{u}_0 \rangle_\Gamma = 2\langle \mathbb{E}(\mathbf{u}_0), \mathbb{E}(\mathbf{u}_0) \rangle_{\mathsf{D}_+}, \\ -\langle \mathbf{t}_{\mathsf{D}_-}(\mathbf{u}_0, \pi_0), \text{Tr}_{\mathsf{D}_-}\mathbf{u}_0 \rangle_\Gamma = 2\langle \mathbb{E}(\mathbf{u}_0), \mathbb{E}(\mathbf{u}_0) \rangle_{\mathsf{D}_-}. \end{aligned} \tag{2.2.39}$$

We add the relations in (2.2.39), while taking into account relations (2.2.38) and (2.2.36), we obtain the identity

$$-\langle \mathfrak{L}\mathcal{V}_\Gamma \boldsymbol{\varphi}_0, \mathcal{V}_\Gamma \boldsymbol{\varphi}_0 \rangle_\Gamma = 2\langle \mathbb{E}(\mathbf{u}_0), \mathbb{E}(\mathbf{u}_0) \rangle_{\mathsf{D}_+} + 2\langle \mathbb{E}(\mathbf{u}_0), \mathbb{E}(\mathbf{u}_0) \rangle_{\mathsf{D}_-}. \tag{2.2.40}$$

Due to the fact that $\mathfrak{L}$ satisfies the non-negativity condition (2.2.1), we have that each side of (2.2.40) is equal to zero and we get $\mathbb{E}(\mathbf{u}_0) = 0$ in $\mathsf{D}_\pm$. Consequently, there exist (see, e.g., [96, Lemma 3.1]) skew-symmetric matrices $A_\pm$ and constants $b_\pm \in \mathbb{R}^3$ such that

$$\mathbf{u}_0 = \mathbf{b}_\pm + \mathbf{A}_\pm \times \mathbf{x}, \text{ in } \mathsf{D}_\pm. \tag{2.2.41}$$

However, by taking into account the expression of the field $\mathbf{u}_0$ given in relation (2.2.37) and the asymptotic formula $\mathbf{V}_\Gamma \boldsymbol{\varphi}_0 = O(|\mathbf{x}|^{-1})$ (see, e.g., [73, relation (3.14)]) that $\mathbf{u}_0(\mathbf{x}) \to 0$, as $|\mathbf{x}| \to \infty$ and hence, $b_- = 0$ and $A_- = 0$. It follows that that $\mathbf{u}_0 = 0$ in D_- and thus, $\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_0 = 0$ on Γ.

Since $\mathbf{u}_0 := \mathbf{V}_\Gamma \boldsymbol{\varphi}_0$, we have

$$\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_0 = \mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_0 = 0, \tag{2.2.42}$$

and $\pi_0 = 0$ in D_-, since $\pi_0(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$, in view of the fact that $\mathcal{Q}^s_\Gamma \boldsymbol{\varphi}_0 = O(|\mathbf{x}|^{-2})$ (see, e.g., [73, relation (3.14)]).

Taking into account relation (2.2.42), we notice that the pair $(\mathbf{u}_0, \pi_0)$ is a solution of the interior Dirichlet problem for the Stokes system in $\mathsf{D}_+ \subset \mathbb{R}^3$. Such a problem has at most one solution (up to an additive constant pressure) in the space $H^1(\mathsf{D}_+)^3 \times L^2(\mathsf{D}_+)$ (see, e.g., [114, Theorem 10.6.2]), which is given by

$$\mathbf{u}_0 = 0, \ \pi_0 = c \text{ in } \mathsf{D}_+, \tag{2.2.43}$$

where $c \in \mathbb{R}$ is a constant.

Until now, we have shown that

$$\mathbf{u}_0 = 0 \text{ in } \mathsf{D}_\pm, \ \pi_0 = c \in \mathbb{R} \text{ in } \mathsf{D}_+, \ \pi_0 = 0 \text{ in } \mathsf{D}_-. \tag{2.2.44}$$

The fields that we have obtained in relation (2.2.44) satisfy the homogenenous version of the transmission problem (2.2.31) and in particular, the second transmission condition of the aforementioned transmission problem. We deduce

$$0 = \mathbf{t}_{\mathsf{D}_+}(\mathbf{u}_0, \pi_0) = -c\boldsymbol{\nu}, \tag{2.2.45}$$

and we get $\pi_0 = 0$. Due to this fact, we obtain

$$\mathbf{u}_0 = \mathbf{V}_\Gamma \varphi_0 = 0, \quad \pi_0 = \mathcal{Q}^s_\Gamma \varphi_0 = 0 \text{ in } \mathsf{D}_\pm. \tag{2.2.46}$$

Now, since $\mathrm{Tr}_{\mathsf{D}_\pm}\mathbf{u}_0 = \mathcal{V}_\Gamma \boldsymbol{\varphi}_\mathbf{0}$, we deduce that

$$\mathcal{V}_\Gamma \boldsymbol{\varphi}_\mathbf{0} = 0. \tag{2.2.47}$$

Relations (2.2.36) and (2.2.47) imply the fact that $\boldsymbol{\varphi}_\mathbf{0} = 0$. This shows that the Fredholm operator of index zero $(2.2.32)_2$ is one-to-one which means that the operator $(2.2.32)_2$ is an isomorphism. Consequently, the operator $(2.2.32)_1$ is an isomorphism as well (see, e.g., [68, Theorem 5.3.7]). This completes our proof. □

Secondly, we state and prove a lemma that shows that our transmission problem (2.2.31) has at most one solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_- - \mathbf{u}_\infty, \pi_-) \in \mathsf{X}_w$, where $\mathbf{u}_\infty \in \mathbb{R}^3$ is a constant vector (see [8, Lemma 4.1], [71, Lemma 4.1]).

Lemma 2.2.5. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied, for $n = 3$. Then, for $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty) \in \mathsf{Y}_w \times \mathbb{R}^3$, the Poisson problem of transmission-type for the Stokes system (2.2.31) has at most one solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)$ which satisfies $(\mathbf{u}_+, \pi_+, \mathbf{u}_- - \mathbf{u}_\infty, \pi_-) \in \mathsf{X}_w$.*

Proof. To prove our result, we use a similar reasoning to that in the proof of Lemma 4.1 of [71]. We consider $(\mathbf{u}_+^0, \pi_+^0, \mathbf{u}_-^0, \pi_-^0) \in \mathsf{X}_w$, the solution of the homogeneous version of our problem (2.2.31). Then, by [114, Proposition 10.6.1] (see also [98, Relations (4.4), (4.5), Lemma 4.2]), the vector fields $\mathbf{u}_\pm^0$ admit the following representation

$$\begin{aligned} \mathbf{u}_+^0 &= \mathbf{V}_\Gamma(\mathbf{t}_{\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0)) - \mathbf{W}_\Gamma(\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+^0), \\ \mathbf{u}_-^0 &= -\mathbf{V}_\Gamma(\mathbf{t}_{\mathsf{D}_-}(\mathbf{u}_-^0, \pi_-^0)) + \mathbf{W}_\Gamma(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-^0). \end{aligned} \tag{2.2.48}$$

Now, we apply the trace operators $\mathrm{Tr}_{\mathsf{D}_+}$ and $\mathrm{Tr}_{\mathsf{D}_-}$, to relations $(2.2.48)_1$ and $(2.2.48)_2$, respectively, while taking into account the jump properties of the Stokes layer potentials (see Lemma 1.3.8, Relations $(1.3.22)_1$ and $(1.3.22)_2$)

$$\begin{aligned} \left(\frac{1}{2}\mathbb{I} + \mathbb{K}_\Gamma\right)(\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+^0) &= \mathcal{V}_\Gamma(\mathbf{t}_{\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0)), \\ \left(-\frac{1}{2}\mathbb{I} + \mathbb{K}_\Gamma\right)(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-^0) &= \mathcal{V}_\Gamma(\mathbf{t}_{\mathsf{D}_-}(\mathbf{u}_-^0, \pi_-^0)). \end{aligned} \tag{2.2.49}$$

If one subtracts $(2.2.49)_2$ from $(2.2.49)_1$, while considering the transmission conditions of the homogeneous version of the transmission problem (2.2.31), we get the following operator equation

$$(\mathbb{I} + \mathcal{V}_\Gamma \mathfrak{L})(\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+^0) = 0. \tag{2.2.50}$$

Now, by Lemma 2.2.4, we have that the operator involved in relation (2.2.50) is an isomorphism and it follows that $\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+^0 = 0$. By taking into account the representation of the vector field $\mathbf{u}_+^0$ (relation $(2.2.48)_1$), we get

$$\mathbf{u}_+^0 = \mathbf{V}_\Gamma(\mathbf{t}_{\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0)) = 0 \text{ in } \mathsf{D}_+. \tag{2.2.51}$$

We apply the trace operator $\mathrm{Tr}_{\mathsf{D}_+}$ to relation (see 2.2.51) and we have

$$\mathcal{V}_\Gamma(\mathbf{t}_{\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0)) = 0 \text{ on } \Gamma. \tag{2.2.52}$$

It follows that (see, e.g., [74, Lemma 3.1]),

$$\mathbf{t}_{\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0) \in \mathrm{Ker}\{\mathcal{V}_\Gamma : H^{-\frac{1}{2}}(\Gamma)^3 \to H^{\frac{1}{2}}(\Gamma)^3\} = \mathbb{R}\nu. \tag{2.2.53}$$

Relations (2.2.51) and (2.2.53), together with the fact that $\mathcal{V}_\Gamma \boldsymbol{\nu} = 0$ in $\mathbb{R}^3$ (see, e.g., [114, Lemma 5.3.1]), imply that

$$\mathbf{u}_+^0 = 0 \text{ in } \mathsf{D}_+. \tag{2.2.54}$$

The transmission conditions of the homogeneous version of problem (2.2.31) give $\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-^0 = 0$ and $\mathbf{t}_{\mathsf{D}_-}(\mathbf{u}_-^0, \pi_-^0) = 0$. So,

$$\mathbf{u}_-^0 = 0 \text{ in } \mathsf{D}_-, \tag{2.2.55}$$

by relation $(2.2.48)_2$.

Using Stokes' equations and relations (2.2.54), (2.2.55), we deduce that

$$\pi_\pm^0 = c_\pm \in \mathbb{R} \text{ in } \mathsf{D}_\pm. \tag{2.2.56}$$

Recall that $\pi_-^0 \in L^2(\mathsf{D}_-)$. Hence $c_0 = 0$ and we have

$$\pi_-^0 = 0 \text{ in } \mathsf{D}_-. \tag{2.2.57}$$

Now, by the second transmission condition of the homogeneous version of the transmission problem, i.e., $(2.2.31)_2$, and relations (2.2.54), (2.2.55), (2.2.57), we get

$$0 = \mathbf{t}_{\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0) = -c_+\boldsymbol{\nu}. \tag{2.2.58}$$

It follows that $c_+ = 0$.

Consequently, we have shown that

$$\mathbf{u}_+^0 = 0,\ \pi_+^0 = 0 \text{ in } \mathsf{D}_+,\ \mathbf{u}_-^0 = 0,\ \pi_-^0 = 0 \text{ in } \mathsf{D}_-, \tag{2.2.59}$$

that is, the transmission problem (2.2.31) has at most one solution. This completes our proof. □

Let us now state and prove the well-posedness result, that we have obtained, for our transmission problem for the Stokes system in complementary Lipschitz domains in $\mathbb{R}^3$ in the case $\mathbf{u}_\infty = 0$ (see [8, Theorem 4.3], [71, Theorem 4.2], [98, Theorem 5.1]).

Theorem 2.2.6. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied, for $n = 3$. Then, for $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) \in \mathsf{Y}_w$, the Poisson problem of transmission-type for the Stokes system (2.2.31) has a unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}$. Moreover, there exists a linear and continuous operator*

$$\mathsf{S} : \mathsf{Y}_w \to \mathsf{X}_w, \tag{2.2.60}$$

that maps the given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) \in \mathsf{Y}_w$ to the unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_w$ of the problem (2.2.31), in the sense that, there is a constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \boldsymbol{\mathfrak{L}}) > 0$ such that

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_w} \leq C||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h})||_{\mathsf{Y}_w}. \tag{2.2.61}$$

In addition, $\mathbf{u}_-$ vanishes at infinity in the sense of Leray.

Proof. First of all, this well-posedness result is a consequence of Theorem 5.1 in [98]. Our objective will be to prove this theorem in a different way, which is inspired by the proof of Theorem 4.2 in [71].

Secondly, by Lemma 2.2.5, the uniqueness of the solution of the transmission problem (2.2.31) is assured.

Now, we devote our attention to the existence of the solution of our problem. To achieve this goal, we will construct a solution of the transmission problem (2.2.31) with the help of the Newtonian, single-layer and double-layer potential. The fields $(\mathbf{u}_\pm, \pi_\pm)$ are sought in the form

$$\begin{aligned} \mathbf{u}_\pm &= \mathcal{N}_{\mathsf{D}_\pm}\mathbf{f}_\pm + \mathbf{V}_\Gamma\boldsymbol{\varphi} + \mathbf{W}_\Gamma\boldsymbol{\phi} \text{ in } \mathsf{D}_\pm, \\ \pi_\pm &= \mathcal{Q}_{\mathsf{D}_\pm}\mathbf{f}_\pm + \mathcal{Q}_\Gamma^s\boldsymbol{\varphi} + \mathcal{Q}_\Gamma^d\boldsymbol{\phi} \text{ in } \mathsf{D}_\pm, \end{aligned} \tag{2.2.62}$$

where the densities $(\boldsymbol{\varphi}, \boldsymbol{\phi}) \in H^{-\frac{1}{2}}(\Gamma)^3 \times H^{\frac{1}{2}}(\Gamma)^3$ are unknown.

Due to our construction and by taking into account the mapping properties of layer potentials involved in relation (2.2.62) (see relations (1.3.9), (1.3.15) and (1.3.20)), we have that

$$(\mathbf{u}_+, \pi_+) \in H^1(\mathsf{D}_+)^3 \times L^2(\mathsf{D}_+), \quad (\mathbf{u}_-, \pi_-) \in \mathcal{H}^1(\mathsf{D}_-)^3 \times L^2(\mathsf{D}_-).$$

Moreover, due to relations (1.3.12), (1.3.14) and (1.3.18), we have that the pairs $(\mathbf{u}_+, \pi_+)$ and $(\mathbf{u}_-, \pi_-)$ satisfy relations $(2.2.31)_1$ and $(2.2.31)_2$, respectively.

Now, we use the transmission condition $(2.2.31)_3$ and relation (1.3.22) of Lemma 1.3.8, in order to obtain the value of the density $\boldsymbol{\phi} \in H^{\frac{1}{2}}(\Gamma)^3$, which is given by

$$\boldsymbol{\phi} = (\mathrm{Tr}_{\mathsf{D}_+}(\mathcal{N}_{\mathsf{D}_+}\mathbf{f}_+) - \mathrm{Tr}_{\mathsf{D}_-}(\mathcal{N}_{\mathsf{D}_-}\mathbf{f}_-)) - \mathbf{g}. \tag{2.2.63}$$

Furthermore, by using the transmission condition $(2.2.31)_3$ and, again, relation (1.3.22) of Lemma 1.3.8 to deduce the equation

$$(\mathbb{I} + \mathfrak{L}\mathcal{V}_\Gamma)\,\boldsymbol{\varphi} = \boldsymbol{\zeta} \in H^{-\frac{1}{2}}(\Gamma)^3, \tag{2.2.64}$$

where $\boldsymbol{\zeta} \in H^{-\frac{1}{2}}(\Gamma)^3$ is given by

$$\begin{aligned} \boldsymbol{\zeta} := \mathbf{h} - (\mathbf{t}_{\mathsf{D}_+}(\mathcal{N}_{\mathsf{D}_+}\mathbf{f}_+, \mathcal{Q}_{\mathsf{D}_+}\mathbf{f}_+, \mathbf{f}_+) - \mathbf{t}_{\mathsf{D}_-}(\mathcal{N}_{\mathsf{D}_-}\mathbf{f}_-, \mathcal{Q}_{\mathsf{D}_-}\mathbf{f}_-, \mathbf{f}_-)) \\ + \mathfrak{L}(\mathrm{Tr}_{\mathsf{D}_+}\mathcal{N}_{\mathsf{D}_+}\mathbf{f}_+) + \mathfrak{L}\left(-\frac{1}{2}\mathbb{I} + \mathbb{K}_\Gamma\right)\boldsymbol{\phi}. \end{aligned} \tag{2.2.65}$$

Note that, the operator $\mathbb{I} + \mathfrak{L}\mathcal{V}_\Gamma : H^{-\frac{1}{2}}(\Gamma)^3 \to H^{-\frac{1}{2}}(\Gamma)^3$ is an isomorphism (see Lemma 2.2.4) and consequently, the solution of equation (2.2.64) is unique and it is given by

$$\boldsymbol{\varphi} = (\mathbb{I} + \mathfrak{L}\mathcal{V}_\Gamma)^{-1}\boldsymbol{\zeta} \in H^{-\frac{1}{2}}(\Gamma)^3, \tag{2.2.66}$$

where the expression of $\boldsymbol{\zeta}$ is provided in relation (2.2.65).

Consequently, the densities $\boldsymbol{\phi}$ and $\boldsymbol{\varphi}$ given by relations (2.2.63) and (2.2.66), together with the layer potential representations (2.2.62) determine a solution of the problem (2.2.31) in the space X_w. This means that the existence of a solution is assured. Moreover, the fields $(\mathbf{u}_-, \pi_-)$ satisfy the Stokes system in the exterior domain. We apply Corollary 1.1.17 in order to deduce that $\mathbf{u}_-$ vanishes at ∞ in the sense of Leray.

Finally, since the transmission problem (2.2.31) has a unique solution, we deduce that the solution operator introduced by relation (2.2.60) is a well-defined, linear and continuous operator that maps the given data to the unique solution of (2.2.31). Moreover, there is a constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \mathfrak{L}) > 0$ such that the inequality (2.2.61) holds. This completes our proof. □

Lastly, we provide the well-posendess result of the transmission problem (2.2.31) in the case $\mathbf{u}_\infty \neq 0$ (see, e.g., [8, Theorem 4.4], [71, Theorem 4.4]).

Theorem 2.2.7. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied for $n = 3$. Then, for $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty) \in \mathsf{Y}_w \times \mathbb{R}^3$, the Poisson problem of transmission-type for the Stokes system (2.2.31) has a unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)$ satisfying the condition $(\mathbf{u}_+, \pi_+, \mathbf{u}_- - \mathbf{u}_\infty, \pi_-) \in \mathsf{X}_w$. Moreover, there is a constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \mathfrak{L}) > 0$ such that*

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_- - \mathbf{u}_\infty, \pi_-)||_{\mathsf{X}_w} \leq C||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty)||_{\mathsf{Y}_w \times \mathbb{R}^3}, \tag{2.2.67}$$

and $\mathbf{u}_- - \mathbf{u}_\infty$ vanishes at infinity in the sense of Leray.

Proof. In order to prove this result, we will follow similar steps to those in the proof of [71, Theorem 4.4]. Consequently, we consider the change of variables

$$\mathbf{v}_+ := \mathbf{u}_+, \quad \mathbf{v}_- := \mathbf{u}_- - \mathbf{u}_\infty,$$

and we obtain

$$\begin{cases} \Delta \mathbf{v}_+ - \nabla \pi_+ = \mathbf{f}_+|_{\mathsf{D}_+} \text{ in } \mathsf{D}_+, \\ \Delta \mathbf{v}_- - \nabla \pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \operatorname{div} \mathbf{v}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \mathrm{Tr}_{\mathsf{D}_+}\mathbf{v}_+ - \mathrm{Tr}_{\mathsf{D}_-}\mathbf{v}_- = \mathbf{g} + \mathbf{u}_\infty \text{ on } \Gamma, \\ \mathbf{t}_{\mathsf{D}_+}(\mathbf{v}_+, \pi_+, \mathbf{f}_+) - \mathbf{t}_{\mathsf{D}_-}(\mathbf{v}_-, \pi_-, \mathbf{f}_-) + \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{v}_+ = \mathbf{h} \text{ on } \Gamma. \end{cases} \tag{2.2.68}$$

Let us note that the transmission problem (2.2.68) is equivalent to the transmission problem (2.2.31) in the unknowns $(\mathbf{v}_+, \pi_+, \mathbf{v}_-, \pi_-) \in \mathsf{X}_w$. The transmission problem (2.2.31) was shown to be well-posed (see Theorem 2.2.6). Moreover, since $\mathbf{u}_\infty$ is present in the right hand side of the transmission condition $(2.2.68)_3$, it will also appear in the estimate (2.2.67). Hence, our assumptions stated in the theorem are all valid. This completes our proof. □

2.3 Transmission problem for the generalized Brinkman and classical Brinkman systems in complementary Lipschitz domains in $\mathbb{R}^3$

In this section we will state and prove a well-posedness result, for a transmission-type problem, which was obtained in the setting of Assumption 1.1.6 for $n = 3$, i.e., complementary Lipschitz domains in $\mathbb{R}^3$. We have considered a generalized version of the Brinkman in the bounded Lipschitz domain D_+ and the Brinkman system in the complementary Lipschitz set D_-. Also, let Assumption 2.2.1 be satisfied, for $n = 3$.

In order to keep our arguments clear, let us introduce the following spaces, namely the space of solutions,

$$\mathsf{X}_{\mathcal{B}} := H^1_{\mathrm{div}}(\mathsf{D}_+)^3 \times L^2(\mathsf{D}_+) \times H^1_{\mathrm{div}}(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-) \tag{2.3.1}$$

and the space of given data,

$$\mathsf{Y}_{\mathcal{B}} := \tilde{H}^{-1}(\mathsf{D}_+)^3 \times \tilde{H}^{-1}(\mathsf{D}_-)^3 \times H^{\frac{1}{2}}(\Gamma)^3 \times H^{-\frac{1}{2}}(\Gamma)^3, \tag{2.3.2}$$

respectively. We point out the fact that the space $\mathfrak{M}(\mathsf{D}_-)$ is introduced in Definition 2.1.1.

Let us emphasize the fact that the presence of the Brinkman system in the exterior Lipschitz domain D_-, provided in Assumption 1.1.6, $n = 3$, leads to *the use of classical Sobolev spaces, instead of the weighted Sobolev spaces* (as in the case of the Stokes system in exterior Lipschitz domains), in order to find the velocity field in D_-. This is a consequence of the behavior of the fundamental solution of the Brinkman system at infinity, in the case $n = 3$.

We turn our attention to the transmission problem for the generalized Brinkman and classical Brinkman system, which is given by

$$\begin{cases} \Delta \mathbf{u}_+ - \mathcal{P}\mathbf{u}_+ - \nabla \pi_+ = \mathbf{f}_+|_{\mathsf{D}_+} \text{ in } \mathsf{D}_+, \\ \Delta \mathbf{u}_- - \alpha\mathbf{u}_- - \nabla \pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \operatorname{div} \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ - \mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_- = \mathbf{g} \text{ on } \Gamma, \\ \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+) - \mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-) + \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ = \mathbf{h} \text{ on } \Gamma, \end{cases} \tag{2.3.3}$$

where unknown fields are $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{\mathcal{B}}$.

Let us state and prove the well-posedness result that we have obtained for problem (2.3.3) (see also [7, Theorem 3.3] and [76, Theorem 4.4] in the case of compact Riemannian manifolds). In addition, the following well-posedness result also provides the far field conditions that our solution satisfies (see Remark 1.3.9).

Theorem 2.3.1. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied for $n = 3$. Let $\alpha > 0$ be a constant. Let $\mathcal{P} \in L^\infty(\mathsf{D}_+)^{3\times 3}$ be such that the condition (1.2.22) holds. Then, for $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) \in \mathsf{Y}_{\mathcal{B}}$ given, the Poisson problem of transmission-type for the generalized and classical Brinkman systems (2.3.3) has a unique solution*

$$(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{\mathcal{B}}. \tag{2.3.4}$$

In addition, the corresponding solution operator,

$$\mathsf{T}_{\mathcal{B}} : \mathsf{Y}_{\mathcal{B}} \to \mathsf{X}_{\mathcal{B}}, \tag{2.3.5}$$

is linear and bounded, and hence, there exists a constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \mathcal{P}, \mathfrak{L}) > 0$ such that the unique solution of (2.3.3) satisfies

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_{\mathcal{B}}} \leq C||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h})||_{\mathsf{Y}_{\mathcal{B}}}. \tag{2.3.6}$$

Moreover, the pair $(\mathbf{u}_-, \pi_-)$ satisfies the following far field conditions

$$\mathbf{u}_-(\mathbf{x}) = O(|\mathbf{x}|^{-2}), \ \nabla\mathbf{u}_-(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \ \pi_-(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \tag{2.3.7}$$

as $|\mathbf{x}| \to \infty$.

Proof. The proof of this result follows similar steps to those in Theorem 4.4 of [76]. Firstly, we concern ourselves with the uniqueness of our solution. We take two different solutions of the problem (2.3.3) and we denote their difference by $(\mathbf{u}_+^0, \pi_+^0, \mathbf{u}_-^0, \pi_-^0) \in \mathsf{X}_{\mathcal{B}}$. We deduce that the fields $(\mathbf{u}_+^0, \pi_+^0, \mathbf{u}_-^0, \pi_-^0)$ satisfy the homogeneous version of our problem (2.3.3) in the space $\mathsf{X}_{\mathcal{B}}$.

By applying now the Green formulas for the classical and generalized Brinkman system (i.e., relations (1.2.18) and (1.2.28)), we deduce that

$$\begin{aligned} \langle \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0), \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+^0\rangle_\Gamma &= 2\langle \mathbb{E}(\mathbf{u}_+^0), \mathbb{E}(\mathbf{u}_+^0)\rangle_{\mathsf{D}_+} + \langle \mathcal{P}\mathbf{u}_+^0, \mathbf{u}_+^0\rangle_{\mathsf{D}_+}, \\ \langle \mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-^0, \pi_-^0), \mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-^0\rangle_\Gamma &= -2\langle \mathbb{E}(\mathbf{u}_-^0), \mathbb{E}(\mathbf{u}_-^0)\rangle_{\mathsf{D}_-} - \alpha\langle \mathbf{u}_-^0, \mathbf{u}_-^0\rangle_{\mathsf{D}_-}. \end{aligned} \tag{2.3.8}$$

If we subtract $(2.3.8)_2$ from $(2.3.8)_1$, we get

$$\begin{aligned} &\langle \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0), \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+^0\rangle_\Gamma - \langle \mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-^0, \pi_-^0), \mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-^0\rangle_\Gamma \\ &= 2\langle \mathbb{E}(\mathbf{u}_+^0), \mathbb{E}(\mathbf{u}_+^0)\rangle_{\mathsf{D}_+} + \langle \mathcal{P}\mathbf{u}_+^0, \mathbf{u}_+^0\rangle_{\mathsf{D}_+} + 2\langle \mathbb{E}(\mathbf{u}_-^0), \mathbb{E}(\mathbf{u}_-^0)\rangle_{\mathsf{D}_-} + \alpha\langle \mathbf{u}_-^0, \mathbf{u}_-^0\rangle_{\mathsf{D}_-}, \end{aligned}$$

and hence

$$\begin{aligned} -\langle \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+^0, \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+^0\rangle_\Gamma &= 2\langle \mathbb{E}(\mathbf{u}_+^0), \mathbb{E}(\mathbf{u}_+^0)\rangle_{\mathsf{D}_+} + \langle \mathcal{P}\mathbf{u}_+^0, \mathbf{u}_+^0\rangle_{\mathsf{D}_+} \\ &\quad + 2\langle \mathbb{E}(\mathbf{u}_-^0), \mathbb{E}(\mathbf{u}_-^0)\rangle_{\mathsf{D}_-} + \alpha\langle \mathbf{u}_-^0, \mathbf{u}_-^0\rangle_{\mathsf{D}_-}. \end{aligned} \tag{2.3.9}$$

Note that each side of relation (2.3.9) is null, by taking into account that the matrix operator $\mathfrak{L}$ satisfies condition (2.2.1). On the other hand, by making use of the fact that matrix operator $\mathcal{P}$ satisfies condition (1.2.22), we deduce that

$$\mathbf{u}_+^0 = 0, \ \pi_+^0 = c \in \mathbb{R} \text{ in } \mathsf{D}_+, \ \mathbf{u}_-^0 = 0, \ \pi_-^0 = d \in \mathbb{R} \text{ in } \mathsf{D}_-. \tag{2.3.10}$$

We apply now Lemma 2.1.3 which states that $\pi^0_-(\mathbf{x}) = O(|\mathbf{x}|^{-1})$ as $|\mathbf{x}| \to \infty$. Consequently, we have that π^0_- vanishes at infinity, or, equivalently, $d = 0$.

Let us now use the definition of the conormal derivative operator for the generalized Brinkman system (see Definition 1.2.14) in order to obtain

$$\mathbf{t}_{\mathcal{P},\nu,\mathsf{D}_+}(\mathbf{u}^0_+, \pi^0_+) = -c\boldsymbol{\nu}. \tag{2.3.11}$$

On the other hand, by using relation $(2.3.3)_4$ and we get

$$\mathbf{t}_{\mathcal{P},\nu,\mathsf{D}_+}(\mathbf{u}^0_+, \pi^0_+) = 0.$$

Hence, we deduce that $c = 0$.

In conclusion, we have shown that

$$\mathbf{u}^0_+ = 0,\ \pi^0_+ = 0 \text{ in } \mathsf{D}_+,\ \mathbf{u}^0_- = 0,\ \pi^0_- = 0 \text{ in } \mathsf{D}_-, \tag{2.3.12}$$

that is, our problem (2.3.3) has at most one solution.

Secondly, we aim to show that a solution of the transmission problem (2.3.3) exists. To this end, we proceed in the following way. Note that, the transmission problem for the Stokes and Brinkman system in complementary Lipschitz domains in $\mathbb{R}^3$ (see relation (2.3.26)) is well-posed (see Theorem 2.3.3). Due to its well-posedness, we are able to introduce the solution operator (see relation (2.3.36))

$$\mathsf{S} : \mathsf{Y}_{\mathcal{B}} \to \mathsf{X}_{\mathcal{B}}. \tag{2.3.13}$$

The role of this operator is to map the given data to the solution of the transmission problem for the Brinkman system in complementary Lipschitz domains in $\mathbb{R}^3$. In view of the well-posedness of the transmission problem for the Stokes and Brinkman system in complementary Lipschitz domains in $\mathbb{R}^3$, the operator $\mathsf{S} : \mathsf{Y}_{\mathcal{B}} \to \mathsf{X}_{\mathcal{B}}$ is well-defined, linear and continuous.

Next, we will view the problem (2.3.3) in terms of the solution operator $\mathsf{S} : \mathsf{Y}_{\mathcal{B}} \to \mathsf{X}_{\mathcal{B}}$. To accomplish this, we rewrite the transmission problem (2.3.3) in the equivalent form

$$\begin{cases} \Delta\mathbf{u}_+ - \nabla\pi_+ = \mathbf{f}_+|_{\mathsf{D}_+} + \mathcal{P}\mathbf{u}_+ \text{ in } \mathsf{D}_+, \\ \Delta\mathbf{u}_- - \alpha\mathbf{u}_- - \nabla\pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \operatorname{div} \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ - \mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_- = \mathbf{g} \text{ on } \Gamma, \\ \mathbf{t}_{\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+ + \mathring{\mathsf{E}}_+(\mathcal{P}\mathbf{u}_+)) - \mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-) + \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ = \mathbf{h} \text{ on } \Gamma, \end{cases} \tag{2.3.14}$$

where, recall that, $\mathring{\mathsf{E}}_+$ denotes the operator of extension by zero outside D_+. Note that, here, we have used Remark 1.2.16 in order to express the $\mathbf{t}_{\mathcal{P},\mathsf{D}_+}$ in terms of $\mathbf{t}_{\mathsf{D}_+}$.

Now, it is possible to write the problem (2.3.14) in the form

$$(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) = \mathsf{S}(\mathbf{f}_+ + \mathring{\mathsf{E}}_+(\mathcal{P}\mathbf{u}_+), \mathbf{f}_-, \mathbf{g}, \mathbf{h}). \tag{2.3.15}$$

If we use the fact that the operator S is linear and

$$\mathsf{S}(\mathbf{f}_+ + \mathring{\mathsf{E}}_+(\mathcal{P}\mathbf{u}_+), \mathbf{f}_-, \mathbf{g}, \mathbf{h}) = \mathsf{S}(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) + \mathsf{S}(\mathring{\mathsf{E}}_+(\mathcal{P}\mathbf{u}_+), 0, 0, 0), \tag{2.3.16}$$

we obtain

$$(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) - \mathsf{S}(\mathring{\mathsf{E}}_+(\mathcal{P}\mathbf{u}_+), 0, 0, 0) = \mathsf{S}(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}), \tag{2.3.17}$$

which is equivalent to relation (2.3.15).

Furthermore, since $\mathcal{P} \in L^\infty(\mathsf{D}_+)^{3\times 3}$, we are able to deduce that the embedding

$$\mathcal{P}\mathbf{u}_+ \in L^2(\mathsf{D}_+) \hookrightarrow \tilde{H}^{-1}(\mathsf{D}_+)^3, \tag{2.3.18}$$

in the sense that

$$\mathring{\mathsf{E}}_+(\mathcal{P}\mathbf{u}_+) \in \tilde{H}^{-1}(\mathsf{D}_+)^3, \tag{2.3.19}$$

is compact. This fact allows us to deduce that the operator $\mathsf{S}_\mathcal{P} : \mathsf{X}_\mathcal{B} \to \mathsf{X}_\mathcal{B}$ given by

$$\mathsf{S}_\mathcal{P}(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) = \mathsf{S}(\mathring{\mathsf{E}}_+(\mathcal{P}\mathbf{u}_+), 0, 0, 0), \tag{2.3.20}$$

is compact in view of relation (2.3.18). Note that $\mathsf{S}_\mathcal{P} : \mathsf{X}_\mathcal{B} \to \mathsf{X}_\mathcal{B}$ is also well-defined, linear and continuous.

Thus, equation (2.3.17) can be written in the equivalent form

$$\mathsf{A}_\mathcal{P}(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) = \mathsf{S}(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}), \tag{2.3.21}$$

where it is obvious that $\mathsf{A}_\mathcal{P} : \mathsf{X}_\mathcal{B} \to \mathsf{X}_\mathcal{B}$ given by

$$\mathsf{A}_\mathcal{P} := \mathbb{I} - \mathsf{S}_\mathcal{P}, \tag{2.3.22}$$

is a Fredholm operator of index zero.

Moreover, we will show that $\mathsf{A}_\mathcal{P} : \mathsf{X}_\mathcal{B} \to \mathsf{X}_\mathcal{B}$ is also one-to-one. To achieve this, we look at the equation

$$\mathsf{A}_\mathcal{P}(\mathbf{u}^0_+, \pi^0_+, \mathbf{u}^0_-, \pi^0_-) = \mathbf{0} \tag{2.3.23}$$

and we notice that the homogeneous version of our transmission problem (2.3.3) is equivalent to this relation (2.3.23). Our problem (2.3.3) has at most one solution, as we have seen in the proof of this result. Then, we have that equation (2.3.23) admits only the trivial solution in the space $\mathsf{X}_\mathcal{B}$, which proves that Ker $\{\mathsf{A}_\mathcal{P} : \mathsf{X}_\mathcal{B} \to \mathsf{X}_\mathcal{B}\} = \{\mathbf{0}\}$.

To conclude, we have shown that the Fredholm operator with index zero, $\mathsf{A}_\mathcal{P} : \mathsf{X}_\mathcal{B} \to \mathsf{X}_\mathcal{B}$ is one-to-one. This means that $\mathsf{A}_\mathcal{P} : \mathsf{X}_\mathcal{B} \to \mathsf{X}_\mathcal{B}$ is an isomorphism. Consequently, equation (2.3.21) has a unique solution in the space $\mathsf{X}_\mathcal{B}$,

$$(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) = (\mathsf{A}_\mathcal{P}^{-1} \circ \mathsf{S})(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}), \tag{2.3.24}$$

which is also the unique solution of the transmission problem (2.3.3).

Finally, the continuity of the operators

$$\mathsf{A}_\mathcal{P}^{-1} : \mathsf{X}_\mathcal{B} \to \mathsf{X}_\mathcal{B}, \quad \mathsf{S} : \mathsf{Y}_\mathcal{B} \to \mathsf{X}_\mathcal{B}, \tag{2.3.25}$$

imply the existence of a constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \mathcal{P}, \mathfrak{L}) > 0$ such that the estimate (2.3.6) holds. This concludes the proof of our theorem. □

2.3.1 Transmission problem for the Stokes and Brinkman systems in complementary Lipschitz domains in $\mathbb{R}^3$

This subsection is dedicated to the study of the transmission problem for the Stokes and Brinkman systems in complementary Lipschitz domains in $\mathbb{R}^3$, i.e., the setting of Assumption 1.1.6 for $n = 3$. Although it can be easily seen that the well-posedness result associated to this particular transmission problem is a consequence of Theorem 2.3.1 (see also [71, Theorem 4.2, Theorem 4.4], [76, Theorem 4.3]), respectively, our aim to to prove the well-posedness of such a boundary value problem. We prove this result in a different manner, that is, we use *a different layer potential approach* based on a combination of single-layer and double-layer potentials with a Newtonian potential in order to construct the solution for this type of boundary value problem.

Let us consider the setting provided by Assumption 1.1.6, in the case $n = 3$. In addition, let Assumption 2.2.1 be satisfied, for $n = 3$. The considered transmission-type problem for the Stokes and Brinkman systems is given by

$$\begin{cases} \Delta \mathbf{u}_+ - \nabla \pi_+ = \mathbf{f}_+|_{\mathsf{D}_+} \text{ in } \mathsf{D}_+, \\ \Delta \mathbf{u}_- - \alpha \mathbf{u}_- - \nabla \pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \operatorname{div} \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \mathrm{Tr}_{\mathsf{D}_+} \mathbf{u}_+ - \mathrm{Tr}_{\mathsf{D}_-} \mathbf{u}_- = \mathbf{g} \text{ on } \Gamma, \\ \mathbf{t}_{\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+) - \mathbf{t}_{\alpha, \mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-) + \boldsymbol{\mathfrak{L}} \mathrm{Tr}_{\mathsf{D}_+} \mathbf{u}_+ = \mathbf{h} \text{ on } \Gamma. \end{cases} \tag{2.3.26}$$

First of all, we have a preliminary result in which we will show that the transmission problem (2.3.26) has at most one solution (see [7, Lemma 3.1], [71, Lemma 4.1]).

Lemma 2.3.2. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied for $n = 3$. Let $\alpha > 0$ be a constant. Then, for the given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) \in \mathsf{Y}_{\mathcal{B}}$, the Poisson problem of transmission-type for the Stokes and Brinkman systems (2.3.26) has at most one solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{\mathcal{B}}$.*

Proof. We prove this result by following similar arguments as in the proof of Lemma 4.1 of [71]. We proceed by considering the difference of two possible solutions of (2.3.26) and we denote this difference by $(\mathbf{u}_+^0, \pi_+^0, \mathbf{u}_-^0, \pi_-^0) \in \mathsf{X}_{\mathcal{B}}$. We notice that the fields $(\mathbf{u}_+^0, \pi_+^0, \mathbf{u}_-^0, \pi_-^0) \in \mathsf{X}_{\mathcal{B}}$ satisfy the homogeneous version of the problem (2.3.26).

Now, if we apply the Green formulas for the Stokes and the Brinkman system (see relations (1.2.15) and (1.2.18)), we get

$$\begin{aligned} &\langle \mathbf{t}_{\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0, 0), \mathrm{Tr}_{\mathsf{D}_+} \mathbf{u}_+^0 \rangle_\Gamma = 2 \langle \mathbb{E}(\mathbf{u}_+^0), \mathbb{E}(\mathbf{u}_+^0) \rangle_{\mathsf{D}_+}, \\ &\langle \mathbf{t}_{\alpha, \mathsf{D}_-}(\mathbf{u}_-^0, \pi_-^0, 0), \mathrm{Tr}_{\mathsf{D}_-} \mathbf{u}_-^0 \rangle_\Gamma = -2 \langle \mathbb{E}(\mathbf{u}_-^0), \mathbb{E}(\mathbf{u}_-^0) \rangle_{\mathsf{D}_-} - \alpha \langle \mathbf{u}_-^0, \mathbf{u}_-^0 \rangle_{\mathsf{D}_-}. \end{aligned} \tag{2.3.27}$$

By subtracting $(2.3.27)_2$ from $(2.3.27)_1$, we obtain the equality

$$\begin{aligned} &\langle \mathbf{t}_{\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0, 0), \mathrm{Tr}_{\mathsf{D}_+} \mathbf{u}_+^0 \rangle_\Gamma - \langle \mathbf{t}_{\alpha, \mathsf{D}_-}(\mathbf{u}_-^0, \pi_-^0, 0), \mathrm{Tr}_{\mathsf{D}_-} \mathbf{u}_-^0 \rangle_\Gamma = \\ &2 \langle \mathbb{E}(\mathbf{u}_+^0), \mathbb{E}(\mathbf{u}_+^0) \rangle_{\mathsf{D}_+} + 2 \langle \mathbb{E}(\mathbf{u}_-^0), \mathbb{E}(\mathbf{u}_-^0) \rangle_{\mathsf{D}_-} + \alpha \langle \mathbf{u}_-^0, \mathbf{u}_-^0 \rangle_{\mathsf{D}_-}. \end{aligned} \tag{2.3.28}$$

Furthermore, by applying the transmission conditions from the homogeneous version of (2.3.26), we obtain the relation

$$2 \langle \mathbb{E}(\mathbf{u}_+^0), \mathbb{E}(\mathbf{u}_+^0) \rangle_{\mathsf{D}_+} + 2 \langle \mathbb{E}(\mathbf{u}_-^0), \mathbb{E}(\mathbf{u}_-^0) \rangle_{\mathsf{D}_-} + \alpha \langle \mathbf{u}_-^0, \mathbf{u}_-^0 \rangle_{\mathsf{D}_-} = -\langle \boldsymbol{\mathfrak{L}} \mathrm{Tr}_{\mathsf{D}_+} \mathbf{u}_+^0, \mathrm{Tr}_{\mathsf{D}_+} \mathbf{u}_+^0 \rangle_\Gamma. \tag{2.3.29}$$

In view of the fact that $\mathfrak{L}$ satisfies the non-negativity condition (2.2.1), we deduce that the left-hand side of relation (2.3.29) is non-negative and the right-hand side of relation (2.3.29) is non-positive. It follows that, both sides are null, that is

$$\begin{aligned} 2\langle \mathbb{E}(\mathbf{u}_+^0), \mathbb{E}(\mathbf{u}_+^0)\rangle_{\mathsf{D}_+} + 2\langle \mathbb{E}(\mathbf{u}_-^0), \mathbb{E}(\mathbf{u}_-^0)\rangle_{\mathsf{D}_-} + \alpha\langle \mathbf{u}_-^0, \mathbf{u}_-^0\rangle_{\mathsf{D}_-} = 0, \\ \langle \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+^0, \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+^0\rangle_\Gamma = 0. \end{aligned} \tag{2.3.30}$$

By taking into account the Brinkman equation and relation (2.3.29), we have that

$$\mathbf{u}_-^0 = 0,\ \pi_-^0 = d \in \mathbb{R} \text{ in } \mathsf{D}_-,\ \mathbb{E}(\mathbf{u}_+^0) = 0,\ \text{in } \mathsf{D}_+, \tag{2.3.31}$$

and due to the behavior of π_-^0 at infinity (see Lemma 2.1.3), we get

$$\mathbf{u}_-^0 = 0,\ \pi_-^0 = 0 \text{ in } \mathsf{D}_-. \tag{2.3.32}$$

On the other hand, by the uniqueness of the solution of the interior Dirichlet problem associated to the Stokes system (see, e.g., [114, Theorem 10.6.2]), we deduce that

$$\mathbf{u}_+^0 = 0,\ \pi_+^0 = c \in \mathbb{R} \text{ in } \mathsf{D}_+. \tag{2.3.33}$$

By employing the definition of the conormal derivative operator for the Stokes system (see Definition 1.2.3), and we get

$$\mathbf{t}_{\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0, 0) = -c\boldsymbol{\nu}. \tag{2.3.34}$$

Now, it remains to apply the second transmission condition of the homogeneous version of the problem (2.3.26) and we obtain

$$\mathbf{t}_{\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0, 0) = 0.$$

which implies that $c = 0$. Consequently, we have that

$$\mathbf{u}_+^0 = 0,\ \pi_+^0 = 0,\ \text{in } \mathsf{D}_+, \tag{2.3.35}$$

which shows that our problem (2.3.26) has at most one solution. This concludes our proof. □

We are now able to state and prove the well-posedness result for the transmission problem (2.3.26) (see, [7, Theorem 3.2], [71, Theorem 4.2]).

Theorem 2.3.3. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied for $n = 3$. Let $\alpha > 0$ be a constant. Then, for the given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) \in \mathsf{Y}_{\mathcal{B}}$, the Poisson problem of transmission-type for the Stokes and Brinkman systems (2.3.26) has a unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{\mathcal{B}}$. In addition, the operator*

$$\mathsf{S} : \mathsf{Y}_{\mathcal{B}} \to \mathsf{X}_{\mathcal{B}}, \tag{2.3.36}$$

which maps the given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) \in \mathsf{Y}_{\mathcal{B}}$ to the corresponding solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{\mathcal{B}}$ of the transmission problem (2.3.26) is linear and continuous. Consequently, there is a constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \mathfrak{L}) > 0$ such that:

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_{\mathcal{B}}} \le C||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h})||_{\mathsf{Y}_{\mathcal{B}}}. \tag{2.3.37}$$

Moreover, $\mathbf{u}_-$, π_- satisfy the far field conditions

$$\mathbf{u}_-(\mathbf{x}) = O(|\mathbf{x}|^{-2}),\ \nabla\mathbf{u}_-(\mathbf{x}) = O(|\mathbf{x}|^{-1}),\ \pi_-(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \tag{2.3.38}$$

as $|\mathbf{x}| \to \infty$.

Proof. The proof of this result follows similar steps to the proof of [71, Theorem 4.2]. First, we emphasize the fact that the problem (2.3.26) has at most one solution, due to Lemma 2.3.2.

Secondly, we turn our attention to the existence argument. We seek a solution of the problem (2.3.26) in the form

$$\begin{aligned} \mathbf{u}_+ &= \mathcal{N}_{\mathsf{D}_+}\mathbf{f}_+ + \mathbf{v}_+,\ \pi_+ = \mathcal{Q}_{\mathsf{D}_+}\mathbf{f}_+ + r_+,\ \text{in } \mathsf{D}_+,\\ \mathbf{u}_- &= \mathcal{N}_{\alpha,\mathsf{D}_-}\mathbf{f}_- + \mathbf{v}_-,\ \pi_- = \mathcal{Q}_{\alpha,\mathsf{D}_-}\mathbf{f}_- + r_-,\ \text{in } \mathsf{D}_-. \end{aligned} \tag{2.3.39}$$

Now, we substitute the fields given in relation (2.3.39) into (2.3.26) and we get a new version of our transmission problem, which is

$$\begin{cases} \Delta\mathbf{v}_+ - \nabla r_+ = 0 \text{ in } \mathsf{D}_+,\\ \Delta\mathbf{v}_- - \alpha\mathbf{v}_- - \nabla r_- = 0 \text{ in } \mathsf{D}_-,\\ \operatorname{div}\mathbf{v}_\pm = 0 \text{ in } \mathsf{D}_\pm,\\ \mathrm{Tr}_{\mathsf{D}_+}\mathbf{v}_+ - \mathrm{Tr}_{\mathsf{D}_-}\mathbf{v}_- = \mathbf{g}_1 \text{ on } \Gamma,\\ \mathbf{t}_{\mathsf{D}_+}(\mathbf{v}_+, r_+) - \mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{v}_-, r_-) + \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{v}_+ = \mathbf{h}_1 \text{ on } \Gamma, \end{cases} \tag{2.3.40}$$

in the unknowns $(\mathbf{v}_+, r_+, \mathbf{v}_-, r_-)$ and the quantities $\mathbf{g}_1$ and $\mathbf{h}_1$ are given by

$$\begin{aligned} \mathbf{g}_1 =&\ \mathbf{g} - (\mathrm{Tr}_{\mathsf{D}_+}(\mathcal{N}_{\mathsf{D}_+}\mathbf{f}_+) - \mathrm{Tr}_{\mathsf{D}_-}(\mathcal{N}_{\alpha,\mathsf{D}_-}\mathbf{f}_-)) \in H^{\frac{1}{2}}(\Gamma)^3,\\ \mathbf{h}_1 =&\ \mathbf{h} - \mathbf{t}_{\mathsf{D}_+}(\mathcal{N}_{\mathsf{D}_+}\mathbf{f}_+, \mathcal{Q}_{\mathsf{D}_+}\mathbf{f}_+, \mathbf{f}_+) + \mathbf{t}_{\alpha,\mathsf{D}_-}(\mathcal{N}_{\alpha,\mathsf{D}_-}\mathbf{f}_-, \mathcal{Q}_{\alpha,\mathsf{D}_-}\mathbf{f}_-, \mathbf{f}_-)\\ &- \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}(\mathcal{N}_{\mathsf{D}_+}\mathbf{f}_+) \in H^{-\frac{1}{2}}(\Gamma)^3. \end{aligned} \tag{2.3.41}$$

Furthermore, we are searching for the fields $\mathbf{v}_\pm$ and $r_\pm$ in the form

$$\begin{aligned} \mathbf{v}_+ &= \mathbf{W}_\Gamma\boldsymbol{\phi} + \mathbf{V}_\Gamma\boldsymbol{\psi},\ r_+ = \mathcal{Q}^d_\Gamma\boldsymbol{\phi} + \mathcal{Q}^s_\Gamma\boldsymbol{\psi},\ \text{in } \mathsf{D}_+,\\ \mathbf{v}_- &= \mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi} + \mathbf{V}_{\alpha,\Gamma}\boldsymbol{\psi},\ r_+ = \mathcal{Q}^d_{\alpha,\Gamma}\boldsymbol{\phi} + \mathcal{Q}^s_{\alpha,\Gamma}\boldsymbol{\psi},\ \text{in } \mathsf{D}_-. \end{aligned} \tag{2.3.42}$$

in the unknown densities $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in H^{\frac{1}{2}}(\Gamma)^3 \times H^{-\frac{1}{2}}(\Gamma)^3$.

Now, we apply relation $(2.3.40)_3$ to the fields $\mathbf{v}_\pm$ and $r_\pm$, which are provided in relation (2.3.42). Then, by employing relations $(1.4.23)_1$ and $(1.4.23)_2$ of Lemma 1.4.8, we obtain the equation

$$(-\mathbb{I} - \mathbb{K}_{\alpha,0,\Gamma})\boldsymbol{\phi} - \mathcal{V}_{\alpha,0,\Gamma}\boldsymbol{\psi} = \mathbf{g}_1. \tag{2.3.43}$$

By analogy, we apply relation $(2.3.40)_4$ to the fields $\mathbf{v}_\pm$ and $r_\pm$. An application of relations $(1.4.23)_3$ and $(1.4.23)_4$ of Lemma 1.4.8 yields the following equation

$$\left(-\mathbb{D}_{\alpha,0,\Gamma} + \mathfrak{L}\left(-\frac{1}{2}\mathbb{I} + \mathbb{K}_\Gamma\right)\right)\boldsymbol{\phi} + (\mathbb{I} - \mathbb{K}^*_{\alpha,0,\Gamma} + \mathfrak{L}\mathcal{V}_\Gamma)\boldsymbol{\psi} = \mathbf{h}_1. \tag{2.3.44}$$

Hence, we have obtained a system of equations which is comprised of relations (2.3.43) and (2.3.44). We will write these equations in matrix form as follows

$$\mathsf{T}_\alpha \begin{bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\psi} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{h}_1 \end{bmatrix}, \tag{2.3.45}$$

where $\mathsf{T}_\alpha : H^{\frac{1}{2}}(\Gamma)^3 \times H^{-\frac{1}{2}}(\Gamma)^3 \to H^{\frac{1}{2}}(\Gamma)^3 \times H^{-\frac{1}{2}}(\Gamma)^3$ is the operator defined by

$$\mathsf{T}_\alpha := \begin{bmatrix} -\mathbb{I} - \mathbb{K}_{\alpha,0,\Gamma} & -\mathcal{V}_{\alpha,0,\Gamma} \\ -\mathbb{D}_{\alpha,0,\Gamma} + \mathfrak{L}(-\frac{1}{2}\mathbb{I} + \mathbb{K}_\Gamma) & \mathbb{I} - \mathbb{K}^*_{\alpha,0,\Gamma} + \mathfrak{L}\mathcal{V}_\Gamma \end{bmatrix}. \tag{2.3.46}$$

Our purpose is to show that the equation (2.3.45) has a unique solution. The strategy that we employ is to show T_α is a Fredholm operator of index zero, which is also one-to-one.

Note that, the operator $\mathsf{T}_\alpha : H^{\frac{1}{2}}(\Gamma)^3 \times H^{-\frac{1}{2}}(\Gamma)^3 \to H^{\frac{1}{2}}(\Gamma)^3 \times H^{-\frac{1}{2}}(\Gamma)^3$ admits the decomposition

$$\mathsf{T}_\alpha = \mathsf{T}_0 + \mathsf{C}_\alpha, \tag{2.3.47}$$

where

$$\mathsf{T}_0 = \begin{bmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix} \tag{2.3.48}$$

and

$$\mathsf{C}_\alpha = \begin{bmatrix} -\mathbb{K}_{\alpha,0,\Gamma} & -\mathcal{V}_{\alpha,0,\Gamma} \\ -\mathbb{D}_{\alpha,0,\Gamma} + \boldsymbol{\mathfrak{L}}(-\frac{1}{2}\mathbb{I} + \mathbb{K}_\Gamma) & -\mathbb{K}^*_{\alpha,0,\Gamma} + \boldsymbol{\mathfrak{L}}\mathcal{V}_\Gamma \end{bmatrix}. \tag{2.3.49}$$

We observe that the operator (2.3.48) is an isomorphism. Regarding the operator (2.3.49), we note that the embeddings $H^{\frac{1}{2}}(\Gamma)^3 \hookrightarrow L^2(\Gamma)^3$ and $L^2(\Gamma)^3 \hookrightarrow H^{-\frac{1}{2}}(\Gamma)^3$ are compact, which imply that the operators

$$\begin{aligned} &\boldsymbol{\mathfrak{L}}\left(-\frac{1}{2}\mathbb{I} + \mathbb{K}_\Gamma\right) : H^{\frac{1}{2}}(\Gamma)^3 \to H^{-\frac{1}{2}}(\Gamma)^3, \\ &\boldsymbol{\mathfrak{L}}\mathcal{V}_\Gamma : H^{-\frac{1}{2}}(\Gamma)^3 \to H^{-\frac{1}{2}}(\Gamma)^3, \end{aligned} \tag{2.3.50}$$

are, in turn, compact. Relations $(2.3.50)_1$ and $(2.3.50)_2$ together with relation (1.4.27) of Lemma 1.4.8 imply that the operator C_α, given by relation (2.3.49), is a compact operator.

We conclude that the operator $\mathsf{T}_\alpha : H^{\frac{1}{2}}(\Gamma)^3 \times H^{-\frac{1}{2}}(\Gamma)^3 \to H^{\frac{1}{2}}(\Gamma)^3 \times H^{-\frac{1}{2}}(\Gamma)^3$ is a Fredholm operator of index zero.

Next, we show that the operator $\mathsf{T}_\alpha : H^{\frac{1}{2}}(\Gamma)^3 \times H^{-\frac{1}{2}}(\Gamma)^3 \to H^{\frac{1}{2}}(\Gamma)^3 \times H^{-\frac{1}{2}}(\Gamma)^3$ is also one-to-one. In order to achieve this goal, we consider the fields $(\boldsymbol{\phi}^0, \boldsymbol{\psi}^0) \in \mathrm{Ker}\ \mathsf{T}_\alpha$, where

$$\mathrm{Ker}\ \mathsf{T}_\alpha := \{(\boldsymbol{\phi}, \boldsymbol{\psi}) \in H^{\frac{1}{2}}(\Gamma)^3 \times H^{-\frac{1}{2}}(\Gamma)^3 : \mathsf{T}_\alpha[\boldsymbol{\phi}, \boldsymbol{\psi}]^t = [0, 0]^t\} \tag{2.3.51}$$

and the superscript t denotes the transpose.

Since $(\boldsymbol{\phi}^0, \boldsymbol{\psi}^0) \in \mathrm{Ker}\ \mathsf{T}_\alpha$, the pair $(\boldsymbol{\phi}^0, \boldsymbol{\psi}^0)$ satisfies

$$(-\mathbb{I} - \mathbb{K}_{\alpha,0,\Gamma})\boldsymbol{\phi}^0 = \mathcal{V}_{\alpha,0,\Gamma}\boldsymbol{\psi}^0, \tag{2.3.52}$$

and

$$\left(-\mathbb{D}_{\alpha,0,\Gamma} + \boldsymbol{\mathfrak{L}}\left(-\frac{1}{2}\mathbb{I} + \mathbb{K}_\Gamma\right)\right)\boldsymbol{\phi}^0 = -(\mathbb{I} - \mathbb{K}^*_{\alpha,0,\Gamma} + \boldsymbol{\mathfrak{L}}\mathcal{V}_\Gamma)\boldsymbol{\psi}^0. \tag{2.3.53}$$

By using the pair $(\boldsymbol{\phi}^0, \boldsymbol{\psi}^0)$, we construct the fields

$$\begin{aligned} &\mathbf{w}^0 = \mathbf{W}_\Gamma\boldsymbol{\phi}^0 + \mathbf{V}_\Gamma\boldsymbol{\psi}^0,\ p^0 = \mathcal{Q}^d_\Gamma\boldsymbol{\phi}^0 + \mathcal{Q}^s_\Gamma\boldsymbol{\psi}^0,\ \text{in } \mathbb{R}^3 \setminus \Gamma, \\ &\mathbf{v}^0 = \mathbf{W}_{\alpha,\Gamma}\boldsymbol{\phi}^0 + \mathbf{V}_{\alpha,\Gamma}\boldsymbol{\psi}^0,\ r^0 = \mathcal{Q}^d_{\alpha,\Gamma}\boldsymbol{\phi}^0 + \mathcal{Q}^s_{\alpha,\Gamma}\boldsymbol{\psi}^0,\ \text{in } \mathbb{R}^3 \setminus \Gamma. \end{aligned} \tag{2.3.54}$$

Consequently, the fields $(\mathbf{w}^0, p^0, \mathbf{v}^0, r^0)$, given by (2.3.54), solve the homogeneous version of the transmission problem (2.3.26). Recall that, due to Lemma 2.3.2 we must have

$$\mathbf{w}^0 = 0,\ p^0 = 0 \text{ in } \mathsf{D}_+,\ \mathbf{v}^0 = 0,\ r^0 = 0 \text{ in } \mathsf{D}_-. \tag{2.3.55}$$

Now, we will prove the following assertion is true. The statement is as follows

$$\begin{aligned} &\mathrm{Tr}_{\mathsf{D}_-}\mathbf{w}^0 = \boldsymbol{\phi}^0,\ \mathrm{Tr}_{\mathsf{D}_+}\mathbf{v}^0 = -\boldsymbol{\phi}^0 \text{ on } \Gamma, \\ &\mathbf{t}_{\mathsf{D}_-}(\mathbf{w}^0, p^0, 0) = -\boldsymbol{\psi}^0,\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{v}^0, r^0, 0) = \boldsymbol{\psi}^0 \text{ on } \Gamma. \end{aligned} \tag{2.3.56}$$

In order to show our claim, we begin by using formulas $(1.4.23)_1$ and $(1.4.23)_2$ together with the linearity of our trace operators $\mathrm{Tr}_{\mathsf{D}_\pm}$ (see Lemma 1.1.18) and again the expressions of the fields $(\mathbf{w}^0, p^0)$ given by (2.3.54) which lead to the fact that

$$\begin{aligned} \mathrm{Tr}_{\mathsf{D}_-}\mathbf{w}^0 &= \mathrm{Tr}_{\mathsf{D}_-}(\mathbf{W}_\Gamma\boldsymbol{\phi}^0 + \mathbf{V}_\Gamma\boldsymbol{\psi}^0) \\ &= [\mathrm{Tr}_{\mathsf{D}_-}(\mathbf{W}_\Gamma\boldsymbol{\phi}^0) - \mathrm{Tr}_{\mathsf{D}_+}(\mathbf{W}_\Gamma\boldsymbol{\phi}^0)] + \mathrm{Tr}_{\mathsf{D}_+}(\mathbf{W}_\Gamma\boldsymbol{\phi}^0) + \mathrm{Tr}_{\mathsf{D}_+}(\mathbf{V}_\Gamma\boldsymbol{\psi}^0) \\ &= \boldsymbol{\phi}^0 + \mathrm{Tr}_{\mathsf{D}_+}(\mathbf{W}_\Gamma\boldsymbol{\phi}^0 + \mathbf{V}_\Gamma\boldsymbol{\psi}^0) \\ &= \boldsymbol{\phi}^0 + \mathrm{Tr}_{\mathsf{D}_+}\mathbf{w}^0. \end{aligned}$$

Now, due to relation (2.3.55), we deduce that $\mathrm{Tr}_{\mathsf{D}_+}\mathbf{w}^0 = 0$ on Γ. Then, we get

$$\mathrm{Tr}_{\mathsf{D}_-}\mathbf{w}^0 = \boldsymbol{\phi}^0, \text{ on } \Gamma. \tag{2.3.57}$$

The formula $\mathrm{Tr}_{\mathsf{D}_-}\mathbf{v}^0 = -\boldsymbol{\phi}^0$ on Γ can be similarly obtained, where the field $\mathbf{v}^0$ is given by (2.3.54).

In order to show the last parts of our claim, we use the formulas $(1.4.23)_3$ and $(1.4.23)_4$ together with the linearity of the conormal derivative operators (see Lemma 1.2.4 and Lemma 1.2.6), we obtain

$$\begin{aligned} \mathbf{t}_{\mathsf{D}_-}(\mathbf{w}^0, p^0) &= \mathbf{t}_{\mathsf{D}_-}(\mathbf{W}_\Gamma\boldsymbol{\phi}^0 + \mathbf{V}_\Gamma\boldsymbol{\psi}^0, \mathcal{Q}^d_\Gamma\boldsymbol{\phi}^0 + \mathcal{Q}^s_\Gamma\boldsymbol{\psi}^0) = \mathbf{t}_{\mathsf{D}_-}(\mathbf{W}_\Gamma\boldsymbol{\phi}^0, \mathcal{Q}^d_\Gamma\boldsymbol{\phi}^0) + \mathbf{t}_{\mathsf{D}_-}(\mathbf{V}_\Gamma\boldsymbol{\psi}^0, \mathcal{Q}^s_\Gamma\boldsymbol{\psi}^0) \\ &= \mathbf{t}_{\mathsf{D}_+}(\mathbf{W}_\Gamma\boldsymbol{\phi}^0, \mathcal{Q}^d_\Gamma\boldsymbol{\phi}^0) + [\mathbf{t}_{\mathsf{D}_-}(\mathbf{V}_\Gamma\boldsymbol{\psi}^0, \mathcal{Q}^s_\Gamma\boldsymbol{\psi}^0) - \mathbf{t}_{\mathsf{D}_+}(\mathbf{V}_\Gamma\boldsymbol{\psi}^0, \mathcal{Q}^s_\Gamma\boldsymbol{\psi}^0)] + \mathbf{t}_{\mathsf{D}_+}(\mathbf{V}_\Gamma\boldsymbol{\psi}^0, \mathcal{Q}^s_\Gamma\boldsymbol{\psi}^0) \\ &= \mathbf{t}_{\mathsf{D}_+}(\mathbf{W}_\Gamma\boldsymbol{\phi}^0, \mathcal{Q}^d_\Gamma\boldsymbol{\phi}^0) - \boldsymbol{\psi}^0 + \mathbf{t}_{\mathsf{D}_+}(\mathbf{V}_\Gamma\boldsymbol{\psi}^0, \mathcal{Q}^s_\Gamma\boldsymbol{\psi}^0) \\ &= -\boldsymbol{\psi}^0 + \mathbf{t}_{\mathsf{D}_+}(\mathbf{W}_\Gamma\boldsymbol{\phi}^0 + \mathbf{V}_\Gamma\boldsymbol{\psi}^0, \mathcal{Q}^d_\Gamma\boldsymbol{\phi}^0 + \mathcal{Q}^s_\Gamma\boldsymbol{\psi}^0) \\ &= -\boldsymbol{\psi}^0 + \mathbf{t}_{\mathsf{D}_+}(\mathbf{w}^0, p^0). \end{aligned}$$

By taking into account the relation (2.3.55), we have that $\mathbf{t}_{\mathsf{D}_+}(\mathbf{w}^0, p^0) = 0$ on Γ. Then, we obtain

$$\mathbf{t}_{\mathsf{D}_-}(\mathbf{w}^0, p^0) = -\boldsymbol{\psi}^0 \text{ on } \Gamma. \tag{2.3.58}$$

The formula $\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{v}^0, r^0) = \boldsymbol{\psi}^0$ on Γ can be similarly obtained.

We apply now the mapping properties of the layer potentials (see also [71, Lemma A.4, Lemma A.6]) in order to deduce that

$$\begin{aligned} &(\mathbf{w}^0|_{\mathsf{D}_+}, p^0|_{\mathsf{D}_+}) \in H^1_{\mathrm{div}}(\mathsf{D}_+)^3 \times L^2(\mathsf{D}_+), \quad (\mathbf{w}^0|_{\mathsf{D}_-}, p^0|_{\mathsf{D}_-}) \in \mathcal{H}^1_{\mathrm{div}}(\mathsf{D}_-)^3 \times L^2(\mathsf{D}_-), \\ &(\mathbf{v}^0|_{\mathsf{D}_+}, r^0|_{\mathsf{D}_+}) \in H^1_{\mathrm{div}}(\mathsf{D}_+)^3 \times L^2(\mathsf{D}_+), \quad (\mathbf{v}^0|_{\mathsf{D}_-}, r^0|_{\mathsf{D}_-}) \in H^1_{\mathrm{div}}(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-). \end{aligned} \tag{2.3.59}$$

Now, we are able to apply Green's formulas (see relations (1.2.15) and (1.2.18)). It is important to underline the fact that even if $r^0|_{\mathsf{D}_-} \in \mathfrak{M}(\mathsf{D}_-)$, the Green formula still holds, cf. [71, p. 25]. The same property can be also obtained if we take into account the behavior at infinity of the single and double layer potentials representations in (2.3.54). Taking into account the expressions obtained in relation (2.3.56), we obtain

$$\begin{aligned} &-\langle\boldsymbol{\psi}^0, \boldsymbol{\phi}^0\rangle_\Gamma = \langle\mathbf{t}_{\mathsf{D}_-}(\mathbf{w}^0, p^0), \mathrm{Tr}_{\mathsf{D}_-}\mathbf{w}^0\rangle_\Gamma = -2\langle\mathbb{E}(\mathbf{w}^0), \mathbb{E}(\mathbf{w}^0)\rangle_{\mathsf{D}_-}, \\ &-\langle\boldsymbol{\psi}^0, \boldsymbol{\phi}^0\rangle_\Gamma = \langle\mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{v}^0, r^0), \mathrm{Tr}_{\mathsf{D}_+}\mathbf{v}^0\rangle_\Gamma = 2\langle\mathbb{E}(\mathbf{v}^0), \mathbb{E}(\mathbf{v}^0)\rangle_{\mathsf{D}_+} + \alpha\langle\mathbf{v}^0, \mathbf{v}^0\rangle_{\mathsf{D}_+}. \end{aligned} \tag{2.3.60}$$

By subtracting relation $(2.3.60)_1$ from $(2.3.60)_2$, we get

$$2\langle \mathbb{E}(\mathbf{v}^0), \mathbb{E}(\mathbf{v}^0)\rangle_{\mathsf{D}_+} + \alpha\langle \mathbf{v}^0, \mathbf{v}^0\rangle_{\mathsf{D}_+} + 2\langle \mathbb{E}(\mathbf{w}^0), \mathbb{E}(\mathbf{w}^0)\rangle_{\mathsf{D}_-} = 0, \tag{2.3.61}$$

which shows that

$$\mathbf{v}^0 = 0 \text{ in } \mathsf{D}_+, \ \mathbb{E}(\mathbf{w}^0) = 0 \text{ in } \mathsf{D}_-. \tag{2.3.62}$$

Now, by using the relation $\mathrm{Tr}_{\mathsf{D}_+}\mathbf{v}^0 = 0$ together with the second relation in (2.3.56) we conclude that $\boldsymbol{\phi}^0 = 0$. Moreover, the first relation in (2.3.56) implies the fact that $\mathrm{Tr}_{\mathsf{D}_-}\mathbf{w}^0 = 0$, and we deduce that the pair $(\mathbf{w}^0, p^0) \in \mathcal{H}^1_{\mathrm{div}}(\mathsf{D}_-)^3 \times L^2(\mathsf{D}_-)$ solves the homogeneous exterior Dirichlet problem for the Stokes system. Due to [56, Theorem 3.4], the solution of this problem is unique, that is,

$$\mathbf{w}^0 = 0, \ p^0 = 0, \ \text{in } \mathsf{D}_-. \tag{2.3.63}$$

It remains to see that the relation $\mathbf{t}_{\mathsf{D}_-}(\mathbf{w}^0, p^0) = 0$ together with the third relation from (2.3.56) imply the fact that $\boldsymbol{\psi}^0 = 0$.

Consequently, we have proved the fact that Ker $\mathsf{T}_\alpha = \{[0, 0]^t\}$, which translates to the fact the operator T_α is an isomorphism. It immediately follows that the equation (2.3.45) has a unique solution and the layer potential representations (2.3.39) and (2.3.42) provide us with the unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{\mathcal{B}}$ (recall Lemma 2.3.2) of our transmission problem (2.3.26).

The linearity and continuity of the single layer potentials, double layer potentials and Newtonian potentials that appear in relations (2.3.39) and (2.3.42) together with the fact that the operator T_α is an isomorphism, lead to the relation

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_{\mathcal{B}}} \leq C||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h})||_{\mathsf{Y}_{\mathcal{B}}}, \tag{2.3.64}$$

where $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \mathfrak{L}) > 0$ is a constant.

Lastly, in view of the well-posedness of the transmission problem (2.3.26) implies that the operator $\mathsf{S} : \mathsf{Y}_{\mathcal{B}} \to \mathsf{X}_{\mathcal{B}}$, which maps the given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) \in \mathsf{Y}_{\mathcal{B}}$ to the unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{\mathcal{B}}$ of the transmission problem (2.3.26) is well-defined, linear and continuous. This concludes the proof of our result. □

2.4 On a Robin-Transmission problem for the Brinkman system

In this section we aim to state and prove a well-posedness result, for a transmission-type problem, which was obtained in the setting of Assumption 1.1.7. Before we state the transmission problem, let us mention that such problems are used to model the fluid flow in the exterior of a cavity or in cavities filled with porous media, in the case of the jump of either tensions or velocity on the interface. Another idea is to analyze the fluid flow in a porous media in reservoirs whose boundary has two parts, the first one that of a solid surface and the second, an interface between the fluid and another fluid or viscoelastic material (for additional details, see, e.g., [72]). From a practical point of view, Baber [16] has analyzed applications of transmission problems, such as the water management in fuel cells or the processing of nutrients between two domains, one containing blood, the other porous tissue.

The transmission-type problem that we wish to treat will be called the Robin-transmission problem for the Brinkman system (see problem (2.4.3)). In addition, let Assumption 2.2.1 be satisfied.

To ensure the clarity of our exposition, we consider the following spaces, namely the space of solutions,

$$\mathsf{X}_{RT} := H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times L^2(\mathsf{D}_+) \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n \times L^2(\mathsf{D}_-), \tag{2.4.1}$$

and the space of given data,

$$\mathsf{Y}_{RT} := \tilde{H}^{-1}(\mathsf{D}_+)^n \times \tilde{H}^{-1}(\mathsf{D}_-)^n \times H^{\frac{1}{2}}_{\nu}(\Gamma_+)^n \times H^{-\frac{1}{2}}(\Gamma_+)^n \times H^{-\frac{1}{2}}(\Gamma_-)^n, \tag{2.4.2}$$

respectively.

We will study the Robin-transmission problem for the Brinkman system, which is given by

$$\begin{cases} \Delta \mathbf{u}_\pm - \alpha \mathbf{u}_\pm - \nabla \pi_\pm = \mathbf{f}_\pm|_{\mathsf{D}_\pm} \text{ in } \mathsf{D}_\pm, \\ \mathrm{div}\ \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \lambda(\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+) - \big(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-\big)\,|_{\Gamma_+} = \mathbf{g}_1 \text{ on } \Gamma_+, \\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_+,\pi_+,\mathbf{f}_+) - \big(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-,\pi_-,\mathbf{f}_-)\big)\,|_{\Gamma_+} = \mathbf{h}_1 \text{ on } \Gamma_+, \\ \big(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-,\pi_-,\mathbf{f}_-)\big)\,|_{\Gamma_-} + \mathfrak{L}\big(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-\big)\,|_{\Gamma_-} = \mathbf{g}_2 \text{ on } \Gamma_-, \end{cases} \tag{2.4.3}$$

where $\alpha > 0$ and $\lambda \in (0,1]$ are given constants. We aim to determine the unknown fields $(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-) \in \mathsf{X}_{RT}$.

Let us state and prove the well-posedness result that was obtained for the Robin-transmission problem (2.4.3) (see also [9, Theorem 1], [75, Theorem 4.1], [82, Theorem 5.8]).

Theorem 2.4.1. *Let $\alpha > 0$ and $\lambda \in (0,1]$ be given constants. Let Assumption 1.1.7 and Assumption 2.2.1 be satisfied. Then, for all data $(\mathbf{f}_+,\mathbf{f}_-,\mathbf{g}_1,\mathbf{h}_1,\mathbf{g}_2) \in \mathsf{Y}_{RT}$, the Poisson problem of Robin-transmission type for the Brinkman system (2.4.3) has a unique solution*

$$(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-) \in \mathsf{X}_{RT}. \tag{2.4.4}$$

In addition, the corresponding solution operator,

$$\mathsf{T}_{RT} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}, \tag{2.4.5}$$

is linear and bounded, and hence, there exists a constant $C \equiv C(\mathsf{D}_+,\mathsf{D}_-,\alpha,\mathfrak{L},\lambda) > 0$ such that the unique solution of (2.4.3) satisfies

$$||(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-)||_{\mathsf{X}_{RT}} \leq C||(\mathbf{f}_+,\mathbf{f}_-,\mathbf{g}_1,\mathbf{h}_1,\mathbf{g}_2)||_{\mathsf{Y}_{RT}}. \tag{2.4.6}$$

Proof. We prove this result in a similar way as to the one used in the proof of Theorem 4.1 in [75] and Theorem 5.8 in [82]. This approach uses layer potential methods. In order to preserve the simplicity of our arguments, let us introduce the space

$$\mathbb{Y} := H^{\frac{1}{2}}_{\nu}(\Gamma_+)^n \times H^{-\frac{1}{2}}(\Gamma_+)^n \times H^{-\frac{1}{2}}(\Gamma_-)^n, \tag{2.4.7}$$

which will appear in the latter.

We divide our arguments into two separate cases. The first case concerns the situation $\lambda \in (0,1)$ and the second case refers to the situation $\lambda = 1$.

Case 1: Assume that $\lambda \in (0,1)$. Firstly, we show that our problem admits a solution, (i.e., the existence of a solution) and we aim to construct it by using a layer potential approach. To this end, let us seek a solution in the form

$$\begin{aligned} \mathbf{u}_+ &= \mathcal{N}_{\alpha,\mathsf{D}_+}\mathbf{f}_+ + \mathbf{W}_{\alpha,\Gamma_+}\mathbf{\Phi} + \mathbf{V}_{\alpha,\Gamma_+}\boldsymbol{\varphi}, \\ \pi_+ &= \mathcal{Q}_{\alpha,\mathsf{D}_+}\mathbf{f}_+ + \mathbf{Q}^d_{\alpha,\Gamma_+}\mathbf{\Phi} + \mathbf{Q}^s_{\alpha,\Gamma_+}\boldsymbol{\varphi}, \\ \mathbf{u}_- &= \mathcal{N}_{\alpha,\mathsf{D}_-}\mathbf{f}_- + \mathbf{W}_{\alpha,\Gamma_+}\mathbf{\Phi} + \mathbf{V}_{\alpha,\Gamma_+}\boldsymbol{\varphi} + \mathbf{V}_{\alpha,\Gamma_-}\boldsymbol{\psi}, \\ \pi_- &= \mathcal{Q}_{\alpha,\mathsf{D}_-}\mathbf{f}_- + \mathbf{Q}^d_{\alpha,\Gamma_+}\mathbf{\Phi} + \mathbf{Q}^s_{\alpha,\Gamma_+}\boldsymbol{\varphi} + \mathbf{Q}^s_{\alpha,\Gamma_-}\boldsymbol{\psi}, \end{aligned} \tag{2.4.8}$$

where $(\mathbf{\Phi}, \boldsymbol{\varphi}, \boldsymbol{\psi}) \in \mathbb{Y}$ are unknown densities and the space $\mathbb{Y}$ is given in relation (2.4.7).

Note that, the mapping properites of the Newtonian, simple and double layer potentials for the Brinkman system (see Theorem 1.4.2, Theorem 1.4.4, Theorem 1.4.7) imply that $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$.

Next, by taking into account the jump formulas for single and double layer potentials (see relation (1.4.23) of Lemma 1.4.8) and by substitution into relation $(2.4.3)_3$, we get

$$\left(\lambda\left(-\frac{1}{2}\mathbb{I} + \mathbb{K}_{\alpha,\Gamma_+}\right) - \left(\frac{1}{2}\mathbb{I} + \mathbb{K}_{\alpha,\Gamma_+}\right)\right)\mathbf{\Phi} + \left(\lambda\mathcal{V}_{\alpha,\Gamma_+} - \mathcal{V}_{\alpha,\Gamma_+}\right)\boldsymbol{\varphi} - \mathcal{V}_{\Gamma_-,\Gamma_+}\boldsymbol{\psi} = \mathbf{g}_{01}, \tag{2.4.9}$$

and $\mathbf{g}_{01}$ is given by

$$\mathbf{g}_{01} = \mathbf{g}_1 - \lambda(\mathrm{Tr}_{\mathsf{D}_+}(\mathcal{N}_{\alpha,\mathsf{D}_+}\mathbf{f}_+)) + (\mathrm{Tr}_{\mathsf{D}_-}(\mathcal{N}_{\alpha,\mathsf{D}_-}\mathbf{f}_-))|_{\Gamma_+}. \tag{2.4.10}$$

Note that, the operator

$$\mathcal{V}_{\Gamma_-,\Gamma_+} : H^{-\frac{1}{2}}(\Gamma_-)^n \to H^{\frac{1}{2}}(\Gamma_+)^n,\ \mathcal{V}_{\Gamma_-,\Gamma_+}\boldsymbol{\psi} := \left(\mathrm{Tr}_{\mathsf{D}_-}\left(\mathbf{V}_{\alpha,\Gamma_-}\boldsymbol{\psi}\right)\right)|_{\Gamma_+}, \tag{2.4.11}$$

is compact, as an integral operator with real analytic kernel (see [36, Theorem A.28, Statement (ii)] which deals with the properties of integral operators with real analytic kernels) and due to the compact embedding $H^1(\Gamma_+)^n \hookrightarrow H^{\frac{1}{2}}(\Gamma_+)^n$. Also, we have $\mathbf{g}_{01} \in H_\nu^{\frac{1}{2}}(\Gamma_+)^n$. This assertion holds true after the application of the Divergence Theorem while taking into account relation (1.4.13).

Now, let us take into account again the jump formulas for single and double layer potentials, and by substitution into relation $(2.4.3)_4$, we get

$$\boldsymbol{\varphi} - \mathbb{K}^*_{\Gamma_-,\Gamma_+}\boldsymbol{\psi} = \mathbf{h}_{01}, \tag{2.4.12}$$

and $\mathbf{h}_{01}$ is given by

$$\mathbf{h}_{01} := \mathbf{h}_1 - \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathcal{N}_{\alpha,\mathsf{D}_+}\mathbf{f}_+, \mathcal{Q}_{\alpha,\mathsf{D}_+}\mathbf{f}_+, \mathbf{f}_+) + (\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathcal{N}_{\alpha,\mathsf{D}_-}\mathbf{f}_-, \mathcal{Q}_{\alpha,\mathsf{D}_-}\mathbf{f}_-, \mathbf{f}_-))|_{\Gamma_+}. \tag{2.4.13}$$

Let us notice that the operator

$$\mathbb{K}^*_{\Gamma_-,\Gamma_+} : H^{-\frac{1}{2}}(\Gamma_-)^n \to H^{-\frac{1}{2}}(\Gamma_+)^n,\ \mathbb{K}^*_{\Gamma_-,\Gamma_+}\boldsymbol{\psi} := \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{V}_{\alpha,\Gamma_-}\boldsymbol{\psi}, \mathbf{Q}_{\alpha,\Gamma_-}\boldsymbol{\psi})\right)|_{\Gamma_+}, \tag{2.4.14}$$

is a compact operator, based on [36, Theorem A.28, Statement (ii)] and the compactness of the embedding $L^2(\Gamma_+)^n \hookrightarrow H^{-\frac{1}{2}}(\Gamma_+)^n$.

It remains now to apply, again, the jump properties of the single-layer and double-layer potentials for the Brinkman system (see Lemma 1.4.8) and by substitution into relation $(2.4.3)_5$ (i.e., the Robin boundary condition), we get the equation

$$\left(\mathbb{D}_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathbb{K}_{\Gamma_+,\Gamma_-}\right)\mathbf{\Phi} + \left(\mathbb{K}^*_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_+,\Gamma_-}\right)\boldsymbol{\varphi} + \left(\frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\alpha,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\alpha,\Gamma_-}\right)\boldsymbol{\psi} = \mathbf{g}_{02}, \tag{2.4.15}$$

and $\mathbf{g}_{02}$ is given by

$$\mathbf{g}_{02} := \mathbf{g}_2 - (\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathcal{N}_{\alpha,\mathsf{D}_-}\mathbf{f}_-, \mathcal{Q}_{\alpha,\mathsf{D}_-}\mathbf{f}_-, \mathbf{f}_-))|_{\Gamma_-} - \mathfrak{L}(\mathrm{Tr}_{\mathsf{D}_-}(\mathcal{N}_{\alpha,\mathsf{D}_-}\mathbf{f}_-))|_{\Gamma_-}. \tag{2.4.16}$$

We emphasise the fact that the following operators

$$\begin{aligned}
&\mathbb{D}_{\Gamma_+,\Gamma_-} : H^{\frac{1}{2}}(\Gamma_+)^n \to H^{-\frac{1}{2}}(\Gamma_-)^n,\ \mathbb{D}_{\Gamma_+,\Gamma_-}\mathbf{\Phi} := \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{W}_{\alpha,\Gamma_+}\mathbf{\Phi}, \mathbf{Q}^d_{\alpha,\Gamma_+}\mathbf{\Phi})\right)|_{\Gamma_-},\\
&\mathbb{K}_{\Gamma_+,\Gamma_-} : H^{\frac{1}{2}}(\Gamma_+)^n \to H^{\frac{1}{2}}(\Gamma_-)^n,\ \mathbb{K}_{\Gamma_+,\Gamma_-}\mathbf{\Phi} := \left(\mathrm{Tr}_{\mathsf{D}_-}\left(\mathbf{W}_{\alpha,\Gamma_+}\mathbf{\Phi}\right)\right)|_{\Gamma_-},\\
&\mathbb{K}^*_{\Gamma_+,\Gamma_-} : H^{-\frac{1}{2}}(\Gamma_+)^n \to H^{-\frac{1}{2}}(\Gamma_-)^n,\ \mathbb{K}^*_{\Gamma_+,\Gamma_-}\boldsymbol{\varphi} := \left(\mathbf{t}_{\alpha,\mathsf{D}_-}\left(\mathbf{V}_{\alpha,\Gamma_+}\boldsymbol{\varphi}, \mathbf{Q}^s_{\alpha,\Gamma_+}\boldsymbol{\varphi}\right)\right)|_{\Gamma_-},\\
&\mathcal{V}_{\Gamma_+,\Gamma_-} : H^{-\frac{1}{2}}(\Gamma_+)^n \to H^{\frac{1}{2}}(\Gamma_-)^n,\ \mathcal{V}_{\Gamma_+,\Gamma_-}\boldsymbol{\varphi} := \left(\mathrm{Tr}_{\mathsf{D}_-}\left(\mathbf{V}_{\alpha,\Gamma_+}\boldsymbol{\varphi}\right)\right)|_{\Gamma_-},
\end{aligned} \tag{2.4.17}$$

which are present in relation (2.4.15) are compact operators. This assertion holds true if we apply [36, Theorem A.28, Statement (ii)] and if we take into account the compactness of the embeddings $H^1(\Gamma_+)^n \hookrightarrow H^{\frac{1}{2}}(\Gamma_+)^n$ and $L^2(\Gamma_+)^n \hookrightarrow H^{-\frac{1}{2}}(\Gamma_+)^n$.

Consequently, the Robin-transmission problem (2.4.3) reduces to the equations given by relations (2.4.9), (2.4.12), (2.4.15). Let us write these equations in matrix form as follows

$$\mathsf{A}(\boldsymbol{\Phi}, \boldsymbol{\varphi}, \boldsymbol{\psi})^t = (\mathbf{g}_{01}, \mathbf{h}_{01}, \mathbf{g}_{02}) \text{ in } \mathbb{Y}, \tag{2.4.18}$$

in the unknown $(\boldsymbol{\Phi}, \boldsymbol{\varphi}, \boldsymbol{\psi})^t \in \mathbb{Y}$. The matrix operator $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ involved in relation (2.4.18) is given by

$$\mathsf{A} := \begin{bmatrix} \lambda\left(-\frac{1}{2}\mathbb{I} + \mathbb{K}_{\alpha,\Gamma_+}\right) - \left(\frac{1}{2}\mathbb{I} + \mathbb{K}_{\alpha,\Gamma_+}\right) & \lambda\mathcal{V}_{\alpha,\Gamma_+} - \mathcal{V}_{\alpha,\Gamma_+} & -\mathcal{V}_{\Gamma_-,\Gamma_+} \\ \mathbf{0} & \mathbb{I} & -\mathbb{K}^*_{\Gamma_-,\Gamma_+} \\ \mathbb{D}_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathbb{K}_{\Gamma_+,\Gamma_-} & \mathbb{K}^*_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_+,\Gamma_-} & \frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\alpha,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\alpha,\Gamma_-} \end{bmatrix}. \tag{2.4.19}$$

Let us write the matrix operator (2.4.19) in the following equivalent form,

$$\mathsf{A} := \begin{bmatrix} -\frac{\lambda+1}{2}\mathbb{I} + (\lambda-1)\mathbb{K}_{\alpha,\Gamma_+} & (\lambda-1)\mathcal{V}_{\alpha,\Gamma_+} & -\mathcal{V}_{\Gamma_-,\Gamma_+} \\ \mathbf{0} & \mathbb{I} & -\mathbb{K}^*_{\Gamma_-,\Gamma_+} \\ \mathbb{D}_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathbb{K}_{\Gamma_+,\Gamma_-} & \mathbb{K}^*_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_+,\Gamma_-} & \frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\alpha,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\alpha,\Gamma_-} \end{bmatrix}. \tag{2.4.20}$$

We claim that the matrix operator $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is an isomorphism. In order to prove this claim, we will prove operator $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is a Fredholm operator of index zero, for $\lambda \in (0, 1]$ and that it is also an injective operator.

Let us proceed by showing, first of all, that $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is a Fredholm operator of index zero. A simple rearrangement allows us to rewrite operator (2.4.20) in the following form

$$\mathsf{A} := \begin{bmatrix} (\lambda-1)\left(\frac{1}{2}\frac{1+\lambda}{1-\lambda}\mathbb{I} + \mathbb{K}_{\alpha,\Gamma_+}\right) & (\lambda-1)\mathcal{V}_{\alpha,\Gamma_+} & -\mathcal{V}_{\Gamma_-,\Gamma_+} \\ \mathbf{0} & \mathbb{I} & -\mathbb{K}^*_{\Gamma_-,\Gamma_+} \\ \mathbb{D}_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathbb{K}_{\Gamma_+,\Gamma_-} & \mathbb{K}^*_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_+,\Gamma_-} & \frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\alpha,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\alpha,\Gamma_-} \end{bmatrix}. \tag{2.4.21}$$

It is immediate that the matrix operator $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is well-defined, linear and continuous.

Let us now recall the definition of the complementary layer-potential operators (see relation (1.4.26)) and with their help we are decompose the matrix operator (2.4.21) as

$$\mathsf{A} = \mathsf{A}_0 + \mathsf{A}_\mathsf{C} : \mathbb{Y} \to \mathbb{Y}, \tag{2.4.22}$$

where the operators $\mathsf{A}_0 : \mathbb{Y} \to \mathbb{Y}$ and $\mathsf{A}_\mathsf{C} : \mathbb{Y} \to \mathbb{Y}$ are defined by

$$\mathsf{A}_0 := \begin{bmatrix} (\lambda-1)\left(\frac{1}{2}\frac{1+\lambda}{1-\lambda}\mathbb{I} + \mathbb{K}_{\Gamma_+}\right) & (\lambda-1)\mathcal{V}_{\Gamma_+} & \mathbf{0} \\ \mathbf{0} & \mathbb{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_-} \end{bmatrix} \tag{2.4.23}$$

and

$$\mathsf{A}_\mathsf{C} := \begin{bmatrix} (\lambda-1)\mathbb{K}_{\alpha,0,\Gamma_+} & (\lambda-1)\mathcal{V}_{\alpha,0,\Gamma_+} & -\mathcal{V}_{\Gamma_-,\Gamma_+} \\ \mathbf{0} & \mathbf{0} & -\mathbb{K}^*_{\Gamma_-,\Gamma} \\ \mathbb{D}_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathbb{K}_{\Gamma_+,\Gamma_-} & \mathbb{K}^*_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_+,\Gamma_-} & \mathbb{K}^*_{\alpha,0,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\alpha,0,\Gamma_-} \end{bmatrix}. \tag{2.4.24}$$

Let us analyze the properties of the operator $\mathsf{A}_0 : \mathbb{Y} \to \mathbb{Y}$ given by relation (2.4.23). Let us take into account the fact the operator

$$\frac{1}{2}\frac{1+\lambda}{1-\lambda}\mathbb{I} + \mathbb{K}_{\Gamma_+} : H^{\frac{1}{2}}(\Gamma_+)^n \to H^{\frac{1}{2}}(\Gamma_+)^n, \tag{2.4.25}$$

is a Fredholm operator of index zero (see, e.g., [114, Corollary 9.1.2], [82, Lemma 5.3]). Next, by the second statement of Lemma 1.3.8, the operator

$$\mathcal{V}_{\Gamma_+} : H^{-\frac{1}{2}}(\Gamma_+)^n \to H^{\frac{1}{2}}(\Gamma_+)^n, \tag{2.4.26}$$

is also a Fredholm operator of index zero. Moreover, the operator

$$\frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_-} : H^{-\frac{1}{2}}(\Gamma_-)^n \to H^{-\frac{1}{2}}(\Gamma_-)^n, \tag{2.4.27}$$

is another Fredholm operator of index zero, since

$$\frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\Gamma_-} : H^{-\frac{1}{2}}(\Gamma_-)^n \to H^{-\frac{1}{2}}(\Gamma_-)^n \tag{2.4.28}$$

is Fredholm operator of index zero and the operator

$$\mathfrak{L}\mathcal{V}_{\Gamma_-} : H^{-\frac{1}{2}}(\Gamma_-)^n \to H^{-\frac{1}{2}}(\Gamma_-)^n \tag{2.4.29}$$

is a compact operator in view of the compact embeddings $H^{\frac{1}{2}}(\Gamma_-)^n \hookrightarrow L^2(\Gamma_-)^n$ and $L^2(\Gamma_-)^n \hookrightarrow H^{-\frac{1}{2}}(\Gamma_-)^n$ (for additional details see, e.g., [75, Theorem 4.1]).

By the arguments in the former, we have shown that the operators (2.4.25), (2.4.26) and (2.4.27) are Fredholm operators of index zero and it follows that the operator $\mathsf{A}_0 : \mathbb{Y} \to \mathbb{Y}$ is Fredholm of index zero.

Let us now focus on the operator $\mathsf{A}_\mathsf{C} : \mathbb{Y} \to \mathbb{Y}$ provided in relation (2.4.24). In view of the compactness of the complementary layer-potential operators $\mathbb{K}_{\alpha,0,\Gamma_+}$, $\mathcal{V}_{\alpha,0,\Gamma_+}$, $\mathbb{K}^*_{\alpha,0,\Gamma_-}$, $\mathcal{V}_{\alpha,0,\Gamma_-}$ (see relation (1.4.27) of Lemma 1.4.9) and also the compactness of the operators (2.4.11), (2.4.14) and (2.4.17), we have that, in turn, the operator $\mathsf{A}_\mathsf{C} : \mathbb{Y} \to \mathbb{Y}$ is a compact operator.

Since our operator $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is a sum of a Fredholm operator of index zero and a compact operator, we deduce that $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is a Fredholm operator of index zero, for $\lambda \in (0, 1)$.

In order to fully prove our claim, we show that the operator $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is injective, or equivalently, we show that the kernel of the operator $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is the null space, i.e.,

$$\mathrm{Ker}\,\{\mathsf{A} : \mathbb{Y} \to \mathbb{Y}\} = \{\mathbf{0}\}. \tag{2.4.30}$$

To achieve this, we consider $(\boldsymbol{\Phi}_0, \boldsymbol{\varphi}_0, \boldsymbol{\psi}_0)^t \in \mathrm{Ker}\,\{\mathsf{A} : \mathbb{Y} \to \mathbb{Y}\}$. Then, we construct the fields $(\mathbf{u}^0_+, \pi^0_+)$ and $(\mathbf{u}^0_-, \pi^0_-)$ as follows

$$\begin{aligned} &\mathbf{u}^0_+ := \mathbf{W}_{\alpha,\Gamma_+}\boldsymbol{\Phi}_0 + \mathbf{V}_{\alpha,\Gamma_+}\boldsymbol{\varphi}_0 \quad \pi^0_+ := \mathbf{Q}^d_{\alpha,\Gamma_+}\boldsymbol{\Phi}_0 + \mathbf{Q}^s_{\alpha,\Gamma_+}\boldsymbol{\varphi}_0 \\ &\mathbf{u}^0_- := \mathbf{W}_{\alpha,\Gamma_+}\boldsymbol{\Phi}_0 + \mathbf{V}_{\alpha,\Gamma_+}\boldsymbol{\varphi}_0 + \mathbf{V}_{\alpha,\Gamma_-}\boldsymbol{\psi}_0 \quad \pi^0_- := \mathbf{Q}^d_{\alpha,\Gamma_+}\boldsymbol{\Phi}_0 + \mathbf{Q}^s_{\alpha,\Gamma_+}\boldsymbol{\varphi}_0 + \mathbf{Q}^s_{\alpha,\Gamma_-}\boldsymbol{\psi}_0. \end{aligned} \tag{2.4.31}$$

Let us note that these fields $(\mathbf{u}^0_+, \pi^0_+)$ and $(\mathbf{u}^0_-, \pi^0_-)$ satisfy

$$\begin{aligned} &\lambda(\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}^0_+) = \left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}^0_-\right)|_{\Gamma_+} \text{ a.e. on } \Gamma_+, \\ &\mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}^0_+, \pi^0_+) = \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}^0_-, \pi^0_-)\right)|_{\Gamma_+} \text{ a.e. on } \Gamma_+, \\ &\left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}^0_-, \pi^0_-)\right)|_{\Gamma_-} + \mathfrak{L}\left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}^0_-\right)|_{\Gamma_-} = 0, \text{ a.e. on } \Gamma_-. \end{aligned} \tag{2.4.32}$$

Now, we apply the Green formula (1.2.18) to the fields introduced in relation (2.4.31) and we get

$$\begin{aligned} 2\left\langle \mathbb{E}(\mathbf{u}^0_+), \mathbb{E}(\mathbf{u}^0_+)\right\rangle_{\mathsf{D}_+} + \alpha\left\langle \mathbf{u}^0_+, \mathbf{u}^0_+\right\rangle_{\mathsf{D}_+} &= \left\langle \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}^0_+, \pi^0_+), \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}^0_+\right\rangle_{\Gamma_+}, \\ 2\left\langle \mathbb{E}(\mathbf{u}^0_-), \mathbb{E}(\mathbf{u}^0_-)\right\rangle_{\mathsf{D}_-} + \alpha\left\langle \mathbf{u}^0_-, \mathbf{u}^0_-\right\rangle_{\mathsf{D}_-} &= -\left\langle \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}^0_-, \pi^0_-)\right)|_{\Gamma_+}, \left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}^0_-\right)|_{\Gamma_+}\right\rangle_{\Gamma_+} \\ &\quad + \left\langle \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}^0_-, \pi^0_-)\right)|_{\Gamma_-}, \left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}^0_-\right)|_{\Gamma_-}\right\rangle_{\Gamma_-}. \end{aligned} \tag{2.4.33}$$

Let us now multiply relation $(2.4.33)_1$ by λ and we add the resulting quantities to relation $(2.4.33)_2$ and by using relations $(2.4.32)_1$, $(2.4.32)_2$ and $(2.4.32)_3$ we have

$$\begin{aligned}&\lambda\left(2\left\langle\mathbb{E}(\mathbf{u}_+^0),\mathbb{E}(\mathbf{u}_+^0)\right\rangle_{\mathsf{D}_+}+\alpha\left\langle\mathbf{u}_+^0,\mathbf{u}_+^0\right\rangle_{\mathsf{D}_+}\right)+2\left\langle\mathbb{E}(\mathbf{u}_-^0),\mathbb{E}(\mathbf{u}_-^0)\right\rangle_{\mathsf{D}_-}+\alpha\left\langle\mathbf{u}_-^0,\mathbf{u}_-^0\right\rangle_{\mathsf{D}_-}\\&\quad=-\left\langle\mathfrak{L}\left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-^0\right)|_{\Gamma_-},\left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-^0\right)|_{\Gamma_-}\right\rangle_{\Gamma_-}.\end{aligned}\tag{2.4.34}$$

Note that, the left hand side of the equality (2.4.34) is non-negative and the right hand side of the equality (2.4.34) is non-positive (due to the fact that $\mathfrak{L}$ satisfies condition (2.2.1)). This leads to the fact that

$$\lambda\left(2\left\langle\mathbb{E}(\mathbf{u}_+^0),\mathbb{E}(\mathbf{u}_+^0)\right\rangle_{\mathsf{D}_+}+\alpha\left\langle\mathbf{u}_+^0,\mathbf{u}_+^0\right\rangle_{\mathsf{D}_+}\right)+2\left\langle\mathbb{E}(\mathbf{u}_-^0),\mathbb{E}(\mathbf{u}_-^0)\right\rangle_{\mathsf{D}_-}+\alpha\left\langle\mathbf{u}_-^0,\mathbf{u}_-^0\right\rangle_{\mathsf{D}_-}=0.\tag{2.4.35}$$

Consequently, we get $\mathbf{u}_\pm^0=\mathbf{0}$ in $\mathsf{D}_\pm$, which, in turn, implies that $\pi_\pm^0=c_\pm^0$, where $c_\pm^0\in\mathbb{R}$ are constants. Also, relations $(2.4.32)_2$ and $(2.4.32)_3$ imply $c_+^0=c_-^0=0$. Hence, we have that

$$\mathbf{u}_\pm^0=\mathbf{0},\ \text{in } \mathsf{D}_\pm,\ \pi_\pm^0=0 \text{ in } \mathsf{D}_\pm.\tag{2.4.36}$$

Let us now apply relation (1.4.23) of Lemma 1.4.8 in order to get

$$\begin{aligned}&\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_+^0=\mathbf{\Phi}_0,\ \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_-^0=-\mathbf{\Phi}_0,\ \text{a.e. on } \Gamma_+,\\&\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_+^0,\pi_+^0)=-\boldsymbol{\varphi}_0,\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_-^0,\pi_-^0)=\boldsymbol{\varphi}_0 \text{ a.e. on } \Gamma_+.\end{aligned}\tag{2.4.37}$$

In addition, the membership $\mathbf{\Phi}_0\in H_\nu^{\frac{1}{2}}(\Gamma_+)^n$ implies that $(\mathbf{W}_{\alpha,\Gamma_+}\mathbf{\Phi}_0)(\mathbf{x})=O(|\mathbf{x}|^{-n})$ as $|\mathbf{x}|\to\infty$, (see [141, Lemma 2.12]). Let us mention that the single-layer potential $\mathbf{V}_{\alpha,\Gamma_+}\boldsymbol{\varphi}_0$ behaves in a similar manner at infinity (see [141, Lemma 2.12]). Hence, $\mathbf{u}_+^0(\mathbf{x})=O(|\mathbf{x}|^{-n})$ as $|\mathbf{x}|\to\infty$. Consequently, the fields $(\mathbf{u}_+^0,\pi_+^0)$ satisfy the Green formula (1.2.18) corresponding to the domain $\mathbb{R}^n\setminus\overline{\mathsf{D}}_+$. Let us apply the Green formula (1.2.18) for the fields $(\mathbf{u}_+^0,\pi_+^0)$ in $\mathbb{R}^n\setminus\overline{\mathsf{D}}_+$, while taking into account relation (2.4.37). We get

$$\begin{aligned}&2\left\langle\mathbb{E}(\mathbf{u}_+^0),\mathbb{E}(\mathbf{u}_+^0)\right\rangle_{\mathbb{R}^n\setminus\overline{\mathsf{D}}_+}+\alpha\left\langle\mathbf{u}_+^0,\mathbf{u}_+^0\right\rangle_{\mathbb{R}^n\setminus\overline{\mathsf{D}}_+}\\&\quad=-\left\langle\mathbf{t}_{\alpha,\nu,\mathbb{R}^n\setminus\overline{\mathsf{D}}_+}(\mathbf{u}_+^0,\pi_+^0),\mathrm{Tr}_{\mathbb{R}^n\setminus\overline{\mathsf{D}}_+}\mathbf{u}_+^0\right\rangle_{\Gamma_+}=\langle\boldsymbol{\varphi}_0,\mathbf{\Phi}_0\rangle_{\Gamma_+}.\end{aligned}\tag{2.4.38}$$

Moreover, we apply the Green formula (1.2.18) for $(\mathbf{u}_-^0,\pi_-^0)$ in D_+, while taking into account relation (2.4.37) and we obtain

$$2\left\langle\mathbb{E}(\mathbf{u}_-^0),\mathbb{E}(\mathbf{u}_-^0)\right\rangle_{\mathsf{D}_+}+\alpha\left\langle\mathbf{u}_-^0,\mathbf{u}_-^0\right\rangle_{\mathsf{D}_+}=\left\langle\mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_-^0,\pi_-^0),\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_-^0\right\rangle_{\Gamma_+}=-\langle\boldsymbol{\varphi}_0,\mathbf{\Phi}_0\rangle_{\Gamma_+}.\tag{2.4.39}$$

Let us now add relations (2.4.38) and (2.4.39). We obtain the following

$$2\left\langle\mathbb{E}(\mathbf{u}_+^0),\mathbb{E}(\mathbf{u}_+^0)\right\rangle_{\mathbb{R}^n\setminus\overline{\mathsf{D}}_+}+\alpha\left\langle\mathbf{u}_+^0,\mathbf{u}_+^0\right\rangle_{\mathbb{R}^n\setminus\overline{\mathsf{D}}_+}+2\left\langle\mathbb{E}(\mathbf{u}_-^0),\mathbb{E}(\mathbf{u}_-^0)\right\rangle_{\mathsf{D}_+}+\alpha\left\langle\mathbf{u}_-^0,\mathbf{u}_-^0\right\rangle_{\mathsf{D}_+}=0,\tag{2.4.40}$$

which shows that

$$\mathbf{u}_+^0=\mathbf{0},\ \pi_+^0=0 \text{ in } \mathbb{R}^n\setminus\overline{\mathsf{D}}_+,\ \mathbf{u}_-^0=\mathbf{0},\ \pi_-^0=0 \text{ in } \mathsf{D}_+.\tag{2.4.41}$$

Let us stress the fact that $\pi_+^0=0$ in $\mathbb{R}^n\setminus\overline{\mathsf{D}}_+$ is a consequence of the fact that the pair $(\mathbf{u}_+^0,\pi_+^0)$ satisfies the homogeneous Brinkman equation in $\mathbb{R}^n\setminus\overline{\mathsf{D}}_+$ and also the fact that $\pi_+^0(\mathbf{x})=O(|\mathbf{x}|^{1-n})$ as $|\mathbf{x}|\to\infty$ (see [73, Relations (3.12), (3.13)]).

Now, by relations (2.4.37) and (2.4.41) we are able to deduce that

$$\mathbf{\Phi}_0 = \mathbf{0},\ \boldsymbol{\varphi}_0 = \mathbf{0}. \tag{2.4.42}$$

Relation (2.4.42) together with the fact that $\mathbf{u}^0_- = \mathbf{0}$ in D_+, implies that $\mathbf{V}_{\alpha,\Gamma_-}\boldsymbol{\psi}_0 = \mathbf{0}$ in D_+. The continuity of the single layer potential for the Brinkman system on Γ_- (see Theorem 1.4.4) implies that

$$\mathbf{V}_{\alpha,\Gamma_-}\boldsymbol{\psi}_0 = \mathbf{0} \text{ in } \mathbb{R}^n \setminus \overline{\mathsf{D}}_+, \tag{2.4.43}$$

while the behavior at infinity of the single layer pressure potential (namely, that $\mathbf{Q}^s_{\alpha,\Gamma_-}\boldsymbol{\psi}_0 = O(|\mathbf{x}|^{1-n})$ for $n \geq 2$, as it can be seen in relation (3.12) of [73]) leads to the fact that

$$\mathbf{Q}^s_{\alpha,\Gamma_-}\boldsymbol{\psi}_0 = 0 \text{ in } \mathbb{R}^n \setminus \overline{\mathsf{D}}_+. \tag{2.4.44}$$

Therefore, by relations (2.4.43) and (2.4.44) we get

$$\mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{V}_{\alpha,\Gamma_-}\boldsymbol{\psi}_0, \mathbf{Q}^s_{\alpha,\Gamma_-}\boldsymbol{\psi}_0) = \mathbf{0}, \text{ on } \Gamma_-, \quad \mathbf{t}_{\alpha,\nu,\mathbb{R}^n\setminus\overline{\mathsf{D}}_+}(\mathbf{V}_{\alpha,\Gamma_-}\boldsymbol{\psi}_0, \mathbf{Q}^s_{\alpha,\Gamma_-}\boldsymbol{\psi}_0) = \mathbf{0}, \text{ on } \Gamma_-. \tag{2.4.45}$$

Let us subtract $(2.4.45)_2$ from $(2.4.45)_1$ and by using the jump formulas (1.4.23) of Lemma 1.4.8 we obtain

$$\boldsymbol{\psi}_0 = \mathbf{0}. \tag{2.4.46}$$

To conclude our argument, we have that, in view of relations (2.4.42) and (2.4.46), we have that property (2.4.30) is satisfied, namely the kernel of the matrix operator $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is the null space, or equivalently, $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is injective.

It follows that our matrix operator $\mathsf{A} : \mathbb{Y} \to \mathbb{Y}$ is an isomorphism and equation (2.4.18) has a unique solution $(\mathbf{\Phi}, \boldsymbol{\varphi}, \boldsymbol{\psi})^t \in \mathbb{Y}$. The unique solution of the equation (2.4.18) together with the layer potential representations provided in relation (2.4.8) give a solution of the Robin-transmission problem (2.4.3) in the space X_{RT}.

Next, we are concerned about the *uniqueness of the solution of the problem* (2.4.3). In order to show this property, let us assume that the problem (2.4.3) admits two solutions and we denote their difference by $(\mathbf{v}^0_\pm, \pi^0_\pm)$. Hence, the fields $(\mathbf{v}^0_+, \pi^0_+, \mathbf{v}^0_-, \pi^0_-) \in \mathsf{X}_{RT}$ satisfy

$$\begin{cases} \Delta\mathbf{v}^0_\pm + \alpha\mathbf{v}^0_\pm - \nabla p^0_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \operatorname{div} \mathbf{v}^0_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \lambda(\mathrm{Tr}_{\mathsf{D}_+}\mathbf{v}^0_+) - \left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{v}^0_-\right)|_{\Gamma_+} = 0 \text{ on } \Gamma_+, \\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{v}^0_+, \pi^0_+) - \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{v}^0_-, \pi^0_-)\right)|_{\Gamma_+} = 0 \text{ on } \Gamma_+, \\ \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{v}^0_-, \pi^0_-)\right)|_{\Gamma_-} + \mathfrak{L}\left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{v}^0_-\right)|_{\Gamma_-} = 0 \text{ on } \Gamma_-, \end{cases} \tag{2.4.47}$$

i.e., the homogenenous version of (2.4.3).

Let us now use Green's formula (1.2.18) in the domains $\mathsf{D}_\pm$ in order to get the following relations

$$\begin{aligned} 2\left\langle \mathbb{E}(\mathbf{v}^0_+), \mathbb{E}(\mathbf{v}^0_+)\right\rangle_{\mathsf{D}_+} + \alpha\left\langle \mathbf{v}^0_+, \mathbf{v}^0_+\right\rangle_{\mathsf{D}_+} &= \left\langle \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{v}^0_+, \pi^0_+), \mathrm{Tr}_{\mathsf{D}_+}\mathbf{v}^0_+\right\rangle_{\Gamma_+} \\ 2\left\langle \mathbb{E}(\mathbf{v}^0_-), \mathbb{E}(\mathbf{v}^0_-)\right\rangle_{\mathsf{D}_-} + \alpha\left\langle \mathbf{v}^0_-, \mathbf{v}^0_-\right\rangle_{\mathsf{D}_-} &= -\left\langle \mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{v}^0_-, \pi^0_-)|_{\Gamma_+}, \left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{v}^0_-\right)|_{\Gamma_+}\right\rangle_{\Gamma_+} \\ &\quad + \left\langle \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{v}^0_-, \pi^0_-)\right)|_{\Gamma_-}, \left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{v}^0_-\right)|_{\Gamma_-}\right\rangle_{\Gamma_-}. \end{aligned} \tag{2.4.48}$$

Let us multiply relation $(2.4.48)_1$ by λ and to the result we will add $(2.4.48)_2$, while taking into account the boundary conditions in problem (2.4.47). After computations, we get

$$\begin{aligned} &\lambda\left(2\left\langle \mathbb{E}(\mathbf{v}^0_+), \mathbb{E}(\mathbf{v}^0_+)\right\rangle_{\mathsf{D}_+} + \alpha\left\langle \mathbf{v}^0_+, \mathbf{v}^0_+\right\rangle_{\mathsf{D}_+}\right) + 2\left\langle \mathbb{E}(\mathbf{v}^0_-), \mathbb{E}(\mathbf{v}^0_-)\right\rangle_{\mathsf{D}_-} + \alpha\left\langle \mathbf{v}^0_-, \mathbf{v}^0_-\right\rangle_{\mathsf{D}_-} \\ &\quad = -\left\langle \mathfrak{L}\left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{v}^0_-\right)|_{\Gamma_-}, \left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{v}^0_-\right)|_{\Gamma_-}\right\rangle_{\Gamma_-}. \end{aligned} \tag{2.4.49}$$

Let us note that, left hand side of (2.4.49) is non-negative and since $\mathfrak{L}$ satisfies condition (2.2.1), the right hand side of (2.4.49) is non-positive. It follows that

$$\mathbf{v}^0_\pm = \mathbf{0},\ \pi^0_\pm = c^0_\pm \in \mathbb{R} \text{ in } \mathsf{D}_\pm. \tag{2.4.50}$$

Now, in view of relation (2.4.50) and the boundary conditions in (2.4.47), we get $c^0_\pm = 0$ in $\mathsf{D}_\pm$. This shows the uniqueness of the solution of the problem (2.4.3).

Finally, the continuity of the potentials involved in relation (2.4.8) implies the existence of some constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \alpha, \mathfrak{L}, \lambda) > 0$, such that the solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$ of the problem (2.4.3) satisfies (2.4.6).

Case 2: Assume that $\lambda = 1$. In this particular case, the matrix operator A in (2.4.20) becomes

$$\mathsf{A} = \begin{bmatrix} -\mathbb{I} & \mathbf{0} & -\mathcal{V}_{\Gamma_-,\Gamma_+} \\ \mathbf{0} & \mathbb{I} & -\mathbb{K}^*_{\Gamma_-,\Gamma_+} \\ \mathbb{D}_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathbb{K}_{\Gamma_+,\Gamma_-} & \mathbb{K}^*_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_+,\Gamma_-} & \frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\alpha,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\alpha,\Gamma_-} \end{bmatrix}. \tag{2.4.51}$$

By using similar steps as presented in the case $\lambda \in (0,1)$, we are able to prove that the Robin-transmission problem (2.4.3) admits a unique solution which depends continuously on the given data for $\lambda = 1$. This concludes our proof. □

2.4.1 The Brinkman system and a related Limiting Robin-Transmission problem in the case $\lambda = 0$

In the latter, let Assumption 1.1.7 be satisfied. We dedicate our efforts to the treatment of the Robin-transmission problem of the Brinkman system (2.4.3) in the special case $\lambda = 0$. This particular choice leads to the problem (2.4.52) which contains a special transmission condition on the boundary Γ_+, namely, that it contains just a trace of the unknown velocity $\mathbf{u}_-$ on Γ_+. Hence, we will call problem (2.4.52) the limiting Robin-transmission problem for the Brinkman system. We treat this case separately due to the fact that the Robin-transmission problem (2.4.3) is not the same problem as the limiting Robin-transmission problem (2.4.52). These problems are different because they have different transmission conditions on the interior boundary. The analysis of the Robin-transmission problem for the Brinkman system (2.4.52) is very useful as its well-posedness provides the well-posedness of the Dirichlet-Robin problem for the same system. This analysis comes from the idea to find well-posedness results for Dirichlet, Neumann, and Robin problems, and of their combination, from well-posedness results for transmission problems (see [82]).

We consider now $\lambda = 0$ in our Robin-transmission problem (2.4.3) and we obtain the following limiting transmission problem

$$\begin{cases} \Delta \mathbf{u}_\pm - \alpha \mathbf{u}_\pm - \nabla \pi_\pm = \mathbf{f}_\pm|_{\mathsf{D}_\pm} \text{ in } \mathsf{D}_\pm, \\ \operatorname{div} \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-\right)|_{\Gamma_+} = -\mathbf{g}_1 \text{ on } \Gamma_+, \\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+) - \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-)\right)|_{\Gamma_+} = \mathbf{h}_1 \text{ on } \Gamma_+, \\ \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-)\right)|_{\Gamma_-} + \mathfrak{L}\left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-\right)|_{\Gamma_-} = \mathbf{g}_2 \text{ on } \Gamma_-, \end{cases} \tag{2.4.52}$$

where $\alpha > 0$ is a given constant. We aim to determine the unknown fields $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$.

In this special case, we have obtained the following well-posedness result (see also [9, Theorem 2], [75, Theorem 4.1], [82, Theorem 6.1]).

Theorem 2.4.2. *Let $\alpha > 0$ be a given constant. Let Assumption 1.1.7 and Assumption 2.2.1 be satisfied. Then, for all data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2) \in \mathsf{Y}_{RT}$, the limiting Poisson problem of Robin-transmission type (2.4.52) has a unique solution*

$$(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}. \tag{2.4.53}$$

In addition, the corresponding solution operator,

$$\mathsf{T}_{lim} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}, \tag{2.4.54}$$

is linear and bounded, and hence, there exists a constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \alpha, \mathfrak{L}, \lambda) > 0$ such that the unique solution of (2.4.52) satisfies

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_{RT}} \leq C||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}}. \tag{2.4.55}$$

Proof. We will prove this result by using similar steps to those in the proof of Theorem 6.1 of [82] and Theorem 2.4.1. Let us begin with *existence of a solution* of the problem (2.4.52). By using a layer potential approach, we seek a solution of the limiting Robin-transmission problem (2.4.52) in the form of the fields $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$ provided in relation (2.4.8) in which we have the unknown densities $(\mathbf{\Phi}, \boldsymbol{\varphi}, \boldsymbol{\psi})^t \in \mathbb{Y}$. Recall that the space $\mathbb{Y}$ was introduced in relation (2.4.7).

Now, we substitute the fields $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$ (given by relation (2.4.8)) into relations $(2.4.52)_3$, $(2.4.52)_4$, $(2.4.52)_5$. Then, if we take into account relation (1.4.23) of Lemma 1.4.8, we obtain the equations

$$\begin{aligned}
&-\left(\frac{1}{2}\mathbb{I} + \mathbb{K}_{\alpha,\Gamma_+}\right)\mathbf{\Phi} - \mathcal{V}_{\alpha,\Gamma_+}\boldsymbol{\varphi} - \mathcal{V}_{\Gamma_-,\Gamma_+}\boldsymbol{\psi} = \mathbf{g}_{lim,01},\\
&\boldsymbol{\varphi} - \mathbb{K}^*_{\Gamma_-,\Gamma_+}\boldsymbol{\psi} = \mathbf{h}_{lim,01},\\
&\left(\mathbb{D}_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathbb{K}_{\Gamma_+,\Gamma_-}\right)\mathbf{\Phi} + \left(\mathbb{K}^*_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_+,\Gamma_-}\right)\boldsymbol{\varphi} + \left(\frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\alpha,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\alpha,\Gamma_-}\right)\boldsymbol{\psi} = \mathbf{g}_{lim,02}.
\end{aligned} \tag{2.4.56}$$

Note that $(\mathbf{g}_{lim,01}, \mathbf{h}_{lim,01}, \mathbf{g}_{lim,02}) \in \mathbb{Y}$, where

$$\begin{aligned}
\mathbf{g}_{lim,01} &= -\mathbf{g}_1 + (\mathrm{Tr}_{\mathsf{D}_-}(\mathcal{N}_{\alpha,\mathsf{D}_-}\mathbf{f}_-))|_{\Gamma_+},\\
\mathbf{h}_{lim,01} &:= \mathbf{h}_1 - \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathcal{N}_{\alpha,\mathsf{D}_+}\mathbf{f}_+, \mathcal{Q}_{\alpha,\mathsf{D}_+}\mathbf{f}_+, \mathbf{f}_+) + (\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathcal{N}_{\alpha,\mathsf{D}_-}\mathbf{f}_-, \mathcal{Q}_{\alpha,\mathsf{D}_-}\mathbf{f}_-, \mathbf{f}_-))|_{\Gamma_+},\\
\mathbf{g}_{lim,02} &:= \mathbf{g}_2 - (\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathcal{N}_{\alpha,\mathsf{D}_-}\mathbf{f}_-, \mathcal{Q}_{\alpha,\mathsf{D}_-}\mathbf{f}_-, \mathbf{f}_-))|_{\Gamma_-} - \mathfrak{L}(\mathrm{Tr}_{\mathsf{D}_-}(\mathcal{N}_{\alpha,\mathsf{D}_-}\mathbf{f}_-))|_{\Gamma_-},
\end{aligned} \tag{2.4.57}$$

while the membership $\mathbf{g}_{lim,01} \in \mathbf{g}_{01} \in H_\nu^{\frac{1}{2}}(\Gamma_+)^n$ is justified in view of relation (1.4.13) and the Flux-Divergence Theorem. Recall that, the compact operators $\mathcal{V}_{\Gamma_-,\Gamma_+}$, $\mathbb{K}^*_{\Gamma_-,\Gamma_+}$, $\mathbb{D}_{\Gamma_+,\Gamma_-}$, $\mathbb{K}_{\Gamma_+,\Gamma_-}$, $\mathbb{K}^*_{\Gamma_+,\Gamma_-}$ and $\mathcal{V}_{\Gamma_+,\Gamma_-}$, which are involved in relation (2.4.56), are introduced in relations (2.4.11), (2.4.14) and (2.4.17), respectively.

In view of relation (2.4.56), we have that the limiting Robin-transmission problem for the Brinkman system (2.4.52) can be rewritten in the matrix form

$$\mathsf{A}_{lim}(\mathbf{\Phi}, \boldsymbol{\varphi}, \boldsymbol{\psi})^t = (\mathbf{g}_{lim,01}, \mathbf{h}_{lim,01}, \mathbf{g}_{lim,02}) \text{ in } \mathbb{Y}, \tag{2.4.58}$$

where $(\mathbf{\Phi}, \boldsymbol{\varphi}, \boldsymbol{\psi})^t \in \mathbb{Y}$ are unknown densities and the matrix operator $\mathsf{A}_{lim} : \mathbb{Y} \to \mathbb{Y}$ is given by

$$\mathsf{A}_{lim} := \begin{bmatrix} -\left(-\frac{1}{2}\mathbb{I} + \mathbb{K}_{\alpha,\Gamma_+}\right) & -\mathcal{V}_{\alpha,\Gamma_+} & -\mathcal{V}_{\Gamma_-,\Gamma_+}\\ \mathbf{0} & \mathbb{I} & -\mathbb{K}^*_{\Gamma_-,\Gamma_+}\\ \mathbb{D}_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathbb{K}_{\Gamma_+,\Gamma_-} & \mathbb{K}^*_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_+,\Gamma_-} & \frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\alpha,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\alpha,\Gamma_-} \end{bmatrix}. \tag{2.4.59}$$

Our next objective is to prove that $\mathsf{A}_{lim} : \mathbb{Y} \to \mathbb{Y}$ is an isomorphism. To achieve this, we show that the operator $\mathsf{A}_{lim} : \mathbb{Y} \to \mathbb{Y}$ is a Fredholm operator of index zero and afterwards, we show that $\mathsf{A}_{lim} : \mathbb{Y} \to \mathbb{Y}$ is one-to-one.

Let us focus on the first claim, namely, that $\mathsf{A}_{lim} : \mathbb{Y} \to \mathbb{Y}$ is Fredholm of index zero. Note that this operator is well-defined, linear and continuous and it admits the decomposition

$$\mathsf{A}_{lim} = \mathsf{A}_{lim,0} + \mathsf{A}_{lim,\mathsf{C}} : \mathbb{Y} \to \mathbb{Y}, \tag{2.4.60}$$

where the operators $\mathsf{A}_{lim,0} : \mathbb{Y} \to \mathbb{Y}$ and $\mathsf{A}_{lim,\mathsf{C}} : \mathbb{Y} \to \mathbb{Y}$ are given by

$$\mathsf{A}_{lim,0} := \begin{bmatrix} -\left(\frac{1}{2}\mathbb{I} + \mathbb{K}_{\Gamma_+}\right) & -\mathcal{V}_{\Gamma_+} & \mathbf{0} \\ \mathbf{0} & \mathbb{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{2}\mathbb{I} + \mathbb{K}^*_{\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_-} \end{bmatrix} \tag{2.4.61}$$

and

$$\mathsf{A}_{lim,\mathsf{C}} := \begin{bmatrix} -\mathbb{K}_{\alpha,0,\Gamma_+} & -\mathcal{V}_{\alpha,0,\Gamma_+} & -\mathcal{V}_{\Gamma_-,\Gamma_+} \\ \mathbf{0} & \mathbf{0} & -\mathbb{K}^*_{\Gamma_-,\Gamma} \\ \mathbb{D}_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathbb{K}_{\Gamma_+,\Gamma_-} & \mathbb{K}^*_{\Gamma_+,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\Gamma_+,\Gamma_-} & \mathbb{K}^*_{\alpha,0,\Gamma_-} + \mathfrak{L}\mathcal{V}_{\alpha,0,\Gamma_-} \end{bmatrix}. \tag{2.4.62}$$

Recall that the operators that appear in $\mathsf{A}_{lim,0}$, i.e., relation (2.4.61), are Fredholm operators of index zero (as in relations (2.4.25), (2.4.26), (2.4.27), (2.4.28) and (2.4.29)), which shows that $\mathsf{A}_{lim,0} : \mathbb{Y} \to \mathbb{Y}$ is Fredholm on index zero. Also, the operators that are involved in $\mathsf{A}_{lim,\mathsf{C}}$, i.e., relation (2.4.62), are all compact operators (see Lemma 1.4.9 and relations (2.4.11), (2.4.14), (2.4.17)). This means that $\mathsf{A}_{lim,\mathsf{C}} : \mathbb{Y} \to \mathbb{Y}$. Consequently, the operator $\mathsf{A}_{lim} = \mathsf{A}_{lim,0} + \mathsf{A}_{lim,\mathsf{C}} : \mathbb{Y} \to \mathbb{Y}$ is Fredholm of index 0.

Let us prove that $\mathsf{A}_{lim} : \mathbb{Y} \to \mathbb{Y}$ is one-to-one, which is equivalent with showing that

$$\operatorname{Ker}\{\mathsf{A}_{lim} : \mathbb{Y} \to \mathbb{Y}\} = \{\mathbf{0}\}. \tag{2.4.63}$$

In order to prove that relation (2.4.63) holds, let $(\boldsymbol{\Phi}_{lim,0}, \boldsymbol{\varphi}_{lim,0}, \boldsymbol{\psi}_{lim,0})^t \in \operatorname{Ker}\{\mathsf{A}_{lim} : \mathbb{Y} \to \mathbb{Y}\}$. Next, we define the fields

$$\begin{aligned} \mathbf{u}^0_+ &:= \mathbf{W}_{\alpha,\Gamma_+}\boldsymbol{\Phi}_{lim,0} + \mathbf{V}_{\alpha,\Gamma_+}\boldsymbol{\varphi}_{lim,0} \\ \pi^0_+ &:= \mathbf{Q}^d_{\alpha,\Gamma_+}\boldsymbol{\Phi}_{lim,0} + \mathbf{Q}^s_{\alpha,\Gamma_+}\boldsymbol{\varphi}_{lim,0} \\ \mathbf{u}^0_- &:= \mathbf{W}_{\alpha,\Gamma_+}\boldsymbol{\Phi}_{lim,0} + \mathbf{V}_{\alpha,\Gamma_+}\boldsymbol{\varphi}_{lim,0} + \mathbf{V}_{\alpha,\Gamma_-}\boldsymbol{\psi}_{lim,0} \\ \pi^0_- &:= \mathbf{Q}^d_{\alpha,\Gamma_+}\boldsymbol{\Phi}_{lim,0} + \mathbf{Q}^s_{\alpha,\Gamma_+}\boldsymbol{\varphi}_{lim,0} + \mathbf{Q}^s_{\alpha,\Gamma_-}\boldsymbol{\psi}_{lim,0}. \end{aligned} \tag{2.4.64}$$

Note that, the fields in (2.4.64) satisfy

$$\begin{aligned} &\left(\operatorname{Tr}_{\mathsf{D}_-}\mathbf{u}^0_-\right)|_{\Gamma_+} = 0 \text{ a.e. on } \Gamma_+, \\ &\mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}^0_+, \pi^0_+) = \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}^0_-, \pi^0_-)\right)|_{\Gamma_+} \text{ a.e. on } \Gamma_+, \\ &\left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}^0_-, \pi^0_-)\right)|_{\Gamma_-} + \mathfrak{L}\left(\operatorname{Tr}_{\mathsf{D}_-}\mathbf{u}^0_-\right)|_{\Gamma_-} = 0, \text{ a.e. on } \Gamma_-. \end{aligned} \tag{2.4.65}$$

Let us apply Green formula (1.2.18) to the pair $(\mathbf{u}^0_-, \pi^0_-)$ introduced in relation (2.4.64). We obtain

$$\begin{aligned} 2\langle \mathbb{E}(\mathbf{u}^0_-), \mathbb{E}(\mathbf{u}^0_-)\rangle_{\mathsf{D}_-} + \alpha\langle \mathbf{u}^0_-, \mathbf{u}^0_-\rangle_{\mathsf{D}_-} = &-\langle(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}^0_-, \pi^0_-))|_{\Gamma_+}, (\operatorname{Tr}_{\mathsf{D}_-}\mathbf{u}^0_-)|_{\Gamma_+}\rangle_{\Gamma_+} \\ &+ \langle(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}^0_-, \pi^0_-))|_{\Gamma_-}, (\operatorname{Tr}_{\mathsf{D}_-}\mathbf{u}^0_-)|_{\Gamma_-}\rangle_{\Gamma_-}. \end{aligned} \tag{2.4.66}$$

In view of relation (2.4.65), we rewrite relation (2.4.66) as

$$2\langle \mathbb{E}(\mathbf{u}_-^0), \mathbb{E}(\mathbf{u}_-^0)\rangle_{\mathsf{D}_-} + \alpha\langle \mathbf{u}_-^0, \mathbf{u}_-^0\rangle_{\mathsf{D}_-} = -\langle \mathfrak{L}(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-^0)|_{\Gamma_-}, (\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-^0)|_{\Gamma_-}\rangle_{\Gamma_-}. \tag{2.4.67}$$

Recall that $\mathfrak{L} \in L^\infty(\Gamma_-)^{n\times n}$ satisfies condition (2.2.1). Then, we have that, the left hand side of (2.4.67) is non-negative and the right hand side of (2.4.67) is non-positive. Consequently, $\mathbf{u}_-^0 = \mathbf{0}$, in D_- and $\pi_-^0 = c_-^0 \in \mathbb{R}$ in D_-. Moreover, $c_-^0 = 0$ in D_-, in view of relation (2.4.65) and the fact that $(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-^0, \pi_-^0))|_{\Gamma_-} = -c_-^0\boldsymbol{\nu}$.

Now, since $\mathbf{u}_-^0 = \mathbf{0}$ and $\pi_-^0 = 0$ in D_-, the second relation of (2.4.65) is equivalent to $\mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0) = 0$. Hence, we apply Green formula (1.2.18) to the pair $(\mathbf{u}_+^0, \pi_+^0)$ introduced in relation (2.4.64) and we obtain

$$2\langle \mathbb{E}(\mathbf{u}_+^0), \mathbb{E}(\mathbf{u}_+^0)\rangle_{\mathsf{D}_+} + \alpha\langle \mathbf{u}_+^0, \mathbf{u}_+^0\rangle_{\mathsf{D}_+} = \langle \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0), \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+^0\rangle_{\Gamma_+}. \tag{2.4.68}$$

In view of the fact that $\mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0) = 0$, we deduce that $\mathbf{u}_+^0 = \mathbf{0}, \pi_+^0 = c_+^0 \in \mathbb{R}$ in D_+. Moreover, since $\mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_+^0, \pi_+^0) = -c_+^0\boldsymbol{\nu}$, we have that $c_+^0 = 0$.

Consequently, we have shown that

$$\mathbf{u}_\pm^0 = \mathbf{0},\ \pi_\pm^0 = 0 \text{ in } \mathsf{D}_\pm. \tag{2.4.69}$$

Now, we can use arguments similar to those presented in the proof of Theorem 2.4.1, relations (2.4.37) through (2.4.46) in order to obtain

$$\boldsymbol{\Phi}_{lim,0} = \mathbf{0},\ \boldsymbol{\varphi}_{lim,0} = \mathbf{0},\ \boldsymbol{\psi}_{lim,0} = \mathbf{0}. \tag{2.4.70}$$

Hence, the operator $\mathsf{A}_{lim} : \mathbb{Y} \to \mathbb{Y}$ is one-to-one, as asserted.

We deduce that our operator $\mathsf{A}_{lim} : \mathbb{Y} \to \mathbb{Y}$ is an isomorphism and the unique solution of equation (2.4.58) together with the potentials given in relation (2.4.8) provide a solution for our problem (2.4.52) in the space X_{RT}. Thus, the existence of a solution of (2.4.52) is established.

Now, in order to show the *uniqueness property*, let us assume that there are two solutions, and we denote their difference by $(\mathbf{u}_\pm^0, \pi_\pm^0)$. Then, the fields $(\mathbf{u}_\pm^0, \pi_\pm^0)$ satisfy relation (2.4.65). In view of the arguments presented in the former, we have that relation (2.4.69) holds, that is the uniqueness property is established.

Finally, the continuity of the potentials (see Theorem 1.4.2, Theorem 1.4.4, Theorem 1.4.7) that are present in relation (2.4.8) assures that the unique solution of the problem (2.4.52) must satisfy the estimate (2.4.55). This concludes our proof. □

2.4.2 The Brinkman system and a related Robin-Dirichlet problem

In this subsection, we aim to emphasize the special role that a transmission-type problem fulfills. In the latter, let $\alpha > 0$ be a given constant and let Assumption 1.1.7 be satisfied. Let us mention that, we will be focusing on the Lipschitz domain D_- and we use similar arguments to those presented in [82, p. 4581]. We point out the fact that the problem (2.4.52) is well-posed, as it was established in Theorem 2.4.2. This means that we get a unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$, of the problem (2.4.52). This solution produces a pair $(\mathbf{u}_-, \pi_-) \in H^1_{\mathrm{div}}(\mathsf{D}_-)^n \times L^2(\mathsf{D}_-)$ that satisfies another boundary value problem, namely, the following Robin-Dirichlet problem for the Brinkman system

$$\begin{cases} \Delta\mathbf{u}_- - \alpha\mathbf{u}_- - \nabla\pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \mathrm{div}\ \mathbf{u}_- = 0 \text{ in } \mathsf{D}_-, \\ (\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-)|_{\Gamma_+} = -\mathbf{g}_1 \text{ on } \Gamma_+, \\ (\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-))|_{\Gamma_-} + \mathfrak{L}(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u})|_{\Gamma_-} = \mathbf{g}_2 \text{ on } \Gamma_-. \end{cases} \tag{2.4.71}$$

In other words, we are able to determine the solution to a boundary value problem (namely, problem (2.4.71)) by extracting it from the solution of a transmission-type problem (namely, problem (2.4.52)). Consequently, the pair $(\mathbf{u}_-, \pi_-)$ is a solution of the Robin-Dirichlet problem (2.4.71).

Moreover, an uniqueness argument, similar to that presented in the proof of Theorem 2.4.2, will lead to the fact that, the Robin-Dirichlet problem for the Brinkman system (2.4.71) is, in turn, well-posed. Under the assumption of Theorem 2.4.2, we obtain the following result (see [9, Corollary 1], [82, p. 4581]).

Corollary 2.4.3. *The Robin-Dirichlet problem for the Brinkman system (2.4.71) has a unique solution* $(\mathbf{u}_-, \pi_-) \in H^1_{\mathrm{div}}(\mathsf{D}_-)^n \times L^2(\mathsf{D}_-)$*, for* $n \in \mathbb{N}$*,* $n \geq 2$*.*

3

Nonlinear Boundary Value Problems of Transmission-type related to the Navier-Stokes and Darcy-Forchheimer-Brinkman systems

This purpose of this chapter is to treat nonlinear transmission-type problems which contain a generalized version of the Darcy-Forchheimer-Brinkman system or the classical Darcy-Forchheimer-Brinkman system (see relation (3.1.1) in Lipschitz domains in Euclidean setting (see Assumption 1.1.6 and Assumption 1.1.7). All these problems are important for their practical applications (see, e.g., [53], [115]). The content of this chapter follows the results that were obtained in the papers [5], [6], [9].

Let us briefly describe the content of this chapter. We give existence and uniqueness results for the following boundary problems. First of all, we analyze Poisson problem of transmission-type for the generalized Darcy-Forchheimer-Brinkman and Stokes systems in complementary Lipschitz domains in $\mathbb{R}^3$. Next, we investigate the Poisson problem of transmission-type for the generalized Darcy-Forchheimer-Brinkman and Brinkman systems in complementary Lipschitz domains in $\mathbb{R}^3$. Lastly, we have the the Poisson problem of Robin-transmission-type for the Darcy-Forchheimer-Brinkman system in Euclidean setting provided by Assumption 1.1.7.

The well-posedness results for the linear problems analyzed in Chapter 2 introduce their solution operators, which are linear and continuous. Taking them into account together with the nonliniarities of the PDEs considered in this chapter (Navier-Stokes equations, Darcy-Forcheimer-Brinkman equations), we reduce the analysis of the boundary value problems for such nonlinear PDEs to the study of certain nonlinear operators and of their fixed points in some special cases. Such nonlinear operators appear from the composition of the linear operators mentioned above and the operators that describe the nonlinearities of the nonlinear PDEs. Their fixed points will provide the solutions of our nonlinear boundary problems (see also [89]).

Let us also take note of some works that concern the investigation of boundary problems which involve nonlinear PDE systems. For example, Choe and Kim [27] have obtained the existence and regularity of solutions for the non-homogeneous Dirichlet problem for the Navier-Stokes system in a bounded Lipschitz domain in $\mathbb{R}^3$, whose boundary data possesses minimal regularity. Kohr, Lanza de Cristoforis and Wendland [74] have obtained an existence and uniqueness result for the Dirchlet problem for the semilinear Darcy-Forchheimer-Brinkman system in a bounded Lipschitz domain in $\mathbb{R}^n$, $n \leq 4$. The authors in [71] have obtained an existence and uniqueness result for a transmission-type problem for the Darcy-Forchheimer-Brinkman and Stokes systems in $\mathbb{R}^3$. Also, in [80], the authors have obtained the existence of solutions of a Dirichlet-transmission problem for the anisotropic Navier-Stokes system in Lipschitz domains in $\mathbb{R}^n$, $n = 2, 3$ (see also [15], [87], [90]).

3.1 The generalized Darcy-Forchheimer-Brinkman system and related results

In this section, let us consider $\mathsf{D} \subset \mathbb{R}^3$ a bounded Lipschitz domain, unless specified otherwise. We present a generalized version of the Darcy-Forchheimer-Brinkman system, which is given by

$$\Delta \mathbf{v} - \mathcal{P}\mathbf{v} - k|\mathbf{v}|\mathbf{v} - \beta(\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla p = \mathbf{g} \text{ in } \mathsf{D}, \ \operatorname{div} \mathbf{v} = 0 \text{ in } \mathsf{D}, \tag{3.1.1}$$

where $\mathcal{P} \in L^\infty(\mathsf{D})^{3\times 3}$ such that condition (1.2.22) holds and $k, \beta : \mathsf{D} \to \mathbb{R}_+$ are given functions, such that $k, \beta \in L^\infty(\mathsf{D}, \mathbb{R}_+)$, i.e., essentially bounded, non-negative functions defined on D (for additional details, see also [57]). We have the following useful remarks.

Remark 3.1.1. *For $\mathcal{P} \equiv \alpha\mathbb{I}$ and $\alpha, k, \beta > 0$ given constants, the system (3.1.1) becomes the classical Darcy-Forchheimer-Brinkman system.*

Remark 3.1.2. *For $\mathcal{P} \equiv 0$, $k = 0$ and for $\beta > 0$ a given constant, the system (3.1.1) becomes the well-known Navier-Stokes system.*

Now, let us state and prove a lemma that we will employ in the proofs of our well-posedness results of this chapter. The lemma reads as follows (see also, [71, Lemma 5.1]).

Lemma 3.1.3. *Let $\mathsf{D} \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded Lipschitz domain and let $k, \beta : \mathsf{D} \to \mathbb{R}_+$ such that $k, \beta \in L^\infty(\mathsf{D}, \mathbb{R}_+)$. Let*

$$\mathsf{J}_{k,\beta,\mathsf{D}}(\mathbf{u}) := \mathring{\mathsf{E}}(k|\mathbf{u}|\mathbf{u} + \beta(\mathbf{u} \cdot \nabla)\mathbf{u}), \tag{3.1.2}$$

where $\mathring{\mathsf{E}}$ is the extension by zero operator outside D. Then, the nonlinear operator

$$\mathsf{J}_{k,\beta,\mathsf{D}} : H^1_{\operatorname{div}}(\mathsf{D})^n \to \tilde{H}^{-1}(\mathsf{D})^n, \tag{3.1.3}$$

is continuous and bounded, in the sense that there exists a constant $c_0 = c_0(\mathsf{D}, k, \beta) > 0$ such that

$$||\mathsf{J}_{k,\beta,\mathsf{D}}(\mathbf{u})||_{\tilde{H}^{-1}(\mathsf{D})^n} \leq c_0 ||\mathbf{u}||^2_{H^1(\mathsf{D})^n}. \tag{3.1.4}$$

In addition, the following Lipschitz-like relation

$$||\mathsf{J}_{k,\beta,\mathsf{D}}(\mathbf{u}) - \mathsf{J}_{k,\beta,\mathsf{D}}(\mathbf{v})|| \leq c_0(||\mathbf{u}||_{H^1(\mathsf{D})^n} + ||\mathbf{v}||_{H^1(\mathsf{D})^n})||\mathbf{u} - \mathbf{v}||_{H^1(\mathsf{D})^n}, \tag{3.1.5}$$

holds, where $c_0 = c_0(\mathsf{D}, k, \beta) > 0$ is the constant that is present in relation (3.1.4).

Proof. The proof of this lemma follows ideas similar to those presented in [71, Lemma 5.1]. Let us describe the main arguments. Since D is a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$, we have that the embeddings

$$H^1(\mathsf{D})^n \hookrightarrow L^q(\mathsf{D})^n, \tag{3.1.6}$$

are continuous, for all q such that $2 \leq q \leq 6$. The embedding (3.1.6) has dense range and we have

$$L^{q'}(\mathsf{D})^n \hookrightarrow \tilde{H}^{-1}(\mathsf{D})^n, \tag{3.1.7}$$

continuously, in the sense that

$$\mathring{\mathsf{E}}\mathbf{u} \in \tilde{H}^{-1}(\mathsf{D})^n, \ \forall\ \mathbf{u} \in L^{q'}(\mathsf{D})^n, \ \frac{6}{5} \leq q' \leq 2, \tag{3.1.8}$$

and there is exists constant $c_q > 0$ such that

$$||\mathring{\mathsf{E}}_{+}\mathbf{u}||_{\tilde{H}^{-1}(\mathsf{D})^n} \leq c_q||\mathbf{u}||_{L^{q'}(\mathsf{D})^n}. \tag{3.1.9}$$

If we take $q = 4$ in relation (3.1.6) and if we apply Hölder's inequality, we deduce that

$$|\mathbf{v}|\mathbf{w} \in L^2(\mathsf{D})^n, \ \forall \ \mathbf{v}, \mathbf{w} \in H^1(\mathsf{D})^n. \tag{3.1.10}$$

Now we define the operator

$$b : H^1(\mathsf{D})^n \times H^1(\mathsf{D})^n \to \tilde{H}^{-1}(\mathsf{D})^n, \ b(\mathbf{v}, \mathbf{w}) := \mathring{\mathsf{E}}_{+}(k|\mathbf{v}|\mathbf{w}), \tag{3.1.11}$$

which is well-defined. This property holds due to the application of relation (3.1.9) in the case $q = 2$.

Let us use relation (3.1.9) in the case $q = 2$ and Hölder's inequality. Consequenly, we deduce that

$$||b(\mathbf{v}, \mathbf{w})||_{\tilde{H}^{-1}(\mathsf{D})^n} \leq c_*||\mathbf{v}||_{H^1(\mathsf{D})^n}||\mathbf{w}||_{H^1(\mathsf{D})^n}, \tag{3.1.12}$$

where $c_* = c_*(\Gamma_+, k) > 0$ is a constant. Thus, the nonlinear operator (3.1.11) is bounded.

We return to relation (3.1.6) and we set $q = 6$. Let us use, again, Hölder's inequality and we get

$$(\mathbf{v} \cdot \nabla)\mathbf{w} \in L^{\frac{3}{2}}(\mathsf{D})^n, \ \forall \ \mathbf{v}, \mathbf{w} \in H^1(\mathsf{D})^n. \tag{3.1.13}$$

Consequently, we have

$$||(\mathbf{v} \cdot \nabla)\mathbf{w}||_{L^{\frac{3}{2}}(\mathsf{D})^n} \leq c'||\mathbf{v}||_{H^1(\mathsf{D})^n}||\mathbf{w}||_{H^1(\mathsf{D})^n}, \tag{3.1.14}$$

where $c' = c'(\mathsf{D}) > 0$ is a constant.

Let us define the operator

$$t : H^1(\mathsf{D})^n \times H^1(\mathsf{D})^n \to \tilde{H}^{-1}(\mathsf{D})^n, \ t(\mathbf{v}, \mathbf{w}) := \mathring{\mathsf{E}}_{+}(\beta(\mathbf{v} \cdot \nabla)\mathbf{w}). \tag{3.1.15}$$

By applying relation (3.1.9) in the case $q = \frac{3}{2}$, we deduce that the operator (3.1.15) is well-defined. Another application of relation (3.1.9) with $q = \frac{3}{2}$ together with relation (3.1.14) yields

$$||t(\mathbf{v}, \mathbf{w})||_{\tilde{H}^{-1}(\mathsf{D})^n} \leq c^*||\mathbf{v}||_{H^1(\mathsf{D})^n}||\mathbf{w}||_{H^1(\mathsf{D})^n}, \tag{3.1.16}$$

where $c^* = c^*(\mathsf{D}, \beta) > 0$ is a constant. This shows that the nonlinear operator (3.1.15) is bounded.

Next, we have that

$$\mathsf{J}_{k,\beta,\mathsf{D}}(\mathbf{v}) = b(\mathbf{v}, \mathbf{v}) + t(\mathbf{v}, \mathbf{v}). \tag{3.1.17}$$

Let us set $c_0 := c_* + c^*$. Then, in view of relations (3.1.12) and (3.1.16), we have that the nonlinear operator (3.1.3) satisfies relation (3.1.4), which shows that it is bounded. In addition, the nonlinear operator (3.1.3) satisfies the Lipschitz-like inequality (3.1.5). This can be shown by using relations (3.1.12) and (3.1.16) and some simple computations. We omit the full arguments for the sake of brevity. Thus, our proof is complete. □

3.2 Transmission problem for the generalized Darcy-Forchheimer-Brinkman and classical Stokes systems in complementary Lipschitz domains in $\mathbb{R}^3$

The purpose of this section is to provide a well-posedness result, for a transmission-type problem, which was obtained in the setting of Assumption 1.1.6 for $n = 3$, i.e., complementary Lipschitz domains in $\mathbb{R}^3$. We have considered a generalized version of the Darcy-Forchheimer-Brinkman system in the bounded Lipschitz domain D_+ and the Stokes system in the complementary Lipschitz set D_-. Also let Assumption 2.2.1 be satisfied, for $n = 3$.

Let us recall the space in which we seek our solution, that is,

$$\mathsf{X}_w := H^1_{\mathrm{div}}(\mathsf{D}_+)^3 \times L^2(\mathsf{D}_+) \times \mathcal{H}^1_{\mathrm{div}}(\mathsf{D}_-)^3 \times L^2(\mathsf{D}_-) \tag{3.2.1}$$

and the space of given data,

$$\mathsf{Y}_w := \tilde{H}^{-1}(\mathsf{D}_+)^3 \times \tilde{\mathcal{H}}^{-1}(\mathsf{D}_-)^3 \times H^{\frac{1}{2}}(\Gamma)^3 \times H^{-\frac{1}{2}}(\Gamma)^3. \tag{3.2.2}$$

We study the following transmission problem of Poisson type for the generalized Darcy-Forchheimer-Brinkman and Stokes systems,

$$\begin{cases} \Delta \mathbf{u}_+ - \mathcal{P}\mathbf{u}_+ - k|\mathbf{u}_+|\mathbf{u}_+ - \beta(\mathbf{u}_+ \cdot \nabla)\mathbf{u}_+ - \nabla \pi_+ = \mathbf{f}_+|_{\mathsf{D}_+} \text{ in } \mathsf{D}_+, \\ \Delta \mathbf{u}_- - \nabla \pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \mathrm{div}\ \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ - \mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_- = \mathbf{g} \text{ on } \Gamma, \\ \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+ + \mathring{\mathsf{E}}_+(k|\mathbf{u}_+|\mathbf{u}_+ + \beta(\mathbf{u}_+ \cdot \nabla)\mathbf{u}_+)) \\ \quad - \mathbf{t}_{\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_-) + \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ = \mathbf{h} \text{ on } \Gamma, \end{cases} \tag{3.2.3}$$

and we aim to determine the unknown fields $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_w$. Once again, since the Stokes system appears in the unbounded Lipschitz domain D_-, we must work with the weighted space $\mathcal{H}^1_{\mathrm{div}}(\mathsf{D}_-)^3$, which is included in the solution space X_w.

The following result regarding the well-posedness of the transmission problem (3.2.3) was obtained, for $\mathbf{u}_\infty \in \mathbb{R}^3$ a given constant (see [6, Theorem 3.3] see also [71, Theorem 5.2] in the case $k, \beta > 0$, $\mathcal{P} \equiv \alpha\mathbb{I}$, where $\alpha > 0$ is a constant).

Theorem 3.2.1. *Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied, for $n = 3$. Let $\mathcal{P} \in L^\infty(\mathsf{D}_+)^{3\times 3}$ such that condition (1.2.22) holds. Let $\mathbf{u}_\infty \in \mathbb{R}^3$ be a constant vector. Then, there exist two constants*

$$\xi = \xi(\mathsf{D}_+, \mathsf{D}_-, \mathcal{P}, k, \beta, \mathfrak{L}) > 0, \quad \eta = \eta(\mathsf{D}_+, \mathsf{D}_-, \mathcal{P}, k, \beta, \mathfrak{L}) > 0, \tag{3.2.4}$$

such that for all given $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty) \in \mathsf{Y}_w \times \mathbb{R}^3$ that satisfy

$$||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty)||_{\mathsf{Y}_w \times \mathbb{R}^3} \leq \xi, \tag{3.2.5}$$

the Poisson problem of transmission-type for the Darcy-Forchheimer-Brinkman and Stokes systems (3.2.3) has a unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_w$ and

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_- - \mathbf{u}_\infty, \pi_-)||_{\mathsf{X}_w} \leq \eta. \tag{3.2.6}$$

In addition, the solution depends continuously on the given data and satisfies the following estimate

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_- - \mathbf{u}_\infty, \pi_-)||_{\mathsf{X}_w} \leq C_0||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty)||_{\mathsf{Y}_w \times \mathbb{R}^3}, \tag{3.2.7}$$

where $C_0 = C_0(\mathsf{D}_+, \mathsf{D}_-, \mathcal{P}, \mathfrak{L}) > 0$ *is a constant and* $\mathbf{u}_- - \mathbf{u}_\infty$ *vanishes at infinity in the sense of Leray.*

Proof. We will follow similar similar steps as those presented in the proof of [71, Theorem 5.2].

Step 1. We begin our proof by concerning ourselves with *the existence of a solution of the problem* (3.2.3). To achieve this, we consider the following change of variables $\mathbf{v}_+ := \mathbf{u}_+$, $\mathbf{v}_- := \mathbf{u}_- - \mathbf{u}_\infty$. Consequently, our problem (3.2.3) reduces to the problem

$$\begin{cases} \Delta \mathbf{v}_+ - \mathcal{P}\mathbf{v}_+ - \nabla \pi_+ = \mathbf{f}_+|_{\mathsf{D}_+} + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{v}_+)|_{\mathsf{D}_+} \text{ in } \mathsf{D}_+, \\ \Delta \mathbf{v}_- - \nabla \pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \text{div } \mathbf{v}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \text{Tr}_{\mathsf{D}_+}\mathbf{v}_+ - \text{Tr}_{\mathsf{D}_-}\mathbf{v}_- = \mathbf{g} + \mathbf{u}_\infty \text{ on } \Gamma, \\ \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{v}_+, \pi_+, \mathbf{f}_+ + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{v}_+)) - \mathbf{t}_{\mathsf{D}_-}(\mathbf{v}_-, \pi_-, \mathbf{f}_-) + \mathfrak{L}\text{Tr}_{\mathsf{D}_+}\mathbf{v}_+ = \mathbf{h} \text{ on } \Gamma. \end{cases} \tag{3.2.8}$$

The technique that we will apply is as follows. We focus on the construction of a nonlinear operator U_+ that maps a closed ball of the space $H^1_{\text{div}}(\mathsf{D}_+)^3$ to the same ball and that is a contraction on the same ball. Consequently, the unique fixed point of the operator U_+ will aid in the determination of a solution of the problem (3.2.8).

We begin the construction of the operator U_+. To do this, we will fix $\mathbf{v}_+ \in H^1_{\text{div}}(\mathsf{D}_+)^3$ and we consider transmission problem for the generalized Brinkman and Stokes systems in the unknowns $(\mathbf{v}^0_+, \pi^0_+, \mathbf{v}^0_-, \pi^0_-)$, which is given by

$$\begin{cases} \Delta \mathbf{v}^0_+ - \mathcal{P}\mathbf{v}^0_+ - \nabla \pi^0_+ = \mathbf{f}_+|_{\mathsf{D}_+} + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{v}_+)|_{\mathsf{D}_+} \text{ in } \mathsf{D}_+, \\ \Delta \mathbf{v}^0_- - \nabla \pi^0_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \text{div } \mathbf{v}^0_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \text{Tr}_{\mathsf{D}_+}\mathbf{v}^0_+ - \text{Tr}_{\mathsf{D}_-}\mathbf{v}^0_- = \mathbf{g} + \mathbf{u}_\infty \text{ on } \Gamma, \\ \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{v}^0_+, \pi^0_+, \mathbf{f}_+ + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{v}_+)) - \mathbf{t}_{\mathsf{D}_-}(\mathbf{v}^0_-, \pi^0_-, \mathbf{f}_-) + \mathfrak{L}\text{Tr}_{\mathsf{D}_+}\mathbf{v}^0_+ = \mathbf{h} \text{ on } \Gamma, \end{cases} \tag{3.2.9}$$

where $\mathbf{g} + \mathbf{u}_\infty \in H^{\frac{1}{2}}(\Gamma)^3$ and the membership $\mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{v}_+) \in \tilde{H}^{-1}(\mathsf{D}_+)^3$ holds due to Lemma 3.1.3.

Now, by employing Theorem 2.2.3, we deduce that the linear transmission problem (3.2.9) has a unique solution, given by the following relation

$$\begin{aligned} (\mathbf{v}^0_+, \pi^0_+, \mathbf{v}^0_-, \pi^0_-) &= (\mathsf{U}_+(\mathbf{v}_+), \mathsf{R}_+(\mathbf{v}_+), \mathsf{U}_-(\mathbf{v}_+), \mathsf{R}_-(\mathbf{v}_+)) \\ &:= \mathsf{T}(\mathbf{f}_+|_{\mathsf{D}_+} + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{v}_+)|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g} + \mathbf{u}_\infty, \mathbf{h}) \in \mathsf{X}_w. \end{aligned} \tag{3.2.10}$$

Note that, the operator $\mathsf{T} : \mathsf{Y}_w \times \mathbb{R}^3 \to \mathsf{X}_w$ which is present in relation (3.2.10) is the solution operator described in relation (2.2.28). It is a well-defined, linear and continuous operator, which maps the given data (belonging to the space $\mathsf{Y}_w \times \mathbb{R}^3$) to the unique solution of the Poisson transmission problem (2.2.4) for the generalized Brinkman and Stokes systems in complementary Lipschitz domains in $\mathbb{R}^3$, in view of Theorem 2.2.3. In addition, the solution operator $\mathsf{T} : \mathsf{Y}_w \times \mathbb{R}^3 \to \mathsf{X}_w$ satisfies the estimate (2.2.29) of Theorem 2.2.3.

Furthermore, if we fix the given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty) \in \mathsf{Y}_w \times \mathbb{R}^3$, we obtain the fact that the nonlinear operator

$$(\mathsf{U}_+, \mathsf{R}_+, \mathsf{U}_-, \mathsf{R}_-) : H^1_{\text{div}}(\mathsf{D}_+)^3 \to \mathsf{X}_w, \tag{3.2.11}$$

is continuous and bounded, in the sense that there is a constant $d_* = d_*(\mathsf{D}_+, \mathsf{D}_-, \mathcal{P}, \mathfrak{L}) > 0$ such that the estimate

$$\begin{aligned} ||(\mathsf{U}_+(\mathbf{v}_+), \mathsf{R}_+(\mathbf{v}_+), \mathsf{U}_-(\mathbf{v}_+), \mathsf{R}_-(\mathbf{v}_+))||_{\mathsf{X}_w} &\le d_*||(\mathbf{f}_+|_{\mathsf{D}_+} + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{v}_+)|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty)||_{\mathsf{Y}_w \times \mathbb{R}^3} \\ &\le d_*||(\mathbf{f}_+|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty)||_{\mathsf{Y}_w \times \mathbb{R}^3} + d_* c_0 ||\mathbf{v}_+||^2_{H^1(\mathsf{D}_+)^3}, \end{aligned} \tag{3.2.12}$$

holds, for all $\mathbf{v}_+ \in H^1_{\text{div}}(\mathsf{D}_+)^3$ and the constant $c_0 = c_0(\mathsf{D}_+, k, \beta) > 0$ is the same constant as in Lemma 3.1.3. Moreover, if we use the operators $(\mathsf{U}_+, \mathsf{R}_+, \mathsf{U}_-, \mathsf{R}_-)$, we are able to rewrite the problem (3.2.8) in the form

$$\begin{cases} \Delta \mathsf{U}_+(\mathbf{v}_+) - \mathcal{P}\mathsf{U}_+(\mathbf{v}_+) - \nabla \mathsf{R}_+(\mathbf{v}_+) = \mathbf{f}_+|_{\mathsf{D}_+} + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{v}_+)|_{\mathsf{D}_+} \text{ in } \mathsf{D}_+, \\ \Delta \mathsf{U}_-(\mathbf{v}_+) - \nabla \mathsf{R}_-(\mathbf{v}_+) = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \text{div } \mathsf{U}_\pm(\mathbf{v}_+) = 0 \text{ in } \mathsf{D}_\pm, \\ \text{Tr}_{\mathsf{D}_+}\mathsf{U}_+(\mathbf{v}_+) - \text{Tr}_{\mathsf{D}_-}\mathsf{U}_-(\mathbf{v}_+) = \mathbf{g} + \mathbf{u}_\infty \text{ on } \Gamma, \\ \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathsf{U}_+(\mathbf{v}_+), \mathsf{R}_+(\mathbf{v}_+), \mathbf{f}_+ + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{v}_+)) - \mathbf{t}_{\mathsf{D}_-}(\mathsf{U}_-(\mathbf{v}_+), \mathsf{R}_-(\mathbf{v}_+), \mathbf{f}_-) \\ \quad + \mathfrak{L}\text{Tr}_{\mathsf{D}_+}\mathsf{U}_+(\mathbf{v}_+) = \mathbf{h} \text{ on } \Gamma. \end{cases} \tag{3.2.13}$$

We underline the fact that we need to show that our operator U_+ has a fixed point $\mathbf{v}_+ \in H^1_{\text{div}}(\mathsf{D}_+)^3$. Indeed, the fixed point of the operator U_+ together with the fields $\mathbf{v}_- = \mathsf{U}_-(\mathbf{v}_+)$ and $\mathsf{R}_\pm = \mathsf{R}_\pm(\mathbf{v}_+)$ will provide a solution of our nonlinear problem (3.2.8).

Our goal is to we show that U_+ has a fixed point, which it to say, that U_+ maps a closed ball $\mathbb{B}_\eta$ in $H^1_{\text{div}}(\mathsf{D}_+)^3$ to the same closed ball $\mathbb{B}_\eta$ in $H^1_{\text{div}}(\mathsf{D}_+)^3$ and that U_+ is a contraction on the ball $\mathbb{B}_\eta$.

We define the following constants

$$\xi := \frac{3}{16c_0 d_*^2} > 0, \quad \eta := \frac{1}{4c_0 d_*} > 0 \tag{3.2.14}$$

and we introduce the closed ball

$$\mathbb{B}_\eta := \{\mathbf{v}_+ \in H^1_{\text{div}}(\mathsf{D}_+)^3 : ||\mathbf{v}_+||_{H^1(\mathsf{D}_+)^3} \le \eta\}. \tag{3.2.15}$$

Now, we assume that the given data satisfies

$$||(\mathbf{f}_+|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty)||_{\mathsf{Y}_w \times \mathbb{R}^3} \le \xi. \tag{3.2.16}$$

Relations (3.2.12), (3.2.14), (3.2.15), (3.2.16) imply the fact that

$$||(\mathsf{U}_+(\mathbf{v}_+), \mathsf{R}_+(\mathbf{v}_+), \mathsf{U}_-(\mathbf{v}_+), \mathsf{R}_-(\mathbf{v}_+))||_{\mathsf{X}_w} \le \frac{1}{4c_0 d_*} = \eta, \tag{3.2.17}$$

for all $\mathbf{u}_+ \in \mathbb{B}_\eta$.

Due to relation (3.2.17), we have that $||\mathsf{U}_+(\mathbf{v}_+)||_{H^1(\mathsf{D}_+)^3} \le \eta$ for all $\mathbf{v}_+ \in \mathbb{B}_\eta$, that is, U_+ maps the ball $\mathbb{B}_\eta$ to the same ball $\mathbb{B}_\eta$ of the space $H^1_{\text{div}}(\mathsf{D}_+)^3$.

Next, we shall prove that U_+ is a contraction on our closed ball $\mathbb{B}_\eta$. In order to achieve this goal, let us fix the data $(\mathbf{f}_+|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty)$, let us consider two arbitrary functions $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{B}_\eta$ and by relation (3.2.10), we get

$$\begin{aligned} ||\mathsf{U}_+(\mathbf{w}_1) - \mathsf{U}_+(\mathbf{w}_2)||_{H^1(\mathsf{D}_+)^3} &\le d_*||\mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{w}_1) - \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{w}_2)||_{\tilde{H}^{-1}(\mathsf{D}_+)^3} \\ &\le d_* c_0 (||\mathbf{w}_1||_{H^1(\mathsf{D}_+)^3} + ||\mathbf{w}_2||_{H^1(\mathsf{D}_+)^3})||\mathbf{w}_1 - \mathbf{w}_2||_{H^1(\mathsf{D}_+)^3} \\ &\le 2d_* c_0 ||\mathbf{w}_1 - \mathbf{w}_2||_{H^1(\mathsf{D}_+)^3} = \frac{1}{2}||\mathbf{w}_1 - \mathbf{w}_2||_{H^1(\mathsf{D}_+)^3}, \end{aligned} \tag{3.2.18}$$

for all $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{B}_\eta$. Note that, the first inequality in relation (3.2.18) holds due to the continuity of the solution operator $\mathsf{T} : \mathsf{Y}_w \times \mathbb{R}^3 \to \mathsf{X}_w$ defined by Theorem 2.2.3 and described in relation (3.2.10). The second inequality of relation (3.2.18) holds due to inequality (3.1.5) of Lemma 3.1.3, while the constants d_*, c_0 are the same constants as in relation (3.2.12). Consequently, it follows that $\mathsf{U}_+ : \mathbb{B}_\eta \to \mathbb{B}_\eta$ is a $\frac{1}{2}$-contraction.

By applying the Banach Fixed Point Theorem, we obtain the existence of a unique fixed point $\mathbf{v}_+ \in \mathbb{B}_\eta$ of the operator U_+. This fixed point $\mathbf{v}_+$, together with the fields $\mathbf{v}_- := \mathsf{U}_-(\mathbf{v}_+)$ and $\pi_\pm := \mathsf{R}_\pm(\mathbf{v}_+)$ given by (3.2.10), provide a solution of the nonlinear problem (3.2.8) in the space X_w. Furthermore, $\mathbf{v}_-$ vanishes at infinity in the sense of Leray, since $\mathbf{v}_- \in \mathcal{H}^1_{\rm div}(\mathsf{D}_-)^3$.

The original fields $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)$ provide us with a solution of our transmission problem (3.2.3) satisfying $(\mathbf{u}_+, \pi_+, \mathbf{u}_- - \mathbf{u}_\infty, \pi_-) \in \mathsf{X}_w$. Relation (3.2.17) together with the expressions of $\mathbf{v}_-$ and $\mathsf{R}_\pm$ in terms of $\mathbf{v}_+$ imply that the estimate (3.2.6) is satisfied by our constructed solution. Furthermore, the quantity $\mathbf{u}_- - \mathbf{u}_\infty$ vanishes at infinity in the sense of Leray.

Also, due to the fact that $\mathbf{v}_+ \in \mathbb{B}_\eta$, we are able to deduce that

$$d_* c_0 ||\mathbf{v}_+||_{H^1(\mathsf{D}_+)^3} \leq \frac{1}{4}$$

and by using relation (3.2.12), we obtain

$$\begin{aligned} &||\mathbf{v}_+||_{H^1(\mathsf{D}_+)^3} + ||\pi_+||_{L^2(\mathsf{D}_+)} + ||\mathbf{v}_-||_{\tilde{H}^1(\mathsf{D}_-)^3} + ||\pi_-||_{L^2(\mathsf{D}_-)} \\ &\quad \leq d_* ||(\mathbf{f}_+|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty)||_{\mathsf{Y}_w \times \mathbb{R}^3} + \frac{1}{4}||\mathbf{v}_+||_{H^1(\mathsf{D}_+)^3}, \end{aligned} \tag{3.2.19}$$

which is equivalent to (cf. [71, Theorem 5.2]),

$$||\mathbf{v}_+||_{H^1(\mathsf{D}_+)^3} \leq \frac{4}{3} d_* ||(\mathbf{f}_+|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}, \mathbf{h}, \mathbf{u}_\infty)||_{\mathsf{Y}_w \times \mathbb{R}^3}. \tag{3.2.20}$$

Now, we substitute relation (3.2.20) into relation (3.2.19) and we obtain the desired estimate (3.2.7) with $C_0 = \frac{4}{3} d_*$. Thus, we conclude our argument for the existence part.

Step 2. Next, we will show that *our problem* (3.2.3) *has a unique solution.* To this end, we consider two solutions of the transmission problem (3.2.3) and we denote them by $(\mathbf{u}^1_+, \pi^1_+, \mathbf{u}^1_-, \pi^1_-)$ and $(\mathbf{u}^2_+, \pi^2_+, \mathbf{u}^2_-, \pi^2_-)$, respectively. Note that $(\mathbf{u}^1_+, \pi^1_+, \mathbf{u}^1_- - \mathbf{u}_\infty, \pi^1_-) \in \mathsf{X}_w$ and $(\mathbf{u}^2_+, \pi^2_+, \mathbf{u}^2_- - \mathbf{u}_\infty, \pi^2_-) \in \mathsf{X}_w$. Note that both solutions satisfy relation (3.2.6).

Now, we consider $(\mathbf{v}^2_+, \mathbf{v}^2_-) = (\mathbf{u}^2_+, \mathbf{u}^2_- - \mathbf{u}_\infty)$, which implies that $\mathbf{u}^2_+ \in \mathbb{B}_\eta$. Due to the fact that $\mathbf{v}^2_+ \in \mathbb{B}_\eta$, it must also follow that $\mathsf{U}_+(\mathbf{v}^2_+) \in \mathbb{B}_\eta$, where $(\mathsf{U}_+(\mathbf{v}^2_+), \mathsf{R}_+(\mathbf{v}^2_+), \mathsf{U}_-(\mathbf{v}^2_+), \mathsf{R}_-(\mathbf{v}^2_+))$ are provided by relation (3.2.10) and these fields satisfy the problem (3.2.13) with $\mathbf{v}_+$ substituted with $\mathbf{v}^2_+$.

Then, we consider the problem (3.2.13), as well as the problem (3.2.8) written in the variables $(\mathbf{v}^2_+, \pi^2_+, \mathbf{v}^2_- - \mathbf{u}_\infty, \pi^2_-)$ and by subtracting one from the other we gen another linear problem, which is

$$\begin{cases} \Delta(\mathsf{U}_+(\mathbf{v}^2_+) - \mathbf{v}^2_+) - \mathcal{P}(\mathsf{U}_+(\mathbf{v}^2_+) - \mathbf{v}^2_+) - \nabla(\mathsf{R}_+(\mathbf{v}^2_+) - \pi^2_+) = \mathbf{0} \text{ in } \mathsf{D}_+, \\ \Delta(\mathsf{U}_-(\mathbf{v}^2_+) - \mathbf{v}^2_-) - \nabla(\mathsf{R}_-(\mathbf{v}^2_+) - \pi^2_-) = \mathbf{0} \text{ in } \mathsf{D}_-, \\ \text{div } (\mathsf{U}_\pm(\mathbf{v}^2_\pm) - \mathbf{v}^2_\pm) = 0 \text{ in } \mathsf{D}_\pm, \\ \mathrm{Tr}_{\mathsf{D}_+}(\mathsf{U}_+(\mathbf{v}^2_+) - \mathbf{v}^2_+) - \mathrm{Tr}_{\mathsf{D}_-}(\mathsf{U}_-(\mathbf{v}^2_+) - \mathbf{v}^2_-) = \mathbf{0} \text{ on } \Gamma, \\ \mathbf{t}_{\mathcal{P},\mathsf{D}_+}((\mathsf{U}_+(\mathbf{v}^2_+) - \mathbf{v}^2_+), (\mathsf{R}_+(\mathbf{v}^2_+) - \pi^2_+)) - \mathbf{t}_{\mathsf{D}_-}((\mathsf{U}_-(\mathbf{v}^2_+) - \mathbf{v}^2_-), (\mathsf{R}_-(\mathbf{v}^2_+) - \pi^2_-)) \\ \quad + \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}(\mathsf{U}_+(\mathbf{v}^2_+) - \mathbf{v}^2_+) = \mathbf{0} \text{ on } \Gamma. \end{cases} \tag{3.2.21}$$

We apply Theorem 2.2.3 in order to conclude that our problem (3.2.21) admits only the trivial solution in X_w. Consequently, $\mathsf{U}_+(\mathbf{v}_+^2) - \mathbf{v}_+^2 = 0$, that is, $\mathbf{v}_+^2$ is a fixed point of the operator U_+. Let us recall that we have proved that $\mathsf{U}_+ : \mathbb{B}_\eta \to \mathbb{B}_\eta$ is a contraction (see relation (3.2.18)), hence, there must be only one unique fixed point $\mathbf{v}_+^1$ in $\mathbb{B}_\eta$. We deduce that $\mathbf{v}_+^2 = \mathbf{v}_+^1$, $\mathbf{v}_-^2 = \mathbf{v}_-^1$ and also $\pi_\pm^2 = \pi_\pm^1$. We have proved that our problem (3.2.3) has at most one solution.

Step 3. We want to show that *the solution of our problem* (3.2.3) *depends continuously on the given data*. Consequently, let us note that, the continuity of the operator U_+ with respect to the given data together with the continuity of the operator $\mathsf{T} : \mathsf{Y}_w \times \mathbb{R}^3 \to \mathsf{X}_w$ (see relation (2.2.28) and Theorem 2.2.3), implies that the solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_w$ depends continuously on the given data and hence, the estimate (3.2.19) is satisfied with the constant $C_0 = \frac{4}{3}d_*$.

This concludes the proof of our result. □

We end this section by stating some important remarks which show the particular situations that are also treated by Theorem 3.2.1.

Remark 3.2.2. *In the case $k = 0$ and $\beta : \mathsf{D}_+ \to \mathbb{R}_+$ such that $\beta \in L^\infty(\mathsf{D}_+, \mathbb{R}_+)$, Theorem 3.2.1 gives a well-posedness result for the nonlinear transmission problem for the generalized Navier-Stokes and Stokes systems.*

Remark 3.2.3. *In the case $k : \mathsf{D}_+ \to \mathbb{R}_+$ such that $k \in L^\infty(\mathsf{D}_+, \mathbb{R}_+)$ and $\beta = 0$, Theorem 3.2.1 gives a well-posedness result for a semilinear transmission problem for a semilinear Darcy-Forchheimer-Brinkman system and Stokes system.*

3.3 Transmission problem for the generalized Darcy-Forchheimer-Brinkman and classical Brinkman systems in complementary Lipschitz domains in $\mathbb{R}^3$

In this section, our goal is to provide a well-posedness result, for a transmission-type problem, which was obtained in the setting of Assumption 1.1.6 for $n = 3$, i.e., complementary Lipschitz domains in $\mathbb{R}^3$. We have considered a generalized version of the Darcy-Forchheimer-Brinkman system in the bounded Lipschitz domain D_+ and the Brinkman system in the complementary Lipschitz set D_-. Also, let Assumption 2.2.1 be satisfied, for $n = 3$.

Let us recall the space

$$\mathsf{X}_{\mathcal{B}} := H^1_{\mathrm{div}}(\mathsf{D}_+)^3 \times L^2(\mathsf{D}_+) \times H^1_{\mathrm{div}}(\mathsf{D}_-)^3 \times \mathfrak{M}(\mathsf{D}_-) \tag{3.3.1}$$

that is, the space in which we seek our solution and

$$\mathsf{Y}_{\mathcal{B}} := \tilde{H}^{-1}(\mathsf{D}_+)^3 \times \tilde{H}^{-1}(\mathsf{D}_-)^3 \times H^{\frac{1}{2}}(\Gamma)^3 \times H^{-\frac{1}{2}}(\Gamma)^3, \tag{3.3.2}$$

the space of given data. Note that $\mathfrak{M}(\mathsf{D}_-)$ is the space provided by Definition 2.1.1.

Since we are dealing with the Brinkman system in the exterior Lipschitz domain D_- (see Assumption 1.1.6 in the case $n = 3$), it follows that we are able to use the classical Sobolev space $H^1_{\mathrm{div}}(\mathsf{D}_-)^3$, instead of the weighted Sobolev space $\mathcal{H}^1_{\mathrm{div}}(\mathsf{D}_-)^3$, as the space in which we seek the velocity field in D_-. This is due to the behavior of the fundamental solution of the Brinkman system at infinity, in the case $n = 3$.

Now, we consider the transmission problem for the generalized Darcy-Forchheimer-Brinkman and classical Brinkman systems, which is given by

$$\begin{cases} \Delta \mathbf{u}_+ - \mathcal{P}\mathbf{u}_+ - k|\mathbf{u}_+|\mathbf{u}_+ - \beta(\mathbf{u}_+\cdot\nabla)\mathbf{u}_+ - \nabla\pi_+ = \mathbf{f}_+|_{\mathsf{D}_+} \text{ in } \mathsf{D}_+,\\ \Delta \mathbf{u}_- - \alpha\mathbf{u}_- - \nabla\pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-,\\ \text{div }\mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm,\\ \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ - \mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_- = \mathbf{g} \text{ on } \Gamma,\\ \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{u}_+,\pi_+,\mathbf{f}_+ + \mathring{\mathsf{E}}_+(k|\mathbf{u}_+|\mathbf{u}_+ + \beta(\mathbf{u}_+\cdot\nabla)\mathbf{u}_+)) - \mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-,\pi_-,\mathbf{f}_-)\\ \quad + \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ = \mathbf{h} \text{ on } \Gamma, \end{cases} \tag{3.3.3}$$

in the unknown fields $(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-)\in\mathsf{X}_{\mathcal{B}}$.

The well-posedness result that we have obtained is as follows (see e.g., [5, Theorem 3.2], and [71, Theorem 5.2] in the case $\mathcal{P}=\alpha\mathbb{I}$, where $\alpha,k,\beta>0$ are constants).

Theorem 3.3.1. *Let $\alpha>0$ be a given constant. Let Assumption 1.1.6 and Assumption 2.2.1 be satisfied, for $n=3$. Let $\mathcal{P}\in L^\infty(\mathsf{D}_+)^{3\times3}$ such that condition (1.2.22) holds. Then, there exist two constants,*

$$\xi=\xi(\mathsf{D}_+,\mathsf{D}_-,\mathcal{P},k,\beta,\mathfrak{L})>0 \quad \eta=\eta(\mathsf{D}_+,\mathsf{D}_-,\mathcal{P},k,\beta,\mathfrak{L})>0 \tag{3.3.4}$$

such that, for all given data $(\mathbf{f}_+,\mathbf{f}_-,\mathbf{g},\mathbf{h})\in\mathsf{Y}_{\mathcal{B}}$ that satisfy the condition

$$||(\mathbf{f}_+,\mathbf{f}_-,\mathbf{g},\mathbf{h})||_{\mathsf{Y}_{\mathcal{B}}}\le\xi, \tag{3.3.5}$$

the Poisson problem of transmission-type for the generalized Darcy-Forchheimer-Brinkman and Stokes systems (3.3.3) has a unique solution $(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-)\in\mathsf{X}_{\mathcal{B}}$ such that

$$||\mathbf{u}_+||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^3}\le\eta. \tag{3.3.6}$$

In addition, the solution depends continuously on the given data, which means that there exists a given constant $C_0=C_0(\mathsf{D}_+,\mathsf{D}_-,\mathcal{P},\mathfrak{L})>0$ such that

$$||(\mathbf{u}_+,\pi_+,\mathbf{u}_-,\pi_-)||_{\mathsf{X}_{\mathcal{B}}}\le C_0||(\mathbf{f}_+,\mathbf{f}_-,\mathbf{g},\mathbf{h})||_{\mathsf{Y}_{\mathcal{B}}}. \tag{3.3.7}$$

Proof. The proof of this result is similar to that of Theorem 5.2 in [71]. Consequently, we will employ similar arguments.

Step 1. We show that *a solution of the transmission problem* (3.3.3) *exists.* To this end, note that our problem (3.3.3) can be rewritten as

$$\begin{cases} \Delta \mathbf{u}_+ - \mathcal{P}\mathbf{u}_+ - \nabla\pi_+ = \mathbf{f}_+|_{\mathsf{D}_+} + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{u}_+) \text{ in } \mathsf{D}_+,\\ \Delta \mathbf{u}_- - \alpha\mathbf{u}_- - \nabla\pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-,\\ \text{div }\mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm,\\ \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ - \mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_- = \mathbf{g} \text{ on } \Gamma,\\ \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{u}_+,\pi_+,\mathbf{f}_+ + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{u}_+)) - \mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-,\pi_-,\mathbf{f}_-) + \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+ = \mathbf{h} \text{ on } \Gamma, \end{cases} \tag{3.3.8}$$

where $\mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{u}_+)\in\tilde{H}^{-1}(\mathsf{D}_+)^3$ is given by Lemma 3.1.3.

Let us now fix $\mathbf{u}_+\in H^1_{\mathrm{div}}(\mathsf{D}_+)^3$ and we write another linear transmission problem for the generalized and classical Brinkman systems in the unknowns $(\mathbf{u}^0_+,\pi^0_+,\mathbf{u}^0_-,\pi^0_-)$ as follows

$$\begin{cases} \Delta \mathbf{u}^0_+ - \mathcal{P}\mathbf{u}^0_+ - \nabla\pi^0_+ = \mathbf{f}_+|_{\mathsf{D}_+} + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{u}_+) \text{ in } \mathsf{D}_+,\\ \Delta \mathbf{u}^0_- - \alpha\mathbf{u}^0_- - \nabla\pi^0_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-,\\ \text{div }\mathbf{u}^0_\pm = 0 \text{ in } \mathsf{D}_\pm,\\ \mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}^0_+ - \mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}^0_- = \mathbf{g} \text{ on } \Gamma,\\ \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathbf{u}^0_+,\pi^0_+,\mathbf{f}_+ + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{u}_+)) - \mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}^0_-,\pi^0_-,\mathbf{f}_-) + \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}^0_+ = \mathbf{h} \text{ on } \Gamma. \end{cases} \tag{3.3.9}$$

Theorem 2.3.1 states that problem (3.3.9) has a unique solution $(\mathbf{u}_+^0, \pi_+^0, \mathbf{u}_-^0, \pi_-^0)$ in the space $\mathsf{X}_{\mathcal{B}}$, which is given by

$$\begin{aligned}(\mathbf{u}_+^0, \pi_+^0, \mathbf{u}_-^0, \pi_-^0) &= (\mathsf{U}_+(\mathbf{u}_+), \mathsf{R}_+(\mathbf{u}_+), \mathsf{U}_-(\mathbf{u}_+), \mathsf{R}_-(\mathbf{u}_+)) \\ &:= \mathsf{T}_{\mathcal{B}}(\mathbf{f}_+|_{\mathsf{D}_+} + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{u}_+)|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}, \mathbf{h}) \in \mathsf{X}_{\mathcal{B}}.\end{aligned} \tag{3.3.10}$$

Let us note that, the operator $\mathsf{T}_{\mathcal{B}} : \mathsf{Y}_{\mathcal{B}} \to \mathsf{X}_{\mathcal{B}}$ which is involved in relation (3.3.10) is the solution operator provided relation (2.3.5). Namely, $\mathsf{T}_{\mathcal{B}} : \mathsf{Y}_{\mathcal{B}} \to \mathsf{X}_{\mathcal{B}}$ is the well-defined, linear and continuous operator, which maps the given data (belonging to the space $\mathsf{Y}_{\mathcal{B}}$) to the unique solution of the Poisson transmission problem (2.3.3) for the generalized Brinkman and classical Brinkman systems in complementary Lipschitz domains in $\mathbb{R}^3$. Also, $\mathsf{T}_{\mathcal{B}} : \mathsf{Y}_{\mathcal{B}} \to \mathsf{X}_{\mathcal{B}}$ satisfies the estimate (2.3.6) of Theorem 2.3.1.

If we fix our given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}) \in \mathsf{Y}_{\mathcal{B}}$, we are able to introduce the following nonlinear operators

$$\mathsf{U}_\pm, \mathsf{R}_\pm : H^1_{\mathrm{div}}(\mathsf{D}_+)^3 \to \mathsf{X}_{\mathcal{B}}, \tag{3.3.11}$$

which are bounded, in the sense that there is a constant $d \equiv d(\mathsf{D}_+, \mathsf{D}_-, \mathcal{P}, \mathfrak{L}) > 0$, such that the following relation

$$\begin{aligned}||(\mathsf{U}_+(\mathbf{u}_+),&\mathsf{R}_+(\mathbf{u}_+), \mathsf{U}_-(\mathbf{u}_+), \mathsf{R}_-(\mathbf{u}_+))||_{\mathsf{X}_{\mathcal{B}}} \\ &\le d||(\mathbf{f}_+|_{\mathsf{D}_+} + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{u}_+)|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}, \mathbf{h})||_{\mathsf{Y}_{\mathcal{B}}} \\ &\le d||(\mathbf{f}_+|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}, \mathbf{h})||_{\mathsf{Y}_{\mathcal{B}}} + dc_0||\mathbf{u}_+||^2_{H^1(\mathsf{D}_+)^3},\end{aligned} \tag{3.3.12}$$

holds for all fields $\mathbf{u}_+ \in H^1_{\mathrm{div}}(\mathsf{D}_+)^3$. Indeed, this inequality is a consequence of Lemma 3.1.3 and $c_0 > 0$ is the constant that appears in Lemma 3.1.3.

The operators $\mathsf{U}_\pm, \mathsf{R}_\pm$ allow us to to write the problem (3.3.9) in the following equivalent form

$$\begin{cases}\Delta\mathsf{U}_+(\mathbf{u}_+) - \mathcal{P}\mathsf{U}_+(\mathbf{u}_+) - \nabla\mathsf{R}_+(\mathbf{u}_+) = \mathbf{f}_+|_{\mathsf{D}_+} + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{u}_+) \text{ in } \mathsf{D}_+, \\ \Delta\mathsf{U}_-(\mathbf{u}_+) - \alpha\mathsf{U}_-(\mathbf{u}_+) - \nabla\mathsf{R}_+(\mathbf{u}_+) = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \mathrm{div}\ \mathsf{U}_\pm(\mathbf{u}_+) = 0 \text{ in } \mathsf{D}_\pm, \\ \mathrm{Tr}_{\mathsf{D}_+}\mathsf{U}_+(\mathbf{u}_+) - \mathrm{Tr}_{\mathsf{D}_-}\mathsf{U}_-(\mathbf{u}_+) = \mathbf{g} \text{ on } \Gamma, \\ \mathbf{t}_{\mathcal{P},\mathsf{D}_+}(\mathsf{U}_+(\mathbf{u}_+), \mathsf{R}_+(\mathbf{u}_+), \mathbf{f}_+ + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{u}_+)) - \mathbf{t}_{\alpha,\mathsf{D}_-}(\mathsf{U}_-(\mathbf{u}_+), \mathsf{R}_-(\mathbf{u}_+), \mathbf{f}_-) \\ \quad + \mathfrak{L}\mathrm{Tr}_{\mathsf{D}_+}\mathsf{U}_+(\mathbf{u}_+) = \mathbf{h} \text{ on } \Gamma.\end{cases} \tag{3.3.13}$$

Next, we aim to show that our nonlinear operator U_+ has a unique fixed point. Indeed, if we show that U_+ has a unique fixed point, then, that fixed point $\mathbf{u}_+ \in H^1_{\mathrm{div}}(\mathsf{D}_+)^3$, together with the fields $\mathbf{u}_- = \mathsf{U}_-(\mathbf{u}_+)$ and with $p_\pm = \mathsf{R}_\pm(\mathbf{u}_+)$ will provide us with a solution for our nonlinear problem (3.3.8).

Let us now show that U_+ maps a closed ball $\mathbb{B}_\eta$ to the same closed ball $\mathbb{B}_\eta$ of the space $H^1_{\mathrm{div}}(\mathsf{D}_+)^3$ and that U_+ is a contraction on $\mathbb{B}_\eta$.

We begin our construction by choosing some constants, namely

$$\xi := \frac{3}{16c_0d^2} > 0, \quad \eta := \frac{1}{4c_0d} > 0, \tag{3.3.14}$$

and we define the closed ball $\mathbb{B}_\eta$ by

$$\mathbb{B}_\eta := \{\mathbf{v}_+ \in H^1_{\mathrm{div}}(\mathsf{D}_+)^3 : ||\mathbf{v}_+||_{H^1(\mathsf{D}_+)^3} \le \eta\}. \tag{3.3.15}$$

and, also, suppose that the given data satisfies

$$||(\mathbf{f}_+|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}, \mathbf{h})||_{\mathsf{Y}_{\mathcal{B}}} \leq \xi. \tag{3.3.16}$$

Now, relations (3.3.12), (3.3.14), (3.3.15), (3.3.16) imply that

$$||(\mathsf{U}_+(\mathbf{u}_+), \mathsf{R}_+(\mathbf{u}_+), \mathsf{U}_-(\mathbf{u}_+), \mathsf{R}_-(\mathbf{u}_+))||_{\mathsf{X}_{\mathcal{B}}} \leq \eta, \tag{3.3.17}$$

for all $\mathbf{u}_+ \in \mathbb{B}_\eta$ and hence $||\mathsf{U}_+(\mathbf{u}_+)||_{H^1(\mathsf{D}_+)^3} \leq \eta$ for all $\mathbf{u}_+ \in \mathbb{B}_\eta$, that is, U_+ maps the ball $\mathbb{B}_\eta$ to itself.

We focus on showing that U_+ is Lipschitz continuous on $\mathbb{B}_\eta$. To achieve this, we fix the given data $(\mathbf{f}_+|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}, \mathbf{h}) \in \mathsf{Y}_{\mathcal{B}}$ and let two arbitrary functions $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{B}_\eta$. We obtain

$$\begin{aligned} ||\mathsf{U}_+(\mathbf{w}_1) - \mathsf{U}_+(\mathbf{w}_2)||_{H^1(\mathsf{D}_+)^3} &\leq d||\mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{w}_1) - \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{w}_2)||_{\tilde{H}^{-1}(\mathsf{D}_+)^3} \\ &\leq dc_0(||\mathbf{w}_1||_{H^1(\mathsf{D}_+)^3} + ||\mathbf{w}_2||_{H^1(\mathsf{D}_+)^3})||\mathbf{w}_1 - \mathbf{w}_2||_{H^1(\mathsf{D}_+)^3} \\ &\leq 2dc_0||\mathbf{w}_1 - \mathbf{w}_2||_{H^1(\mathsf{D}_+)^3} = \frac{1}{2}||\mathbf{w}_1 - \mathbf{w}_2||_{H^1(\mathsf{D}_+)^3}, \end{aligned} \tag{3.3.18}$$

for all $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{B}_\eta$, where we have taken into account the continuity of the operator $\mathsf{T}_{\mathcal{B}} : \mathsf{Y}_{\mathcal{B}} \to \mathsf{X}_{\mathcal{B}}$, which is introduced in relation (3.3.10), and inequality (3.1.5), while the constants d and c_0 are the same constants as in relation (3.3.12). Based on the arguments presented in relation (3.3.18), we have that $\mathsf{U}_+ : \mathbb{B}_\eta \to \mathbb{B}_\eta$ is a $\frac{1}{2}$-contraction.

Banach's fixed point theorem implies that there is a unique fixed point $\mathbf{u}_+ \in \mathbb{B}_\eta$ of the operator U_+, which, together with the fields given by $\mathbf{u}_- = \mathsf{U}_-(\mathbf{u}_+)$ and $\pi_\pm = \mathsf{R}_\pm(\mathbf{u}_+)$, produces a solution of our transmission problem (3.3.8).

Now, since $\mathbf{u}_+ \in \mathbb{B}_\eta$, we get

$$dc_0||\mathbf{u}_+||_{H^1(\mathsf{D}_+)^3} \leq \frac{1}{4},$$

and by using relation (3.3.12), we have that

$$\begin{aligned} &||\mathbf{u}_+||_{H^1(\mathsf{D}_+)^3} + ||\pi_+||_{L^2(\mathsf{D}_+)} + ||\mathbf{u}_-||_{H^1(\mathsf{D}_-)^3} + ||\pi_-||_{\mathfrak{M}(\mathsf{D}_-)} \\ &\qquad \leq d||(\mathbf{f}_+|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}, \mathbf{h})||_{\mathsf{Y}_{\mathcal{B}}} + \frac{1}{4}||\mathbf{u}_+||_{H^1(\mathsf{D}_+)^3}, \end{aligned} \tag{3.3.19}$$

and, consequently,

$$||\mathbf{u}_+||_{H^1(\mathsf{D}_+)^3} \leq \frac{4}{3}d||(\mathbf{f}_+|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}, \mathbf{h})||_{\mathsf{Y}_{\mathcal{B}}}. \tag{3.3.20}$$

If we substitute relation (3.3.20) into relation (3.3.19) we get the desired estimate (3.3.7) where $C_0 = \frac{4}{3}d$.

Step 2. We aim to show that *the solution of the problem* (3.3.3) *is unique*. Since the arguments are similar to those presented in the proof of Theorem 3.2.1, let us provide the main ideas that are used in the proof of this step. Let $(\mathbf{u}_+^1, \pi_+^1, \mathbf{u}_-^1, \pi_-^1)$ and $(\mathbf{u}_+^2, \pi_+^2, \mathbf{u}_-^2, \pi_-^2)$ be two solutions of the transmission problem (3.3.3). We note that these fields belong to the space $\mathsf{X}_{\mathcal{B}}$ and both satisfy relation (3.3.6).

Using the fields $(\mathbf{u}_+^2, \pi_+^2, \mathbf{u}_-^2, \pi_-^2)$, we get the linear and homogeneous transmission problem for the classical and generalized Brinkman systems in the setting of Assumption 1.1.6, $n = 3$, in the unknowns $(\mathsf{U}_+(\mathbf{u}_+^2) - \mathbf{u}_+^2, \mathsf{R}_+(\mathbf{u}_+^2) - \pi_+^2, \mathsf{U}_-(\mathbf{u}_+^2) - \mathbf{u}_-^2, \mathsf{R}_-(\mathbf{u}_+^2) - \pi_-^2)$. Theorem 2.3.1 guarantees that this particular problem admits only the trivial solution in $\mathsf{X}_{\mathcal{B}}$. Hence, $\mathsf{U}_+(\mathbf{u}_+^2) - \mathbf{u}_+^2 = 0$, that is, $\mathbf{u}_+^2$ is a fixed point of the nonlinear operator U_+. Recall that $\mathsf{U}_+ : \mathbb{B}_\eta \to \mathbb{B}_\eta$ is a $\frac{1}{2}$-contraction, and hence, there is a unique fixed point $\mathbf{u}_+^1$ in $\mathbb{B}_\eta$. Consequently, $\mathbf{u}_+^2 = \mathbf{u}_+^1$, $\mathbf{u}_-^2 = \mathbf{u}_-^1$ and $\pi_\pm^2 = \pi_\pm^1$. The uniqueness is thus, established.

Step 3. It remains to show that *the solution of our problem* (3.3.3) *depends continuously on the given data.* We have that the continuity of the operator U_+ with respect to the given data and the continuity of the operator $\mathsf{T}_{\mathcal{B}} : \mathsf{Y}_{\mathcal{B}} \to \mathsf{X}_{\mathcal{B}}$ (see relation (2.3.5) and Theorem 2.3.1) imply that the solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_w$ depends continuously on the given data and hence, the estimate (3.3.7) is satisfied with the constant $C_0 = \frac{4}{3} d_*$. This concludes the proof of our result. □

We end this section by stating some useful remarks that are derived from our well-posedness result, that is, Theorem 3.3.1.

Remark 3.3.2. *If* $k = 0$ *and* $\beta : \mathsf{D}_+ \to \mathbb{R}_+$ *such that* $\beta \in L^\infty(\mathsf{D}_+, \mathbb{R}_+)$ *in Theorem* 3.3.1, *then we get the well-posedness result for the nonlinear transmission problem for the generalized Navier-Stokes and Brinkman systems in complementary Lipschitz domains in* $\mathbb{R}^3$.

Remark 3.3.3. *If* $k : \mathsf{D}_+ \to \mathbb{R}_+$ *such that* $k \in L^\infty(\mathsf{D}_+, \mathbb{R}_+)$ *and* $\beta = 0$ *in Theorem* 3.3.1, *then we get the well-posedness result for a semilinear transmission problem for a semilinear Darcy-Forchheimer-Brinkman system and the Brinkman system in complementary Lipschitz domains in* $\mathbb{R}^3$.

3.4 On a Robin-Transmission problem for the Darcy-Forchheimer-Brinkman system

In this section, we give an existence and uniqueness result for a transmission-type problem, which was obtained in the setting of Assumption 1.1.7. This particular transmission-type problem that we study will be called the Robin-transmission problem for the Darcy-Forchheimer-Brinkman system (see problem (3.4.3)). In addition, let $\lambda \in (0, 1]$ be a constant and let Assumption 2.2.1 be satisfied, for $n = 2, 3$.

Let us recall the space in which we seek our solution,

$$\mathsf{X}_{RT} := H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times L^2(\mathsf{D}_+) \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n \times L^2(\mathsf{D}_-), \tag{3.4.1}$$

and the space of given data,

$$\mathsf{Y}_{RT} := \tilde{H}^{-1}(\mathsf{D}_+)^n \times \tilde{H}^{-1}(\mathsf{D}_-)^n \times H^{\frac{1}{2}}_{\nu}(\Gamma_+)^n \times H^{-\frac{1}{2}}(\Gamma_+)^n \times H^{-\frac{1}{2}}(\Gamma_-)^n. \tag{3.4.2}$$

The Robin-transmission problem for the Darcy-Forchheimer-Brinkman system is given by

$$\begin{cases} \Delta \mathbf{u}_\pm - \alpha \mathbf{u}_\pm - k|\mathbf{u}_\pm|\mathbf{u}_\pm - \beta(\mathbf{u}_\pm \cdot \nabla)\mathbf{u}_\pm - \nabla \pi_\pm = \mathbf{f}_\pm|_{\mathsf{D}_\pm} \text{ in } \mathsf{D}_\pm, \\ \mathrm{div}\ \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \lambda \left(\mathrm{Tr}_{\mathsf{D}_+} \mathbf{u}_+\right) - \left(\mathrm{Tr}_{\mathsf{D}_-} \mathbf{u}_-\right)|_{\Gamma_+} = \mathbf{g}_1 \text{ on } \Gamma_+, \\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+ + \mathring{\mathsf{E}}_+(k|\mathbf{u}_+|\mathbf{u}_+ + \beta(\mathbf{u}_+ \cdot \nabla)\mathbf{u}_+)) \\ \quad - \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_- + \mathring{\mathsf{E}}_-(k|\mathbf{u}_-|\mathbf{u}_- + \beta(\mathbf{u}_- \cdot \nabla)\mathbf{u}_-))\right)|_{\Gamma_+} = \mathbf{h}_1 \text{ on } \Gamma_+, \\ \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_- + \mathring{\mathsf{E}}_-(k|\mathbf{u}_-|\mathbf{u}_- + \beta(\mathbf{u}_- \cdot \nabla)\mathbf{u}_-))\right)|_{\Gamma_-} \\ \quad + \mathfrak{L}\left(\mathrm{Tr}_{\mathsf{D}_-} \mathbf{u}_-\right)|_{\Gamma_-} = \mathbf{g}_2 \text{ on } \Gamma_-, \end{cases} \tag{3.4.3}$$

in the unknown fields $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{XT}$. Note that $\mathring{\mathsf{E}}_\pm$ is the extension by zero-operator outside $\overline{\mathsf{D}}_\pm$.

We have obtained the following well-posedness result (see also, [71, Theorem 5.2]).

Theorem 3.4.1. *Let $\alpha > 0$, $k, \beta \in \mathbb{R}^*$ and $\lambda \in (0,1]$ be given constants. Let Assumption* 1.1.7 *and Assumption 2.2.1 be satisfied, for $n = 2, 3$. Then, there exist two constants,*

$$\xi \equiv \xi(\mathsf{D}_+, \mathsf{D}_-, \alpha, k, \beta, \lambda, \mathfrak{L}) > 0, \quad \eta \equiv \eta(\mathsf{D}_+, \mathsf{D}_-, \alpha, k, \beta, \lambda, \mathfrak{L}) > 0, \tag{3.4.4}$$

such that, for every $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2) \in \mathsf{Y}_{RT}$, which satisfies the condition

$$||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}} \leq \xi, \tag{3.4.5}$$

the Poisson problem of Robin-transmission type (3.4.3) for the Darcy-Forchheimer-Brinkman system has a unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$ with the property

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_{RT}} \leq \eta. \tag{3.4.6}$$

Moreover, there exists a constant $C_0 \equiv C_0(\mathsf{D}_+, \mathsf{D}_-, \alpha, \mathfrak{L}, \lambda) > 0$ such that the unique solution satisfies

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_{RT}} \leq C_0||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{h}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}}. \tag{3.4.7}$$

Proof. We prove this result by employing similar arguments to those presented in the proof of [71, Theorem 5.2]. We divide our arguments into three steps.

Step 1. We will show that *a solution of the problem* (3.4.3) *exists.* We rewrite the nonlinear transmission problem (3.4.3) as

$$\begin{cases} \Delta \mathbf{u}_\pm - \alpha \mathbf{u}_\pm - \nabla \pi_\pm = \mathbf{f}_\pm|_{\mathsf{D}_\pm} + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_\pm)|_{\mathsf{D}_\pm} \text{ in } \mathsf{D}_\pm, \\ \text{div } \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \lambda\left(\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}_+\right) - \left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-\right)|_{\Gamma_+} = \mathbf{g}_1 \text{ on } \Gamma_+, \\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+ + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_+)) - \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_-))\right)|_{\Gamma_+} \\ \quad = \mathbf{h}_1 \text{ on } \Gamma_+, \\ \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_-))\right)|_{\Gamma_-} + \mathfrak{L}\left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-\right)|_{\Gamma_-} = \mathbf{g}_2 \text{ on } \Gamma_-. \end{cases} \tag{3.4.8}$$

Next, we aim to construct a nonlinear operator H that maps a closed ball B_η of the space $H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n$ into itself, and also is a contraction on B_η. Hence, the unique fixed point of H will provide a solution of the problem (3.4.8).

Let us construct our nonlinear operator in the following way. Recall that the given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2) \in \mathsf{Y}_{RT}$ which appears in (3.4.8) is fixed. In addition, we fix

$$(\mathbf{u}_+, \mathbf{u}_-) \in H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n. \tag{3.4.9}$$

Let us consider the following linear Poisson problem of transmission type for the Brinkman system in the unknowns $(\mathbf{u}^0_+, \pi^0_+, \mathbf{u}^0_-, \pi^0_-)$

$$\begin{cases} \Delta \mathbf{u}^0_\pm - \alpha \mathbf{u}^0_\pm - \nabla \pi^0_\pm = \mathbf{f}_\pm|_{\mathsf{D}_\pm} + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_\pm)|_{\mathsf{D}_\pm} \text{ in } \mathsf{D}_\pm, \\ \text{div } \mathbf{u}^0_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \lambda\left(\mathrm{Tr}_{\mathsf{D}_+}\mathbf{u}^0_+\right) - \left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}^0_-\right)|_{\Gamma_+} = \mathbf{g}_1 \text{ on } \Gamma_+, \\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}^0_+, \pi^0_+, \mathbf{f}_+ + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_+)) - \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}^0_-, \pi^0_-, \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_-))\right)|_{\Gamma_+} \\ \quad = \mathbf{h}_1 \text{ on } \Gamma_+, \\ \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}^0_-, \pi^0_-, \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_-))\right)|_{\Gamma_-} + \mathfrak{L}\left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}^0_-\right)|_{\Gamma_-} = \mathbf{g}_2 \text{ on } \Gamma_-. \end{cases} \tag{3.4.10}$$

In addition, the membership $\mathring{\mathsf{E}}(k|\mathbf{u}_\pm|\mathbf{u}_\pm + \beta(\mathbf{u}_\pm \cdot \nabla)\mathbf{u}_\pm) \in \tilde{H}^{-1}(\mathsf{D}_\pm)^n$ holds in view of Lemma 3.1.3.

Let us apply Theorem 2.4.1. Consequently we deduce that the transmission problem (3.4.10) has a unique solution

$$\begin{aligned}(\mathbf{u}_+^0, \pi_+^0, \mathbf{u}_-^0, \pi_-^0) &:= \mathsf{T}_{RT}(\mathbf{f}_+|_{\mathsf{D}_+} + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{u}_+)|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-} + \mathsf{J}_{k,\beta,\mathsf{D}_-}(\mathbf{u}_-)|_{\mathsf{D}_-}, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2) \in \mathsf{X}_{RT} \\ &= (\mathsf{U}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_-(\mathbf{u}_+, \mathbf{u}_-)).\end{aligned} \tag{3.4.11}$$

Let us note that, the operator $\mathsf{T}_{RT} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}$ which is involved in relation (3.4.11) is the solution operator given by relation (2.4.5). Let us recall that $\mathsf{T}_{RT} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}$ is the well-defined, linear and continuous operator, which maps the given data (belonging to the space Y_{RT}) to the unique solution of the Poisson problem of Robin-transmission type (2.4.3) for the Brinkman system in the setting of Assumption 1.1.7, for $n = 2, 3$. Also, $\mathsf{T}_{RT} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}$ satisfies the estimate (2.4.6) of Theorem 2.4.1.

Furthermore, by Lemma 3.1.3 and Theorem 2.4.1 and for $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2) \in \mathsf{Y}_{RT}$, the nonlinear operators given by relation (3.4.11),

$$(\mathsf{U}_+, \mathsf{R}_+, \mathsf{U}_-, \mathsf{R}_-) : H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n \to \mathsf{X}_{RT}, \tag{3.4.12}$$

are continuous and there exists a constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \alpha, \lambda, \mathfrak{L}) > 0$ such that

$$\begin{aligned}&||(\mathsf{U}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_-(\mathbf{u}_+, \mathbf{u}_-))||_{\mathsf{X}_{RT}} \\ &\le C||(\mathbf{f}_+|_{\mathsf{D}_+} + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{u}_+)|_{\mathsf{D}_+}, \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_-}(\mathbf{u}_-)|_{\mathsf{D}_-}, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}} \\ &\le C||(\mathbf{f}_+|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}} + ||\mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{u}_+)||_{\tilde{H}^{-1}(\mathsf{D}_+)^n} + ||\mathsf{J}_{k,\beta,\mathsf{D}_-}(\mathbf{u}_-)||_{\tilde{H}^{-1}(\mathsf{D}_-)^n} \\ &\le C||(\mathbf{f}_+|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}} + c_1^+ C||\mathbf{u}_+||^2_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n} + c_1^- C||\mathbf{u}_-||^2_{H^1_{\mathrm{div}}(\mathsf{D}_-)^n},\end{aligned} \tag{3.4.13}$$

for all $(\mathbf{u}_+, \mathbf{u}_-) \in H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n$, where c_1^+ and c_1^- are the constants provided by Lemma 3.1.3, corresponding to D_+ and D_-, respectively.

By taking into account (3.4.10), we have

$$\begin{cases}\Delta \mathsf{U}_\pm(\mathbf{u}_+, \mathbf{u}_-) - \alpha \mathsf{U}_\pm(\mathbf{u}_+, \mathbf{u}_-) - \nabla \mathsf{R}_\pm(\mathbf{u}_+, \mathbf{u}_-) \\ \quad = \mathbf{f}_\pm|_{\mathsf{D}_\pm} + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_\pm)|_{\mathsf{D}_\pm} \text{ in } \mathsf{D}_\pm, \\ \operatorname{div} \mathsf{U}_\pm(\mathbf{u}_+, \mathbf{u}_-) = 0 \text{ in } \mathsf{D}_\pm, \\ \lambda \left(\mathrm{Tr}_{\mathsf{D}_+} \mathsf{U}_+(\mathbf{u}_+, \mathbf{u}_-)\right) - \left(\mathrm{Tr}_{\mathsf{D}_-} \mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-)\right)|_{\Gamma_+} = \mathbf{g}_1 \text{ on } \Gamma_+, \\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathsf{U}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_+(\mathbf{u}_+, \mathbf{u}_-), \mathbf{f}_+ + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_+)) \\ \quad - \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_-(\mathbf{u}_+, \mathbf{u}_-), \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_-))\right)|_{\Gamma_+} \\ \quad = \mathbf{h}_1 \text{ on } \Gamma_+, \\ \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_-(\mathbf{u}_+, \mathbf{u}_-), \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_-))\right)|_{\Gamma_-} \\ \quad + \mathfrak{L}\left(\mathrm{Tr}_{\mathsf{D}_-} \mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-)\right)|_{\Gamma_-} = \mathbf{g}_2 \text{ on } \Gamma_-.\end{cases} \tag{3.4.14}$$

Let us introduce the nonlinear operator

$$\mathsf{H} : H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n \to H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n$$

by

$$\mathsf{H}(\mathbf{u}_+, \mathbf{u}_-) := (\mathsf{U}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-)). \tag{3.4.15}$$

Now, if we prove that the nonlinear operator H possesses a fixed point $(\mathbf{u}_+.\mathbf{u}_-) \in H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n$, this fixed point will solve the equation $\mathsf{H}(\mathbf{u}_+, \mathbf{u}_-) = (\mathbf{u}_+, \mathbf{u}_-)$ and together with $\pi_\pm = \mathsf{R}_\pm(\mathbf{u}_+, \mathbf{u}_-)$ provides a solution of the problem (3.4.8) in X_{RT}.

In order to justify our claim, we show that H maps a closed ball $\mathsf{B}_\eta \subseteq H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n$ to itself and also is a contraction on the ball B_η.

Let us introduce the constants

$$\xi := \frac{3}{16C^2 \max\{c_1^+, c_1^-\}} > 0, \quad \eta := \frac{1}{4C \max\{c_1^+, c_1^-\}} > 0, \tag{3.4.16}$$

and the closed ball

$$\mathsf{B}_\eta := \{(\mathbf{u}_+, \mathbf{u}_-) \in H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n : ||(\mathbf{u}_+, \mathbf{u}_-)||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n} \leq \eta\}, \tag{3.4.17}$$

while the constants c_1^+ and c_1^+ are the same constants that appear in relation (3.4.13). In addition, we assume that the given data satisfies

$$||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}} \leq \xi. \tag{3.4.18}$$

In view of relations (3.4.13), (3.4.16), (3.4.17), (3.4.18), we get

$$||(\mathsf{U}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-))||_{\mathsf{X}_{RT}} \leq \eta, \tag{3.4.19}$$

for all $(\mathbf{u}_+, \mathbf{u}_-) \in \mathsf{B}_\eta$, which shows that $||\mathsf{H}(\mathbf{u}_+, \mathbf{u}_-)||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n} \leq \eta$. Consequently H maps B_η to B_η.

Let us prove that H is a contraction on B_η. To achieve this, let us fix the given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2) \in \mathsf{Y}_{RT}$. If $(\mathbf{v}_+, \mathbf{v}_-), (\mathbf{w}_+, \mathbf{w}_-) \in \mathsf{B}_\eta$ are arbitrary fields, we obtain

$$\begin{aligned}
&||\mathsf{H}(\mathbf{v}_+, \mathbf{v}_-) - \mathsf{H}(\mathbf{w}_+, \mathbf{w}_-)||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n} \\
&\leq C||(\mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{v}_+) - \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{w}_+), \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{v}_-) - \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{w}_-))||_{\tilde{H}^{-1}(\mathsf{D}_+)^n \times \tilde{H}^{-1}(\mathsf{D}_-)^n} \\
&\leq Cc_1^+(||\mathbf{v}_+||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n} + ||\mathbf{w}_+||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n})||\mathbf{v}_+ - \mathbf{w}_+||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n} \\
&\qquad + Cc_1^-(||\mathbf{v}_-||_{H^1_{\mathrm{div}}(\mathsf{D}_-)^n} + ||\mathbf{w}_-||_{H^1_{\mathrm{div}}(\mathsf{D}_-)^n})||\mathbf{v}_- - \mathbf{w}_-||_{H^1_{\mathrm{div}}(\mathsf{D}_-)^n} \\
&\leq 2\eta C \max\{c_1^+, c_1^-\}||(\mathbf{v}_+ - \mathbf{w}_+, \mathbf{v}_- - \mathbf{w}_-)||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n} \\
&= \frac{1}{2}||(\mathbf{v}_+ - \mathbf{w}_+, \mathbf{v}_- - \mathbf{w}_-)||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n}.
\end{aligned} \tag{3.4.20}$$

In (3.4.20) we have used the linearity and continuity of the operator $\mathsf{T}_{RT} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}$ (see relation (2.4.5)) together with relation (3.1.5) of Lemma 3.1.3. Hence we have that the operator $\mathsf{H} : \mathsf{B}_\eta \to \mathsf{B}_\eta$ is a $\frac{1}{2}$-contraction.

Due to Banach's fixed point theorem we get the existence of a unique fixed point $(\mathbf{u}_+, \mathbf{u}_-) \in \mathsf{B}_\eta$ of the operator H, namely, $\mathsf{H}(\mathbf{u}_+, \mathbf{u}_-) = (\mathbf{u}_+, \mathbf{u}_-)$. The pair $(\mathbf{u}_+, \mathbf{u}_-)$ together with the functions $\pi_\pm = \mathsf{R}_\pm(\mathbf{u}_+, \mathbf{u}_-)$ given by (3.4.11), determine a solution of the nonlinear problem (3.4.8) in the space X_{RT}. Hence, $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)$ is a solution of the nonlinear transmission problem (3.4.3) in X_{RT}.

In view of the membership $(\mathbf{u}_+, \mathbf{u}_-) \in \mathsf{B}_\eta$, we get

$$Cc_1^+||\mathbf{u}_+||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n} \leq Cc_1^+\eta \leq \frac{1}{4}, \quad Cc_1^-||\mathbf{u}_-||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n} \leq Cc_1^-\eta \leq \frac{1}{4}. \tag{3.4.21}$$

Then, we apply inequality (3.4.13) to obtain

$$\begin{aligned}
&||\mathbf{u}_+||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n} + ||\pi_+||_{L^2(\mathsf{D}_+)} + ||\mathbf{u}_-||_{H^1_{\mathrm{div}}(\mathsf{D}_-)^n} + ||\pi_-||_{L^2(\mathsf{D}_-)} = ||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_{RT}} \\
&\leq C||(\mathbf{f}_+|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}} + \frac{1}{4}||\mathbf{u}_+||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n} + \frac{1}{4}||\mathbf{u}_-||_{H^1_{\mathrm{div}}(\mathsf{D}_-)^n},
\end{aligned} \tag{3.4.22}$$

hence

$$||\mathbf{u}_+||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n} + ||\mathbf{u}_-||_{H^1_{\mathrm{div}}(\mathsf{D}_-)^n} \leq \frac{4}{3}C||(\mathbf{f}_+|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}}. \tag{3.4.23}$$

By substituting relation (3.4.23) into relation (3.4.22), we get the desired estimate (3.4.7) with $C_0 = \frac{4}{3}C$.

Step 2. We want to show the *uniqueness property of the solution of the nonlinear transmssion problem* (3.4.3). The Banach fixed point theorem implies the uniqueness property of the solution of problem (3.4.3) inside the ball B_η. Since the arguments that are involved in the proof of this step are similar to those in the proof of Theorem 3.2.1, we omit them for the sake of brevity.

Step 3. It remains to show that *the solution of our problem* (3.4.3) *depends continuously on the given data.* To this end, the continuity of the nonlinear operator $\mathsf{H} : \mathsf{B}_\eta \to \mathsf{B}_\eta$ and the continuity of the solution operator $\mathsf{T}_{RT} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}$ (see relation (2.4.5)) show that the unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$ depends continuously on the given data and the estimate (3.4.7) holds with the choice of constant $C_0 = \frac{4}{3}C$. This concludes our proof. □

3.4.1 The Darcy-Forchheimer-Brinkman system and a related Limiting Robin-Transmission Problem in the case $\lambda = 0$

In this subsection, we will work in the setting of Assumption 1.1.7. We wish to discuss a special Robin-transmission problem of the Darcy-Forchheimer-Brinkman system. This new transmission-type problem is obtained by choosing $\lambda = 0$ in the transmission problem (3.4.3). Consequently, we get the problem (3.4.24) which includes a particular transmission condition on the boundary Γ_+, that is, it contains just a trace of the unknown velocity $\mathbf{u}_-$ on Γ_+. Due to this fact, problem (3.4.24) will be called the limiting Robin-transmission problem for the Darcy-Forchheimer-Brinkman system. Note that, this limiting Robin-transmission problem contains a Robin-Dirichlet boundary value problem for the Darcy-Forchheimer-Brinkman system in D_-. Our purpose is to state the well-posedness of the limiting Robin-transmission problem for the Darcy-Forchheimer-Brinkman system and, as a consequence, obtain a well-posedness result for the Robin-Dirichlet problem for the Darcy-Forchheimer-Brinkman system. Equivalently, we isolate the solution of the Robin-Dirichlet problem from the solution of the limiting Robin-transmission problem. This original method emphasizes the fact that the solutions of certain boundary value problems can be determined by considering, first of all, particular transmission problems.

Let us consider $\lambda = 0$ in the Robin-transmission problem for the Darcy-Forchheimer-Brinkman system (3.4.3). We get the following limiting Robin-transmission problem for the Darcy-Forchheimer-Brinkman system,

$$\begin{cases} \Delta\mathbf{u}_\pm - \alpha\mathbf{u}_\pm - k|\mathbf{u}_\pm|\mathbf{u}_\pm - \beta(\mathbf{u}_\pm\cdot\nabla)\mathbf{u}_\pm - \nabla\pi_\pm = \mathbf{f}_\pm|_{\mathsf{D}_\pm} \text{ in } \mathsf{D}_\pm, \\ \text{div } \mathbf{u}_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-\right)|_{\Gamma_+} = -\mathbf{g}_1 \text{ on } \Gamma_+, \\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+ + \mathring{\mathsf{E}}_+(k|\mathbf{u}_+|\mathbf{u}_+ + \beta(\mathbf{u}_+\cdot\nabla)\mathbf{u}_+)) \\ \quad - \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_- + \mathring{\mathsf{E}}_-(k|\mathbf{u}_-|\mathbf{u}_- + \beta(\mathbf{u}_-\cdot\nabla)\mathbf{u}_-))\right)|_{\Gamma_+} \\ \quad = \mathbf{h}_1 \text{ on } \Gamma_+, \\ \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_- + \mathring{\mathsf{E}}_-(k|\mathbf{u}_-|\mathbf{u}_- + \beta(\mathbf{u}_-\cdot\nabla)\mathbf{u}_-))\right)|_{\Gamma_-} \\ \quad + \mathfrak{L}\left(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-\right)|_{\Gamma_-} = \mathbf{g}_2 \text{ on } \Gamma_-, \end{cases} \tag{3.4.24}$$

in the unknown fields $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{XT}$. Note that, $\mathring{\mathsf{E}}_{\pm}$ is (the extension by zero)-operator outside $\overline{\mathsf{D}}_{\pm}$. In addition, let Assumption 2.2.1 be fulfilled, for $n = 2, 3$.

Before we state the well-posedness result that we have obtained for the limiting transmission problem (3.4.24), let us mention the fact that the proof of this result, namely Theorem 3.4.2 follows very closely the arguments that are presented in the proofs of Theorem 3.2.1 and Theorem 3.4.1. We highlight an important aspect, namely, the solution operator $\mathsf{T}_{lim} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}$ and its properties (introduced in Theorem 2.4.2) are used in the proof of Theorem 3.4.2. The well-posedness result that was obtained is as follows (see, e.g., [71, Theorem 5.2]).

Theorem 3.4.2. *Let $\alpha > 0$, $k, \beta \in \mathbb{R}^*$ be given constants. Let Assumption 1.1.7 and Assumption 2.2.1 be satisfied for $n = 2, 3.s$ Then, there exist two constants,*

$$\xi \equiv \xi(\mathsf{D}_+, \mathsf{D}_-, \alpha, k, \beta, \mathfrak{L}) > 0, \quad \eta \equiv \eta(\mathsf{D}_+, \mathsf{D}_-, \alpha, k, \beta, \mathfrak{L}) > 0, \tag{3.4.25}$$

such that, for every $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2) \in \mathsf{Y}_{RT}$, which satisfies the condition

$$||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}} \leq \xi, \tag{3.4.26}$$

the limiting Poisson problem of Robin-transmission type (3.4.24) for the Darcy-Forchheimer-Brinkman system has a unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$ with the property

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_{RT}} \leq \eta. \tag{3.4.27}$$

Moreover, there exists a constant $C_0 \equiv C_0(\mathsf{D}_+, \mathsf{D}_-, \alpha, \mathfrak{L}, \lambda) > 0$ such that the unique solution satisfies the estimate

$$||(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-)||_{\mathsf{X}_{RT}} \leq C_0 ||(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}}. \tag{3.4.28}$$

Proof. In order to prove our result, we use a similar technique to that of [71, Theorem 5.2] and Theorem 3.4.1. This proof consists of three steps.

Step 1. *Existence of a solution of problem* (3.4.24). To this end we rewrite the limiting nonlinear Robin-transmission problem (3.4.24) as

$$\begin{cases} \Delta \mathbf{u}_{\pm} - \alpha \mathbf{u}_{\pm} - \nabla \pi_{\pm} = \mathbf{f}_{\pm}|_{\mathsf{D}_{\pm}} + \mathsf{J}_{k,\beta,\mathsf{D}_{\pm}}(\mathbf{u}_{\pm})|_{\mathsf{D}_{\pm}} \text{ in } \mathsf{D}_{\pm}, \\ \text{div } \mathbf{u}_{\pm} = 0 \text{ in } \mathsf{D}_{\pm}, \\ \left(\mathrm{Tr}_{\mathsf{D}_-} \mathbf{u}_-\right)|_{\Gamma_+} = -\mathbf{g}_1 \text{ on } \Gamma_+, \\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}_+, \pi_+, \mathbf{f}_+ + \mathsf{J}_{k,\beta,\mathsf{D}_{\pm}}(\mathbf{u}_+)) - \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_{\pm}}(\mathbf{u}_-))\right)|_{\Gamma_+} \\ \quad = \mathbf{h}_1 \text{ on } \Gamma_+, \\ \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_{\pm}}(\mathbf{u}_-))\right)|_{\Gamma_-} + \mathfrak{L}\left(\mathrm{Tr}_{\mathsf{D}_-} \mathbf{u}_-\right)|_{\Gamma_-} = \mathbf{g}_2 \text{ on } \Gamma_-. \end{cases} \tag{3.4.29}$$

Next, we aim to construct a nonlinear operator

$$\mathsf{H} : \mathsf{B}_\eta \to \mathsf{B}_\eta, \tag{3.4.30}$$

where B_η is a closed ball of the space $H^1_{\text{div}}(\mathsf{D}_+)^n \times H^1_{\text{div}}(\mathsf{D}_-)^n$. We show that the nonlinear operator H maps the ball B_η to itself and then, we show that H is a contraction. Hence, the unique fixed point of the nonlinear operator H gives a solution of the nonlinear problem (3.4.29).

In order to introduce the nonlinear operator H, we proceed as follows. We note that the given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2) \in \mathsf{Y}_{RT}$, which is present in (3.4.29), is fixed. We also fix $(\mathbf{u}_+, \mathbf{u}_-) \in$

$H^1_{\text{div}}(\mathsf{D}_+)^n \times H^1_{\text{div}}(\mathsf{D}_-)^n$. Then, we write the following linear limiting Robin-transmission problem for the Brinkman system

$$\begin{cases} \Delta \mathbf{u}^0_\pm - \alpha \mathbf{u}^0_\pm - \nabla \pi^0_\pm = \mathbf{f}_\pm|_{\mathsf{D}_\pm} + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_\pm)|_{\mathsf{D}_\pm} \text{ in } \mathsf{D}_\pm, \\ \text{div } \mathbf{u}^0_\pm = 0 \text{ in } \mathsf{D}_\pm, \\ \left(\text{Tr}_{\mathsf{D}_-} \mathbf{u}^0_-\right)|_{\Gamma_+} = -\mathbf{g}_1 \text{ on } \Gamma_+, \\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathbf{u}^0_+, \pi^0_+, \mathbf{f}_+ + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_+)) - \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}^0_-, \pi^0_-, \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_-))\right)|_{\Gamma_+} \\ \quad = \mathbf{h}_1 \text{ on } \Gamma_+, \\ \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}^0_-, \pi^0_-, \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_-))\right)|_{\Gamma_-} + \mathfrak{L}\left(\text{Tr}_{\mathsf{D}_-} \mathbf{u}^0_-\right)|_{\Gamma_-} = \mathbf{g}_2 \text{ on } \Gamma_-, \end{cases} \tag{3.4.31}$$

where $(\mathbf{u}^0_+, \pi^0_+, \mathbf{u}^0_-, \pi^0_-)$ is the unknown, while, by Lemma 3.1.3, we obtain the membership $\mathring{\mathsf{E}}(k|\mathbf{u}_\pm|\mathbf{u}_\pm + \beta(\mathbf{u}_\pm \cdot \nabla)\mathbf{u}_\pm) \in \tilde{H}^{-1}(\mathsf{D}_\pm)^n$.

We apply Theorem 2.4.2 in order to deduce that our transmission problem (3.4.31) has a unique solution

$$\begin{aligned} (\mathbf{u}^0_+, \pi^0_+, \mathbf{u}^0_-, \pi^0_-) &:= \mathsf{T}_{lim}(\mathbf{f}_+|_{\mathsf{D}_+} + \mathsf{J}_{k,\beta,\mathsf{D}_+}(\mathbf{u}_+)|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-} + \mathsf{J}_{k,\beta,\mathsf{D}_-}(\mathbf{u}_-)|_{\mathsf{D}_-}, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2) \in \mathsf{X}_{RT} \\ &= (\mathsf{U}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_-(\mathbf{u}_+, \mathbf{u}_-)). \end{aligned} \tag{3.4.32}$$

In relation (3.4.32), the operator $\mathsf{T}_{lim} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}$ is the solution operator provided by relation (2.4.54). Recall that, in view of Theorem 2.4.2, the operator $\mathsf{T}_{lim} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}$ is well-defined, linear and continuous. It maps the given data to the unique solution of the limiting Robin-transmission problem (2.4.52) for the Brinkman system, under Assumption 1.1.7, $n = 2, 3$. Moreover, the operator $\mathsf{T}_{lim} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}$ satisfies the estimate (2.4.55), given in Theorem 2.4.2.

Now, for the fixed given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2) \in \mathsf{Y}_{RT}$ and in view of Lemma 3.1.3 and Theorem 2.4.2, we deduce that the operators

$$(\mathsf{U}_+, \mathsf{R}_+, \mathsf{U}_-, \mathsf{R}_-) : H^1_{\text{div}}(\mathsf{D}_+)^n \times H^1_{\text{div}}(\mathsf{D}_-)^n \to \mathsf{X}_{RT}, \tag{3.4.33}$$

introduced in relation (3.4.32) are continuous operators. Moreover, in view of Lemma 3.1.3, there exists a constant $C \equiv C(\mathsf{D}_+, \mathsf{D}_-, \alpha, \mathfrak{L}) > 0$ such that

$$\begin{aligned} &||(\mathsf{U}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_-(\mathbf{u}_+, \mathbf{u}_-))||_{\mathsf{X}_{RT}} \\ &\leq C||(\mathbf{f}_+|_{\mathsf{D}_+}, \mathbf{f}_-|_{\mathsf{D}_-}, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2)||_{\mathsf{Y}_{RT}} + c_1^+ C||\mathbf{u}_+||^2_{H^1_{\text{div}}(\mathsf{D}_+)^n} + c_1^- C||\mathbf{u}_-||^2_{H^1_{\text{div}}(\mathsf{D}_-)^n}, \end{aligned} \tag{3.4.34}$$

for all fields $(\mathbf{u}_+, \mathbf{u}_-) \in H^1_{\text{div}}(\mathsf{D}_+)^n \times H^1_{\text{div}}(\mathsf{D}_-)^n$ (see also relation (3.4.13)). Note that the constants c_1^+ and c_1^- which are present in relation (3.4.34) are obtained by employing relation (3.1.4) of Lemma 3.1.3, in the sets D_+ and D_-, respectively.

Now let us use relation (3.4.32) in order to rewrite problem (3.4.31) as

$$\begin{cases} \Delta \mathsf{U}_\pm(\mathbf{u}_+, \mathbf{u}_-) - \alpha \mathsf{U}_\pm(\mathbf{u}_+, \mathbf{u}_-) - \nabla \mathsf{R}_\pm(\mathbf{u}_+, \mathbf{u}_-) \\ \quad = \mathbf{f}_\pm|_{\mathsf{D}_\pm} + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_\pm)|_{\mathsf{D}_\pm} \text{ in } \mathsf{D}_\pm, \\ \text{div } \mathsf{U}_\pm(\mathbf{u}_+, \mathbf{u}_-) = 0 \text{ in } \mathsf{D}_\pm, \\ \left(\text{Tr}_{\mathsf{D}_-} \mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-)\right)|_{\Gamma_+} = -\mathbf{g}_1 \text{ on } \Gamma_+, \\ \mathbf{t}_{\alpha,\mathsf{D}_+}(\mathsf{U}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_+(\mathbf{u}_+, \mathbf{u}_-), \mathbf{f}_+ + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_+)) \\ \quad - \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_-(\mathbf{u}_+, \mathbf{u}_-), \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_-))\right)|_{\Gamma_+} \\ \quad = \mathbf{h}_1 \text{ on } \Gamma_+, \\ \left(\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-), \mathsf{R}_-(\mathbf{u}_+, \mathbf{u}_-), \mathbf{f}_- + \mathsf{J}_{k,\beta,\mathsf{D}_\pm}(\mathbf{u}_-))\right)|_{\Gamma_-} \\ \quad + \mathfrak{L}\left(\text{Tr}_{\mathsf{D}_-} \mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-)\right)|_{\Gamma_-} = \mathbf{g}_2 \text{ on } \Gamma_-. \end{cases} \tag{3.4.35}$$

We introduce the nonlinear operator

$$\mathsf{H} : H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n \to H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n, \tag{3.4.36}$$

which is given by

$$\mathsf{H}(\mathbf{u}_+, \mathbf{u}_-) := (\mathsf{U}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-)). \tag{3.4.37}$$

Then, a solution of the problem (3.4.29) in the space X_{RT} can be found, if we prove that our nonlinear operator (3.4.36) admits a fixed point $(\mathbf{u}_+, \mathbf{u}_-)$ which belongs to the space $H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n$. This fixed point satisfies $\mathsf{H}(\mathbf{u}_+, \mathbf{u}_-) = (\mathbf{u}_+, \mathbf{u}_-)$ and together with $\pi_\pm = \mathsf{R}(\mathbf{u}_+, \mathbf{u}_-)$ determine a solution of (3.4.29).

To show that the nonlinear operator (3.4.36) admits a fixed point, we show that the operator (3.4.36) maps a closed ball of the space $H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n$ to the same ball and is a contraction on that ball.

Now, we define the constants $\xi > 0$ and $\eta > 0$ by relation (3.4.16). Let us choose the close ball B_η as in relation (3.4.17) and we assume the given data satisfy inequality (3.4.18). Then, in view of relation (3.4.34), we have that

$$||\mathsf{U}_+(\mathbf{u}_+, \mathbf{u}_-), \mathsf{U}_-(\mathbf{u}_+, \mathbf{u}_-)||_{\mathsf{X}_{RT}} \leq \eta, \ \forall \ (\mathbf{u}_+, \mathbf{u}_-) \in \mathsf{B}_\eta, \tag{3.4.38}$$

which shows that the nonlinear operator H maps the closed ball B_η to itself, as claimed.

Our next aim is to show that the operator $\mathsf{H} : \mathsf{B}_\eta \to \mathsf{B}_\eta$ is a contraction on the ball B_η. Then, for fixed given data $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}_1, \mathbf{h}_1, \mathbf{g}_2) \in \mathsf{Y}_{RT}$ and $(\mathbf{v}_+, \mathbf{v}_-), (\mathbf{w}_+, \mathbf{w}_-) \in \mathsf{B}_\eta$, we have that (see relation (3.4.20))

$$\begin{aligned} &||\mathsf{H}(\mathbf{v}_+, \mathbf{v}_-) - \mathsf{H}(\mathbf{w}_+, \mathbf{w}_-)||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n} \\ &\leq \frac{1}{2} ||(\mathbf{v}_+ - \mathbf{w}_+, \mathbf{v}_- - \mathbf{w}_-)||_{H^1_{\mathrm{div}}(\mathsf{D}_+)^n \times H^1_{\mathrm{div}}(\mathsf{D}_-)^n}. \end{aligned} \tag{3.4.39}$$

In order to obtain the estimate in relation (3.4.39), we use relation (3.1.5) and the linearity and continuity of the operator $\mathsf{T}_{lim} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}$ (see relation (2.4.54)). Hence, the nonlinear operator $\mathsf{H} : \mathsf{B}_\eta \to \mathsf{B}_\eta$ is a $\frac{1}{2}$-contraction.

Let us apply Banach's fixed point theorem in order to obtain the existence of a unique fixed point $(\mathbf{u}_+, \mathbf{u}_-) \in \mathsf{B}_\eta$, that is

$$\mathsf{H}(\mathbf{u}_+, \mathbf{u}_-) = (\mathbf{u}_+, \mathbf{u}_-). \tag{3.4.40}$$

Then, the fields $(\mathbf{u}_+, \mathbf{u}_-)$ together with $\pi_\pm = \mathsf{R}_\pm(\mathbf{u}_+, \mathbf{u}_-)$ given in relation (3.4.32) provide a solution for the nonlinear problem (3.4.29) in the space X_{RT}. Hence, the fields $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$ give a solution of the nonlinear limiting Robin-transmission problem (3.4.24).

Moreover, by using arguments similar to those employed in relations (3.4.21), (3.4.22) and (3.4.23) of Theorem 3.4.1, we have that the estimate (3.4.28) holds, for $C_0 = \frac{4}{3}C$.

Step 2. *Uniqueness of a solution of problem* (3.4.24). By using similar ideas to those presented in Theorem 3.4.1 and Theorem 3.2.1, we are able to prove that the solution of the problem (3.4.24) is unique, in view of the uniqueness property of the fixed point belonging to B_η, which is guaranteed by Banach's fixed point theorem.

Step 3. *Continuous depedence of the solution of the problem* (3.4.24) *on the given data.* As in Theorem 3.4.1 and Theorem 3.2.1, we note that the continuity of the operators $\mathsf{H} : \mathsf{B}_\eta \to \mathsf{B}_\eta$ and $\mathsf{T}_{lim} : \mathsf{Y}_{RT} \to \mathsf{X}_{RT}$ (see relation (2.4.54)) implies that the unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$ of the problem (3.4.24) depends continuously on the given data and estimate (3.4.28) holds with $C_0 = \frac{4}{3}C$. Thus, our proof is complete.

□

3.4.2 The Darcy-Forchheimer-Brinkman system and a related Robin-Dirichlet problem

The goal of this subsection is to highlight the particular role that a transmission-type problem satisfies. In the latter, let $\alpha, k, \beta > 0$ be given constants and let Assumption 1.1.7 be satisfied. Note that, we consider the Lipschitz domain D_- and we use similar arguments as those described in [82, p. 4581]. Let us proceed by stating the fact that the problem (3.4.24) is well-posed (see Theorem 3.4.2). Consequently, we obtain a unique solution $(\mathbf{u}_+, \pi_+, \mathbf{u}_-, \pi_-) \in \mathsf{X}_{RT}$ of the problem (3.4.24). From it, we extract the pair $(\mathbf{u}_-, \pi_-) \in H^1_{\mathrm{div}}(\mathsf{D}_-)^n \times L^2(\mathsf{D}_-)$ and we note that this particular pair satisfies another boundary value problem, namely, the following Robin-Dirichlet problem for the Darcy-Forchheimer-Brinkman system in D_-. This boundary value problem is given by

$$\begin{cases} \Delta\mathbf{u}_- - \alpha\mathbf{u}_- - k|\mathbf{u}_-|\mathbf{u}_- - \beta(\mathbf{u}_-\cdot\nabla)\mathbf{u}_- - \nabla\pi_- = \mathbf{f}_-|_{\mathsf{D}_-} \text{ in } \mathsf{D}_-, \\ \mathrm{div}\ \mathbf{u}_- = 0 \text{ in } \mathsf{D}_-, \\ (\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u}_-)|_{\Gamma_+} = -\mathbf{g}_1 \text{ on } \Gamma_+, \\ (\mathbf{t}_{\alpha,\mathsf{D}_-}(\mathbf{u}_-, \pi_-, \mathbf{f}_- + \mathring{\mathsf{E}}_-(k|\mathbf{u}_-|\mathbf{u}_- + \beta(\mathbf{u}_-\cdot\nabla)\mathbf{u}_-)))|_{\Gamma_-} + \mathfrak{L}(\mathrm{Tr}_{\mathsf{D}_-}\mathbf{u})|_{\Gamma_-} = \mathbf{g}_2, \text{ on } \Gamma_-. \end{cases} \tag{3.4.41}$$

To summarize, we can obtain the solution for a boundary value problem (that is, problem (3.4.41)) by extracting it from the solution of a transmission-type problem (that is, problem (3.4.24)). It follows that the pair $(\mathbf{u}_-, \pi_-)$ is a solution of the Robin-Dirichlet problem (3.4.41) for the Darcy-Forchheimer-Brinkman system.

Let $(\mathbf{f}_-, \mathbf{g}_1, \mathbf{g}_2) \in \tilde{H}^{-1}(\mathsf{D}_-)^n \times H^{\frac{1}{2}}_{\nu}(\Gamma_+)^n \times H^{-\frac{1}{2}}(\Gamma_-)^n$ satisfying conditon (3.4.26) of Theorem 3.4.2. Then, we have the following consequence (see [82, p. 4581]).

Corollary 3.4.3. *The Robin-Dirichlet problem for the Darcy-Forchheimer-Brinkman system (3.4.41) has a solution in the space* $H^1_{\mathrm{div}}(\mathsf{D}_-)^n \times L^2(\mathsf{D}_-)$*, where* $n = 2, 3$.

4

A numerical approach related to the Darcy-Forchheimer-Brinkman system with Robin-Dirichlet conditions

The aim of this chapter is to study, numerically, the Robin-Dirichlet problem for the Darcy-Forchheimer-Brinkman system, namely problem (3.4.41). In addition, we have an existence result for the problem (3.4.41), which is Corollary 3.4.3. We solve numerically a lid-driven cavity problem. This problem consists of a square cavity which contains a solid square. Consequently, we have an interior boundary (that is, the boundary of the internal solid square) and an exterior boundary (that is, the exterior walls of the cavity). The interior walls are considered to be fixed. The exterior walls slide at different constant velocities. In addition, the domain contained between the exterior and the interior boundary is filled with a porous media and is saturated by a viscous Newtonian incompressible fluid, which is modelled by the Darcy-Forchheimer-Brinkman system (see Relation (4.1.1)). The geometry is given in Figure 4.1. The content of this chapter follows the results that were obtained in the paper [9].

We note that our previous approaches in Chapter 2 and Chapter 3 have focused on obtaining a unique solution for our transmission-type problems. Indeed, we have used layer potential theory to construct a solution in the linear problems. We have also used the Banach fixed point Theorem in order to get a solution in the non-linear setting. In addition, we have seen that we may obtain a solution to other boundary value problems by extracting it from a transmission problem. We present another approach to finding a solution for a boundary value problem which is rooted in some devices that stem from Numerical Analysis.

In the latter, we take note of some past works that concern the lid-driven cavity flow problem. Firstly, let us emphasize the contribution of Ghia, Ghia and Shin [55]. The authors have obtained numerical results for a driven flow in a square cavity. These results provide a useful test case by which other numerical methods can be checked against. In [88], the authors note that the lid-driven cavity flow problem is a test problem, in two or three dimensions, through which diverse numerical schemes can be validated or invalidated. The attractiveness of such a problem consists of its simple geometry and its perceived flow structure. Gutt and Groşan [62] have investigated numerically a mixed Dirichlet-Robin boundary problem for the Darcy-Brinkman system in the setting of the lid-driven porous cavity problem. They also analyze the influence of various parameters on the fluid flow. Papuc [118] has investigated a lid-driven porous cavity flow problem with an internal square block. The author has analyzed this problem both theoretically and numerically by investigating a Dirchlet problem for the Darcy-Forchheimer-Brinkman system.

4.1 Numerical study of the lid-driven cavity flow problem in a 2-dimensional cavity with Navier slip boundary condition in the presence of a solid body

4.1.1 Statement of the problem and remarks

Let us describe the mathematical model of our lid-driven problem in a two-dimensional cavity with Navier slip boundary condition, in the presence of a solid body. Our goal is to study the flow of a viscous Newtonian incompressible fluid in a porous medium in a special Lipschitz domain denoted by D_-, as seen in Figure 4.1, while we consider Dirichlet boundary condition on the interior boundary and Robin boundary conditions on the exterior boundary. Let us describe the geometry of our problem. We consider $\mathsf{D} \subset \mathbb{R}^2$ a square cavity of length L which contains a solid square obstacle, denoted by D_+, of length l such that $l < L$. Let us define $\mathsf{D}_- := \mathsf{D} \setminus \mathsf{D}_+$. The interior boundary, denoted by Γ_+ is considered to be fixed, while the exterior boundary Γ_-, consist of four walls Γ_-^t, Γ_-^l, Γ_-^b, Γ_-^r which are sliding at different constant velocities (see Figure 4.1).

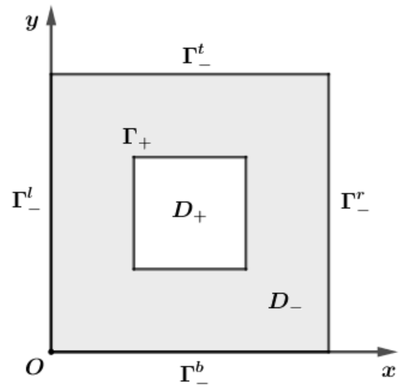


Figure 4.1: The porous cavity with internal square block

4.1.2 Mathematical model of the problem

Inside the porous cavity, i.e., Fig 4.1, the fluid flow is described by the Darcy-Forchheimer-Brinkman system (see, e.g., [4], [60], [144]). On the exterior boundary Γ_-, we impose the Navier-slip condition which is a Robin type boundary condition (see, [66], [125]) and on the interior boundary Γ_+, we impose the Dirichlet boundary condition. The mathematical model for our problem is

$$\begin{cases} \Delta \mathbf{u}_- - \dfrac{\kappa}{K}\mathbf{u}_- - \dfrac{\kappa}{\nu\rho}\nabla\pi_- = \dfrac{1}{\nu}(\mathbf{u}_- \cdot \nabla)\dfrac{\mathbf{u}_-}{\kappa} + \dfrac{\kappa C_f}{\nu\sqrt{K}}|\mathbf{u}_-|\mathbf{u}_- & \text{in } \mathsf{D}_- \\ \operatorname{div} \mathbf{u}_- = 0 & \text{in } \mathsf{D}_- \\ \mathbf{u}_- = \mathbf{g}_1 & \text{on } \Gamma_+ \\ \mathbf{u}_- + s_l \dfrac{\partial \mathbf{u}_-}{\partial \mathbf{n}_-} = \mathbf{g}_2 & \text{on } \Gamma_-. \end{cases} \tag{4.1.1}$$

Let us describe the quantities that are involved in (4.1.1). We have

- $\mathbf{u}_- = (u_x, u_y)|_{\mathsf{D}_-}$ is the two-dimensional velocity field

- π_- is the pressure function
- κ is the porosity of the medium
- K is the permeability of the medium
- $C_f = \frac{1.75}{\sqrt{150\kappa^3}}$ is the friction coefficient (see, e.g., [139])
- s_l is the slip length parameter
- $\mathbf{n}_-$ is the outward unit normal to Γ_-.

In addition, the values of $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{n}_-$ are given by

$$\mathbf{g}_1 = (0,0) \text{ on } \Gamma_+, \qquad \mathbf{g}_2 = \begin{cases} (u^t, 0) & \text{on } \Gamma_-^t \\ (0, u^r) & \text{on } \Gamma_-^r \\ (u^b, 0) & \text{on } \Gamma_-^b \\ (0, u^l) & \text{on } \Gamma_-^l, \end{cases} \tag{4.1.2}$$

and

$$\mathbf{n}_- = \begin{cases} \mathbf{n}_-^t = (0,1) & \text{on } \Gamma_-^t \\ \mathbf{n}_-^r = (1,0) & \text{on } \Gamma_-^r \\ \mathbf{n}_-^b = (0,-1) & \text{on } \Gamma_-^b \\ \mathbf{n}_-^l = (-1,0) & \text{on } \Gamma_-^l, \end{cases} \tag{4.1.3}$$

where Γ_-^t, Γ_-^r, Γ_-^b and Γ_-^l are the top side, the right side, the bottom side and the left side of the exterior square in Figure 4.1, respectively. Consequently, by employing relations (4.1.2) and (4.1.3) we can rewrite the boundary condition $(4.1.1)_4$ as

$$\mathbf{u}_- = \begin{cases} \left(u^t - s_l \dfrac{\partial u_x}{\partial y}, 0\right) & \text{on } \Gamma_-^t \\ \left(0, u^r - s_l \dfrac{\partial u_y}{\partial x}\right) & \text{on } \Gamma_-^r \\ \left(u^b + s_l \dfrac{\partial u_y}{\partial x}, 0\right) & \text{on } \Gamma_-^b \\ \left(0, u^l + s_l \dfrac{\partial u_y}{\partial x}\right) & \text{on } \Gamma_-^l. \end{cases} \tag{4.1.4}$$

Now, in order to conduct the non-dimensional analysis, let us replace the dimensional variables in (4.1.1) and (4.1.4) with the dimensionless variables

$$X = \frac{x}{L}, \qquad Y = \frac{y}{L}, \qquad S_l = \frac{s_l}{L}, \qquad U_x = \frac{u_x}{u^t}, \qquad U_y = \frac{u_y}{u^t}, \qquad \Pi = \frac{\pi}{\rho(u^t)^2}.$$

. Hence, we obtain

$$\begin{cases} \Delta \mathbf{U}_- - \dfrac{\kappa}{Da}\mathbf{U}_- - Re\,\kappa\nabla\Pi = Re\,(\mathbf{U}_-\cdot\nabla)\dfrac{\mathbf{U}_-}{\kappa} + \dfrac{Re\,\kappa\,C_f}{\sqrt{Da}}|\mathbf{U}_-|\mathbf{U}_- & \text{in } \mathsf{D}_- \\ \operatorname{div}\mathbf{U}_- = 0 & \text{in } \mathsf{D}_- \\ \mathbf{U}_- = (0,0) & \text{on } \Gamma_+ \\ \mathbf{U}_- + S_l \dfrac{\partial \mathbf{U}_-}{\partial \mathbf{n}_-} = \mathbf{G}_2 & \text{on } \Gamma_- \end{cases} \tag{4.1.5}$$

and

$$\mathbf{G}_2 = \begin{cases} (1,0) & \text{on } \Gamma^t_- \\ (0,U^r) & \text{on } \Gamma^r_- \\ (U^b,0) & \text{on } \Gamma^b_- \\ (0,U^l) & \text{on } \Gamma^l_-. \end{cases} \tag{4.1.6}$$

In addition, in view of the non-dimensional analysis, in (4.1.5) we have that

- $Re = \frac{U_0 L}{\nu}$ - the Reynolds number
- $Da = \frac{K}{L^2}$ - the Darcy number

and the right hand side of (4.1.6) contains the values

$$U^r = \frac{u^r}{u^t},\ U^b = \frac{u^b}{u^t},\ U^l = \frac{u^l}{u^t}, \tag{4.1.7}$$

which are the ratios between the sliding velocities of the outer right, bottom and left walls and the velocity of the upper wall.

In our analysis, we also consider the stream function Ψ which is given by

$$U_x = \frac{\partial \Psi}{\partial Y},\ U_y = -\frac{\partial \Psi}{\partial X}. \tag{4.1.8}$$

We use this function to compute the maximum stream function value reached inside the cavity, Ψ_{max}. Also, we use the stream function Ψ in order to visualize the fluid flow pattern, which is observed in the form of the stream lines.

4.1.3 Numerical method and validation of the model

We use the finite element based software COMSOL Multiphysics (see [145]) in order to solve the system (4.1.5) together with the equation

$$\Delta \Psi = \frac{\partial U_x}{\partial Y} - \frac{\partial U_y}{\partial X}, \tag{4.1.9}$$

Note that equation (4.1.9) is derived from relation (4.1.8).

In order to discretize the domain in Figure 4.1, we consider a free quad mesh. The mesh was constructed as follows. We starting with a fixed number of elements, N, which established on either side of Γ_-. On the side of the Γ_+ we have $N\dfrac{L}{l}$ elements. The maximum size of an element inside the cavity is set to $\dfrac{1}{N}$. To get a numerical solution, the nonlinear solver iterates until the relative error is less than $\epsilon = 10^{-6}$.

Next, we perform a convergence test for the maximum value of the stream function, Ψ_{max}, depending on the refinement level of the mesh. This is done in order to determine the dependence of the solution on the chosen mesh. Moreover, it allows us to find the optimal grid from the perspective of computational cost and as well as the accuracy of the results. Then, for our problem (4.1.5) together with (4.1.9) we have the following default settings

$$L = 1,\ l = 0.4,\ \kappa = 0.3,\ Re = 100,\ Da = 0.01,\ U^r = U^b = -0.1,\ U^l = 0.1. \tag{4.1.10}$$

N (elements on exterior side)	Ψ_{max}	**Error**$_{\Psi_{max}}$
20	0.04366673	
40	0.04358508	0.000081465
60	0.04358398	0.0000011
80	0.04358332	0.00000066

Table 4.1: Mesh dependence

In view of (4.1.10) we have obtained Table 4.1, which contains the computed values of Ψ_{max} for different values N. From Table 4.1 we determine that the choice of the mesh containing 80 elements on each side of Γ_- of in Figure 4.1 is appropriate for our simulations.

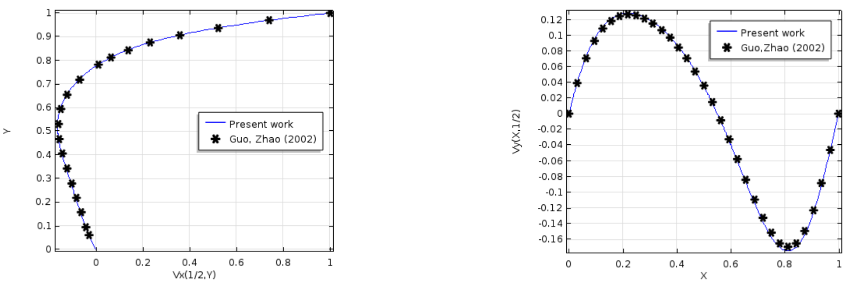
Now we compare our numerical solutions with previous established results in order to validate our approach. To this end, we have the following settings

$$U^r = U^b = U^l = 0,\ \ \mu = 1,\ \ l = 0,\ \ S_l = 0, \tag{4.1.11}$$

which is the case of the porous lid-driven square cavity problem with vanishing obstacle and no-slip boundary condition. Next, for the values

$$\kappa = 0.1,\ \ Re = 10,\ \ Da = 0.01, \tag{4.1.12}$$

we plot the x component of the velocity, U_x, along the vertical line through the cavity center and the y component of the velocity, U_y, along the horizontal line through the cavity center. We compare the obtained velocity profiles that we determined with the data obtained in [60]. Both graphs in Figure 4.2 show a good agreement.



(a) U_x along vertical centerline

(b) U_y along horizontal centerline

Figure 4.2: The components of the velocity along vertical and horizontal center-lines of the squared cavity, compared with [60].

4.1.4 Results and discussion

We aim to determine the impact of the dimensionless slip length, S_l, on the fluid flow inside the porous cavity. To this aim, we set the parameters

$$l = 0.4,\ \ \kappa = 0.3,\ \ Re = 100,\ \ Da = 0.01,\ \ U^r = U^b = -0.1,\ \ U^l = 0.1 \tag{4.1.13}$$

and we study the flow properties for $S_l \in (0, 0.003)$.

S_l	Ψ_{max}
0	0.04371041
0.0005	0.04317579
0.001	0.04290319
0.0015	0.04262631
0.002	0.04234790
0.0025	0.04206969
0.003	0.04179283

Table 4.2: Maximum stream function values for different S_l

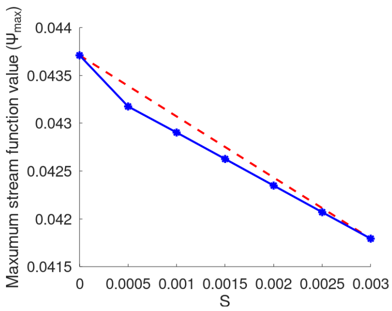
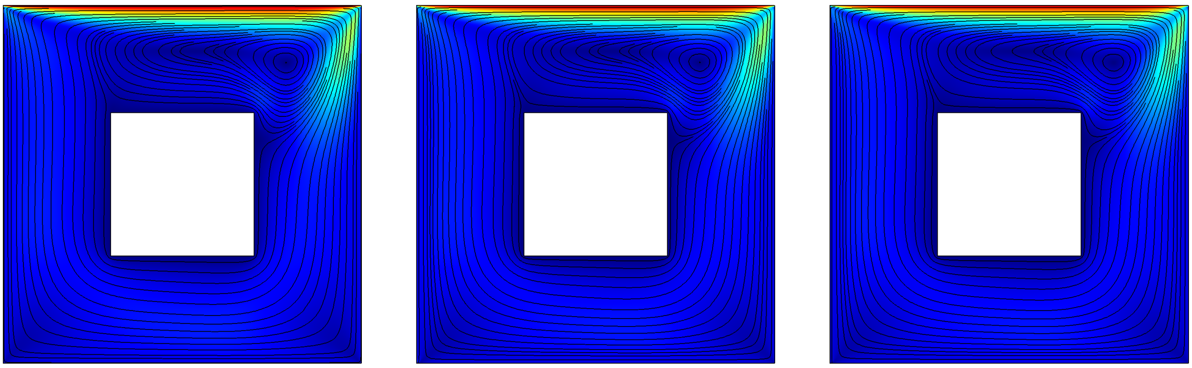


Figure 4.3: Dimensionless slip length effect

The computed values Ψ_{max} inside the cavity for different values of the dimensionless parameter $S_l \in (0, 0.003)$ are displayed in Table 4.2. These values are also represented in Figure 4.3. Figure 4.3 shows the linear decrease of Ψ_{max} between $S_l = 0.0005$ and $S_l = 0.003$. The fluid displacement inside the porous cavity is highlighted in Figure 4.4. An important remark that can be made here is that the variation of the dimensionless slip parameter S_l does not suddenly change the flow pattern. This can be seen in the similarity of all three images in Figure 4.4 being quite similar. Even if the stream lines and the velocity profile are different in each case, these differences are negligible and not so obvious.



(a) $S^l = 0$ (b) $S^l = 0.0015$ (c) $S^l = 0.003$

Figure 4.4: Streamlines and Velocity profiles for different values of slidding parameter S_l.

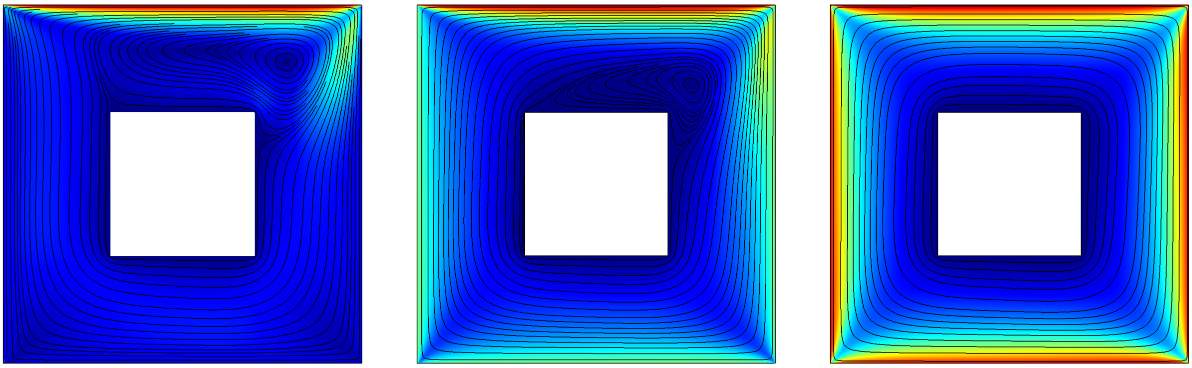
We continue our analysis and we set $S_l = 0.0005$. We consider

$$U^r = U^b = U, \; U^l = -U, \tag{4.1.14}$$

where U is a constant which takes the values

$$U = 0.1, 0.3, 0.5, 0.7, 0.9, \tag{4.1.15}$$

respectively. Hence, we want to see the how fluid flow behaves inside the cavity, whether the velocity of the vertical walls and the bottom one increases towards the velocity of the top lid. The other parameters remain the same as in relation (4.1.13). In this situation, the stream lines and the velocity profiles for the fluid particles for $U = 0.1, 0.5, 0.9$ are provided in Figure 4.5. Let us note that, for increasing values of U, the center of the secondary vortex, which is initially close to the top side, tends to approach the center of the cavity, eventually being assimilated by the main vortex rotating around the obstacle. This is due to the balance of forces generated by the four walls arranged symmetrically. In Table 4.3 we see how Ψ_{max} varies, while its minimum is reached for $U = 0.3$. Beyond $U = 0.3$, Ψ_{max} tends to increase as U approaches the velocity of the upper wall, reaching a maximum value for $U = 0.9$.



(a) $U = 0.1$ (b) $U = 0.5$ (c) $U = 0.9$

Figure 4.5: Streamlines and Velocity profiles for $S_l = 0.0005$ and $U = 0.1, 0.5, 0.9$.

Finally, let us consider the default values given by (4.1.13) and we fix $S_l = 0.0005$. Let us modify one by one, the sliding direction of the left, bottom and right walls, in order to verify the influence on the flow pattern.

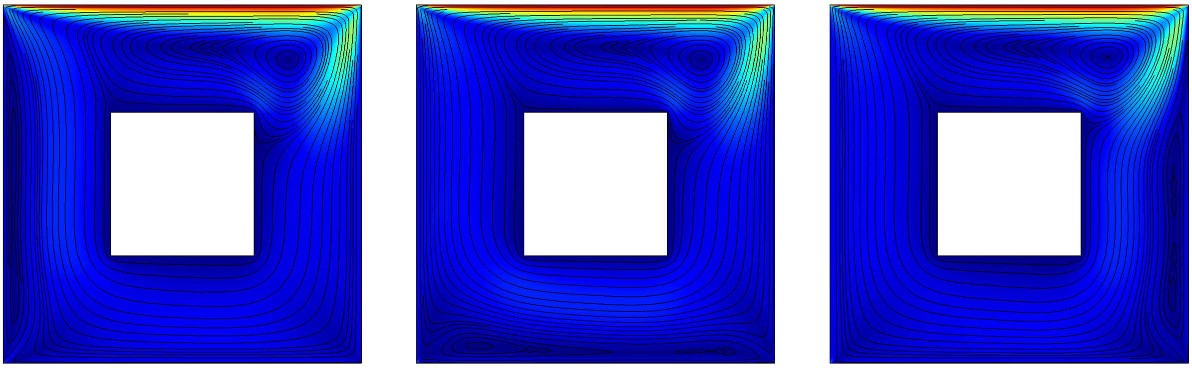
In Table 4.4 we see that lowest value of Ψ_{max} is obtained when the right wall has an upward sliding direction and the highest when the bottom wall moves to the right. Let us note that, in all three situations, high values are obtained inside the cavity, similar to those encountered in the case where the walls form a movement similar to the clockwise rotation motion. This means that the sliding of a wall in the opposite direction does not slow the flow inside the cavity, but rather enhances it through the vortices it forms. These vortices can be seen in Figure 4.6 having the direction of flow opposite to the one of the main vortex.

U	Ψ_{max}
0.1	0.04317579
0.3	0.04286458
0.5	0.04294202
0.7	0.04330634
0.9	0.04392932

Table 4.3: Maximum stream function values for
variation of U

(U^l, U^b, U^r)	Ψ_{max}
(-0.1,-0.1, -0.1)	0.04347869
(0.1, 0.1, -0.1)	0.04356940
(0.1,-0.1, 0.1)	0.04298635

Table 4.4: Maximum stream function values for different sliding directions of U^l, U^b, U^r (right)



(a) $U^l = -0.1$ (b) $U^b = 0.1$ (c) $U^r = 0.1$

Figure 4.6: Streamlines and Velocity profiles for $S_l = 0.0005$ and different slidding directions of the three lower walls.

Further research directions

We would like to point out some research directions that could be followed after this monograph.

Extension of the obtained results

As a first future direction, we aim to extend the original results that were presented to more general function spaces such as L^p-based Sobolev space, for $p \in (1, \infty)$, Besov spaces, Bessel potential spaces, Triebel-Lizorkin spaces. We can also consider our boundary value problems in certain domains whose geometry is more general or more complex, for example, polyhedral domains, domains with cusps. In addition, we intend to obtain such results by using other techniques such variational methods and the fixed point index theory. In addition, we can pursue a practical study such as the investigation of the correlation between physical parameters (for example, the Reynolds number) and the existence of vortexes in some viscous fluid flows in the presence of solid obstacles. In such a study we can formulate boundary value problems which are similar to the ones which we have investigated.

Variable coefficients

In recent years, a great deal of work has been devoted (see, e.g., [78], [79], [86]) to the generalization of the Stokes equations. Namely, instead of the Laplacian, one can consider another divergence form, second-order elliptic differential operator. Consequently, this approach leads to the anisotropic Stokes system and anisotropic Navier-Stokes system, respectively. These generalizations account of the possibility of the modeling of a incompressible fluid with variable viscosity.

This new perspective leads to the future idea of studying boundary problems for more general Brinkman or Darcy-Forchheimer-Brinkman equations, in various configurations, while all the coefficients that appear in these systems are variable (see, e.g., [85]).

Bidisperse (Multidisperse) Porous Media Models

Another possible development that can be pursued is the theoretical and/or numerical study of bidisperse porous media.

The authors in [84] have developed a theoretical analysis for a general system of coupled Navier-Stokes-type equations in the incompressible case in the setting of a bounded domain, where a homogeneous Dirichlet condition was considered. Their approach is based on the model proposed by Nield and Kuznetsov in the papers [116] and [117]. Kohr and Precup [85] have studied a general class of coupled anisotropic Navier-Stokes-like equations with variable coefficients that describe viscous fluid flows in multidisperse anisotropic porous media. They have considered also non-homogeneous reaction-type terms in the incompressible case. The authors have employed a variational technique and fixed point index theory in order to obtain existence results.

The papers [84] and [85] suggest an a possible research direction, that of the investigation of other models that appear in the study of flows in anisotropic bidisperse (or multidisperse) porous media with the goal of obtaining existence results for other boundary problems associated to the underlying PDE systems.

Moreover, another point of exploration can be the diversification of the numerical methods that can be employed in the study of boundary problems suggested by applications in Fluid Mechanics and porous media. Let us mention that, in addition to the classical approaches as finite difference methods (e.g., employed in [58]), finite volume methods (see also the monograph of Hoffmann and Chiang [67]), there are powerful PDE solvers such as FreeFem++, Ansys, Comsol that can be employed in order to obtain numerical results for future studies of various boundary value problems.

Boundary value problems on manifolds

Finally, we want to specify the results included in this monograph have all been obtained in the Euclidean setting of $\mathbb{R}^n$. There are also many works devoted to the investigation of boundary problems on compact manifolds (see, e.g., [76], [82], [86], [112], [113]). A natural step would be to consider similar boundary value problems, with those that we have treated, but in the setting of compact Riemannian manifolds or non-compact Riemannian manifolds.

More recently, a new concept has been developed. We want to highlight a contribution made by Kohr, Nistor and Wendland in [81], in which they obtained the results needed to introduce and investigate layer potentials on manifolds with conical or cylindrical ends. They devoted their study to the introduction of classes of pseudodifferential operators that are defined on these manifolds, called 'translation invariant at infinity' and 'essentially translation invariant' operators and studied their properties, having in view applications to the Stokes system. As a future research direction that can be inferred, the work [81] (see also [111]) provides an opening for the analysis of various boundary problems for other elliptic PDE systems in the setting of manifolds with cylindrical ends.

Conclusions

The aim of this book is to provide existence and uniqueness results for transmission-type boundary value problems for certain constant-coefficient and variable-coefficient elliptic systems. Some of these systems can be found in the field of Fluid Mechanics, while others are involved in certain models of porous media. The aforementioned transmission-type problems are investigated in the Euclidean setting using the means of potential theory and fixed point methods and we complement the theoretical results with a numerical investigation of a boundary value problem.

We begin by describing all notions that we use throughout this monograph. We discuss Lipschitz domains, L^2-based Sobolev spaces, weighted Sobolev spaces (see Section 1.1). We continue with the introduction of the (Gagliardo) trace operator (see Theorem 1.1.18) in the classical Sobolev spaces as well in the weighted Sobolev spaces (see Remark 1.1.19). Next, we analyze the Stokes, Brinkman and generalized Brinkman equations (see relation (1.2.21)) and we provide their associated conormal derivative operators (see Definition 1.2.3, Lemma 1.2.4, Definition 1.2.5, Lemma 1.2.6, Definition 1.2.14, Lemma 1.2.15). For the Stokes and Brinkman systems, respectively, we give their respective fundamental solution (see Subsection 1.3.1, Subsection 1.4.1), we introduce their associated single layer, double layer and Newtonian potentials (see Definition 1.3.1, Definition 1.3.3, Definition 1.3.5, Definition 1.4.1, Definition 1.4.3, Definition 1.4.5). For each of these potentials we have given their mapping properties (see Theorem 1.3.2, Theorem 1.3.4, Theorem 1.3.7, Theorem 1.4.2, Theorem 1.4.4, Theorem 1.4.7), their jump properties (see Lemma 1.3.8, Lemma 1.4.8) and their growth conditions at infinity (see relation (1.3.26), relation (1.4.28)).

The following chapter is concerned with existence and uniqueness results for transmission problem for linear PDE systems. First, we have a well-posedness result for the exterior Dirichlet problem for the Brinkman system in $\mathbb{R}^3$ (see Theorem 2.1.2). This is an auxiliary result that we use in our monograph. Next, by using a layer potential analysis and Fredholm operator theory, a well-posedness result for the transmission problem for the generalized Brinkman and Stokes systems is obtained in the setting of weighted Sobolev spaces in complementary Lipschitz domains in $\mathbb{R}^3$ (see Theorem 2.2.2). Another existence and uniqueness result for the transmission problem for the generalized Brinkman and classical Brinkman systems is also obtained in the setting of classical Sobolev spaces in complementary Lipschitz domains in $\mathbb{R}^3$ (see Theorem 2.3.1). This is possible by making use of a layer potential technique. Each of the well-posedness results mentioned in the former is accompanied by a well-posedness result for some transmission-type problems which contain constant coefficient systems (see Theorem 2.2.6 and Theorem 2.3.3, respectively). Moreover, in the Euclidean setting of $\mathbb{R}^n$, $n \geq 2$, in the geometric configuration given in Assumption 1.1.7, we have a well-posedness result for the Robin-transmission problem for the classical Brinkman system (see Theorem 2.4.1). This result was established with the help of potential theory and Fredholm operator theory. A similar approach is adopted in order to show that, in $\mathbb{R}^n$, $n \geq 2$, the limiting Robin-transmission for the classical Brinkman system is also well-posed (see Theorem 2.4.2) and in view of its well-posedness, we obtain, as a consequence, the well-posedness of the Robin-Dirichlet problem for the Brinkman system (see Corollary 2.4.3).

The next chapter follows a similar structure to the second chapter. It contains the generalization of the Darcy-Forchheimer-Brinkman system (see relation (3.1.1)) and a lemma (see Lemma 3.1.3) that we use in this particular chapter. The main technique of this chapter is that of combining the results that were obtained in the previous chapter with a fixed point theorem in order to obtain existence and uniqueness results. Here, we have the well-posedness result for the generalized Darcy-Forchheimer-Brinkman and Stokes systems in the setting of Assumption 1.1.6 in weighted Sobolev spaces in $\mathbb{R}^3$ (see Theorem 3.2.1). Next, we have the well-posedness result for the generalized Darcy-Forchheimer-Brinkman and Brinkman systems in the setting of Assumption 1.1.6 in $\mathbb{R}^3$ (see Theorem 3.3.1). Also, another well-posedness result is obtained for the Robin-transmission problem for the classical Darcy-Forchheimer-Brinkman systems under the Assumption 1.1.7, in $\mathbb{R}^n$, $n \geq 2$ (see Theorem 3.4.1). In addition, the limiting Robin-transmission problem for the classical Darcy-Forchheimer-Brinkman in the setting of Assumption 1.1.7 is also well-posed (see Theorem 3.4.2). This previous well-posedness result gives an existence result for the Robin-Dirichlet problem for the classical Darcy-Forchheimer-Brinkman system in D_- in the setting of Assumption 1.1.7 (see Corollary 3.4.3).

The final chapter consists of a numerical investigation for the lid-driven cavity flow problem in two dimensions for the Darcy-Forchheimer-Brinkman system. For this problem we have considered Dirichlet boundary conditions on the interior wall and Robin boundary conditions on the exterior wall (see Figure 4.1). We analyze the impact of the dimensionless slip length on the behavior of the fluid flow inside the porous cavity.

Lastly, this work employs layer potential methods and a fixed point theorem in order to obtain well-posedness results for transmission problems which contain PDE systems that appear in Fluid Mechanics and Porous Media. This study is complemented by a numerical investigation of a boundary value problem. All results are obtained in an Euclidean setting.

Appendix

A Agmon-Douglis-Nirenberg elliptic systems

The goal of this section is to give the notion of an Agmon-Douglis-Nirenberg elliptic system. Let us note that, the linear PDE systems that are involved in our work, such as the Stokes or Brinkman systems are examples of Agmon-Douglis-Nirenberg systems. In order to describe such a system, we provide, first of all, some useful concepts (see [42] and see also [68], [70], [143]).

Let $n \in \mathbb{N}$, $n \geq 2$. Let us begin by considering a differential operator of order $r \in \mathbb{N}$,

$$\mathfrak{F}(y, D) = \sum_{|\alpha| \leq r} f_\alpha(y) D^\alpha, \tag{A.1}$$

where $\alpha \in \mathbb{Z}_+^n$ is a multi-index, D^α is given by relation (1.1.1) and $|\alpha| = \sum_{j=1}^n |\alpha|$. Recall that D_k is given by relation (1.1.2) and let $\zeta^\alpha := \zeta_1^{\alpha_1} ... \zeta_n^{\alpha_n}$. Now, assume that $f_\alpha \in \mathcal{C}^\infty(\mathbb{R}^n)$. Consequently, the symbol of $\mathfrak{F}(y, D)$ is the polynomial

$$f(y, \zeta) := \sum_{|\alpha| \leq r} f_\alpha(y) \zeta^\alpha. \tag{A.2}$$

The principal symbol of $\mathfrak{F}(y, D)$ is given by

$$(\sigma_r(f))(y, \zeta) := \sum_{|\alpha| = r} f_\alpha(y) \zeta^\alpha. \tag{A.3}$$

We provide the following useful definition.

Definition A.1. *For $r \in \mathbb{R}$, the space S^r of symbols of order r is the space of functions $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that the following condition*

$$|D_\zeta^\alpha D_y^\beta f(y, \zeta)| \leq d_{\alpha,\beta}(1 + |\zeta|)^{r-|\alpha|}, \ \forall \ y, \zeta \in \mathbb{R}^n \tag{A.4}$$

holds for all multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$.

Next, we discuss pseudodifferential operators. This notion is used in the description of a Agmon-Douglis-Nirenberg system. We have the following proposition.

Proposition A.2. *Let $f \in S^r$. If $v \in \mathcal{S}(\mathbb{R}^n)$, then $\mathfrak{F}(y, D)v(y) \in \mathcal{S}(\mathbb{R}^n)$, where*

$$\mathfrak{F}(y, D)v(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{iy \cdot \zeta} f(y, \zeta) \hat{v}(\zeta) \mathrm{d}\zeta. \tag{A.5}$$

Definition A.3. *An operator $\mathfrak{F}(y,D)$ given by relation (A.5) is called a pseudodifferential operator of order r on $\mathbb{R}^n$.*

Let us give the definition that allows us to introduce elliptic symbols.

Definition A.4. *Let $f \in S^r$. The symbol f is elliptic if there exist $d, \mathfrak{R} > 0$ two constants, such that*

$$|f(y,\zeta)| \geq d|\zeta|^r, \ \forall\ \zeta \in \mathbb{R}^n \text{ such that } |\zeta| > \mathfrak{R}. \tag{A.6}$$

In the latter, we use Definition A.3. We consider

$$\mathfrak{F}(y,D) = (\mathfrak{F}_{ij}(y,D))_{i,j=\overline{1,m}}, \ y \in \mathbb{R}^n, \tag{A.7}$$

a matrix of pseudodifferential operators $\mathfrak{F}_{ij}(y,D)$, whose symbols are denoted by $\mathfrak{F}^{ij} = \mathfrak{F}^{ij}(y,D)$. In addition, we assume that there are a_i, b_j for $i,j = \overline{1,m}$ such that $\mathfrak{F}^{ij} \in S^{a_i+b_j}$.

In our description, let us now deal with the Agmon-Douglis-Nirenberg system of PDEs

$$\sum_{j=1}^{m} \sum_{|\beta|=0}^{a_i+b_j} \mathfrak{F}^{ij}_{\beta}(y) D^{\beta} v_j(x) = f_i(x), \ j = \overline{1,m}. \tag{A.8}$$

In this case, the matrix of pseudodifferential operators $\mathfrak{F} = (\mathfrak{F}_{ij})_{i,j=\overline{1,m}}$ is given by

$$\mathfrak{F}_{ij} := \sum_{|\beta|=0}^{a_i+b_j} \mathfrak{F}^{ij}_{\beta}(y) D^{\beta} \tag{A.9}$$

and its corresponding symbols $\mathfrak{F}^{ij}_{\beta}(y) \in S^{a_i+b_j}$. Next, we assume that $a_i \leq 0$ for $i = \overline{1,m}$.

Now, for the system (A.8) we have its symbol matrix $(\mathfrak{F}^{ij}(y,\zeta))_{i,j=\overline{1,m}}$, which is given by

$$\mathfrak{F}^{ij}(y,\zeta) := \sum_{|\beta|=0}^{a_i+b_j} \mathfrak{F}^{ij}_{\beta}(y)(i\zeta)^{\beta} \tag{A.10}$$

and its principal part is given by

$$\mathfrak{F}^{ij}_{a_i+b_j}(y,\zeta) := \sum_{|\beta|=0}^{a_i+b_j} \mathfrak{F}^{ij}_{\beta}(y)(i\zeta)^{\beta}, \tag{A.11}$$

where $\mathfrak{F}^{ij}_{a_i+b_j}(y,\zeta)$ is zero if $\mathfrak{F}^{ij}(y,\zeta)$ is of order at most $a_i + b_j$.

Let us state the definition of an elliptic system in the sense of Agmon-Douglis-Nirenberg.

Definition A.5. *Let $(\mathfrak{F}_{ij})_{i,j=\overline{1,m}}$ be a matrix of pseudodifferential operators. Then, the system (A.8) $((\mathfrak{F}_{ij}))$ is elliptic in the sense of Agmon-Douglis-Nirenberg if its characteristic determinant $\sigma(y,\zeta)$ satisfies*

$$\sigma(y,\zeta) \neq 0, \ \forall\ y \in \mathbb{R}^n, \zeta \in \mathbb{R}^n \setminus \{0\}, \tag{A.12}$$

where

$$\sigma(y,\zeta) := \det\left[|\zeta|^{a_i+b_j} \mathfrak{F}^{jk}_{a_i+b_j}\left(y, \frac{\zeta}{|\zeta|}\right)\right]_{m\times m}. \tag{A.13}$$

We end this section with the following remark.

Remark A.6. *The Stokes operator (1.2.1) is elliptic in the sense of Agmon-Douglis-Nirenberg. In addition the Brinkman operator (1.2.4) is elliptic in the sense of Agmon-Douglis-Nirenberg as well (see [2], [68]).*

Let us end this section by mentioning the fact that even in the anisotropic case, the anisotropic Stokes and anisotropic Brinkman systems are elliptic in the sense of Agmon-Douglis-Nirenberg. The authors in [77] have proved this claim by using a variational argument.

B Fredholm operators

The aim of this section is to provide the reader with very useful results regarding Fredholm operator theory, which have appeared in the proof of our results, in Chapter 2 and Chapter 3, respectively. We will introduce the notion of a Freholm operator and two results that we have used throughout this work (see, e.g., [2], [68], [143]).

To this end, assume that X and Y are two Banach spaces. Recall that, for a linear operator $A : X \to Y$, we have

$$\mathrm{Ker}A := \{x \in X \mid Ax = 0\}, \tag{B.1}$$

the kernel (or null space) of A,

$$\mathrm{Im}\ A := \{y \in Y \mid \exists\ x \in X \text{ such that } Ax = y\}, \tag{B.2}$$

the image of A. We will also need the quotient space

$$Y/\mathrm{Im}\ A := \{[y] = y + \mathrm{Im}\ A \mid y \in Y\}. \tag{B.3}$$

Also, let us denote by

$$LC(X,Y) := \{A : X \to Y \mid A \text{ is linear and continuous}\}, \tag{B.4}$$

the set of all linear and continuous operators defined on X with values in Y. This space is a normed space and its norm is given by

$$||A||_{LC(X,Y)} := \sup_{x\in X, ||x||_X \le 1} |\langle A, x\rangle|. \tag{B.5}$$

We have the following definition (see [143, Definition 9.1]).

Definition B.1. *Let $A \in LC(X,Y)$. Then, A is a Fredholm operator if*

(i) $n_0 := \dim(\mathrm{Ker}A) < \infty$.

(ii) $n_1 := \dim(\mathrm{coker}A) := \dim Y/\mathrm{Im}\ A < \infty$

(iii) $\mathrm{Im}\ A := A(X)$ *is a closed subspace of* X*.*

Moreover, we define the index of A by

$$\mathrm{ind}A := \dim(\mathrm{Ker}A) - \dim(\mathrm{coker}A) = n_0 - n_1 < \infty. \tag{B.6}$$

Note that if condition (*ii*) of Definition (B.1) holds, then condition (*iii*) of Definition (B.1) holds as well (see [143, Definition 9.1]).

Now, let us recall a useful definition.

Definition B.2. *Assume that X and Y are normed spaces and let $A : X \to Y$ be a linear operator. Then, $A : X \to Y$ is a compact operator if it maps bounded subsets of X into relatively compact subsets of Y.*

In the latter, we give some results that are involved in the proof of our well-posedness results in Chapter 2 and Chapter 3.

Lemma B.3. *Assume that X and Y are Banach spaces and let $A : X \to Y$ be a Fredholm operator. Let $K : X \to Y$ be a linear and compact operator. Then the operator $A + K : X \to Y$ is a Fredholm operator as well and* $\operatorname{ind}(A + K) = \operatorname{ind}A$.

Corollary B.4. *Assume that X and Y are Banach spaces. Let $K : X \to Y$ be a linear and compact operator. Then the operator $\mathbb{I} + K : X \to Y$ is a Fredholm operator of index zero.*

Corollary B.5. *Assume that X and Y are Banach spaces. Let $A : X \to Y$ be a Fredholm operator of index zero. Then, the operator $A : X \to Y$ is an isomorphism if either one of the following condition holds*

(i) A is injective

(ii) A is surjective.

Now, we consider X a real Banach space. The dual space of X, denoted by X' is given by

$$X' := LC(X, \mathbb{R}). \tag{B.7}$$

Note that, X' is also a Banach space and its norm is given by

$$||g||_{X'} = \sup_{x_0 \in X, ||x_0||_X = 1} |\langle g, x_0 \rangle|. \tag{B.8}$$

Assume that X and Y are Banach spaces. Let $A \in LC(X, Y)$. Then, the adjoint of $A \in LC(X, Y)$ is the operator $A' \in LC(Y', X')$ defined by

$$\langle Ax, y' \rangle = \langle x, B'y' \rangle, \ \forall \ x \in X, y' \in Y'. \tag{B.9}$$

Let us end this section by providing the following result.

Lemma B.6. *Let $A : X \to Y$ be a Fredholm operator. Then $A' : Y' \to X'$ is also a Fredholm operator and*

$$\operatorname{ind}A' = -\operatorname{ind}A. \tag{B.10}$$

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